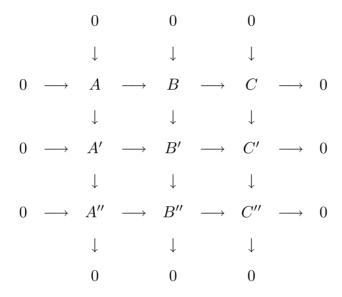
1. (5-Lemma) If

is a commutative diagram of modules with exact rows, show that if  $\beta_1$  and  $\beta_2$  are isomorphisms,  $\alpha_1$  is an epimorphism, and  $\alpha_2$  is a monomorphism, then  $\gamma$  is an isomorphism. This is often applied when  $A_1$ ,  $B_1$ ,  $A_5$  and  $B_5$  are zero (short 5-lemma).

2. Suppose that the following diagram



is a commutative diagram of modules with exact columns and exact middle row. Show that if either  $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$  or  $0 \longrightarrow A'' \longrightarrow B'' \longrightarrow C'' \longrightarrow 0$  is exact, then both are.

3. Let  $\alpha: F \longrightarrow G$  be a map of complexes and  $M(\alpha)$  be the mapping cone of  $\alpha$ . Show that the natural inclusion makes G into a subcomplex of  $M(\alpha)$  and  $M(\alpha)/G \cong F[-1]$ , so that there is a short exact sequence

$$0 \longrightarrow G \longrightarrow M(\alpha) \longrightarrow F[-1] \longrightarrow 0$$

of complexes.

- 4. Show that the following are equivalent.
  - (i) B is an injective R-module.
  - (ii)  $\operatorname{Hom}_R(-,B)$  is an exact functor.
  - (iii)  $\operatorname{Ext}_R^i(A, B)$  vanishes for all  $i \neq 0$  and all A.
  - (iv)  $\operatorname{Ext}_R^1(A,B)$  vanishes for all A.

- 5. Show that the following are equivalent for every R-module B.
  - (i) B is flat.
  - (ii)  $\operatorname{Tor}_n^R(A,B) = 0$  for all  $n \neq 0$  and all A. (iii)  $\operatorname{Tor}_1^R(A,B) = 0$  for all A.
- 6. Show that if  $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$  is exact and both B and C are flat, then A is also flat.
- 7. Show that the following are equivalent for every R-module B.
  - (i) B is a flat R-module.
  - (ii)  $B^* = \operatorname{Hom}_{\mathbf{Z}}(B, \mathbf{Q}/\mathbf{Z})$  is an injective R-module. (iii)  $I \otimes_R B \cong IB$  for every ideal I of R.

  - (iv)  $\operatorname{Tor}_{1}^{R}(R/I, B) = 0$  for every ideal I of R.