1. (5-Lemma) If

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \alpha_2} \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

   is a commutative diagram of modules with exact rows, show that if β_1 and β_2 are isomorphisms, α_1 is an epimorphism, and α_2 is a monomorphism, then γ is an isomorphism. This is often applied when A_1, B_1, A_5 and B_5 are zero (short 5-lemma).

2. Suppose that the following diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A'' & \longrightarrow & B'' & \longrightarrow & C'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

   is a commutative diagram of modules with exact columns and exact middle row. Show that if either $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ or $0 \longrightarrow A'' \longrightarrow B'' \longrightarrow C'' \longrightarrow 0$ is exact, then both are.

3. Let $\alpha : F \longrightarrow G$ be a map of complexes and $M(\alpha)$ be the mapping cone of α. Show that the natural inclusion makes G into a subcomplex of $M(\alpha)$ and $M(\alpha)/G \cong F[-1]$, so that there is a short exact sequence

$$0 \longrightarrow G \longrightarrow M(\alpha) \longrightarrow F[-1] \longrightarrow 0$$

   of complexes.

4. Show that the following are equivalent.
   (i) B is an injective R-module.
   (ii) $\mathrm{Hom}_R(-, B)$ is an exact functor.
   (iii) $\mathrm{Ext}^i_R(A, B)$ vanishes for all $i \neq 0$ and all A.
   (iv) $\mathrm{Ext}^1_R(A, B)$ vanishes for all A.

5. Show that the following are equivalent for every R-module B.

   (i) B is flat.
   (ii) $\mathrm{Tor}_n^R(A, B) = 0$ for all $n \neq 0$ and all A.
   (iii) $\mathrm{Tor}_1^R(A, B) = 0$ for all A.

6. Show that if $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is exact and both B and C are flat, then A is also flat.

7. Show that the following are equivalent for every R-module B.

   (i) B is a flat R-module.
   (ii) $B^* = \mathrm{Hom}_{\mathbf{Z}}(B, \mathbf{Q}/\mathbf{Z})$ is an injective R-module.
   (iii) $I \otimes_R B \cong IB$ for every ideal I of R.
   (iv) $\mathrm{Tor}_1^R(R/I, B) = 0$ for every ideal I of R.