1. Suppose $R = A \times B$ is the product of two rings A and B. Let S be equal to $\{1\} \times B$. Show that $S^{-1}R$ is isomorphic to A.

2. Let $S = \mathbf{Z} - \{0\}$ and let M be the direct sum $\bigoplus_{n>1} \mathbf{Z}/n\mathbf{Z}$ where n is a natural number. Show that $S^{-1}M = 0$, that M is a faithful $\mathbf{Z}$-module and that the canonical mapping
$$S^{-1}\mathrm{End}_{\mathbf{Z}}(M) \longrightarrow \mathrm{End}_{S^{-1}\mathbf{Z}}(S^{-1}M)$$
is *not* injective.

3. Let $S = \mathbf{Z} - \{0\}$ and M be a free $\mathbf{Z}$-module with infinite basis. Show that the canonical mapping
$$S^{-1}\mathrm{End}_{\mathbf{Z}}(M) \longrightarrow \mathrm{End}_{S^{-1}\mathbf{Z}}(S^{-1}M)$$
is *not* surjective.

4. Let A be a non-zero ring and let Γ be the set of all multiplicatively closed subsets S of A such that $0 \notin S$. Show that Γ has maximal elements, and that $S \in \Gamma$ is maximal if and only if $A - S$ is a minimal prime ideal of A.

5. Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be ring homomorphisms. If $g \circ f$ is flat and g is faithfully flat, then f is flat.

6. Let A be a ring and $\mathfrak{p}$ be a prime ideal of A. Then the canonical image of $\mathrm{Spec}(A_{\mathfrak{p}})$ in $\mathrm{Spec}(A)$ is equal to the intersection of all the open neigborhoods of $\mathfrak{p}$ in $\mathrm{Spec}(A)$.

7. Let A be a ring, and let F be the A-module A^n. Let x_1, x_2, ..., x_n be a set of generators and let e_1, e_2, ..., e_n denote the canonical basis of F. Define $\phi : F \longrightarrow F$ by $\phi(e_i) = x_i$ and let N be the kernel of ϕ. Then we have the short exact sequence
$$0 \longrightarrow N \longrightarrow F \xrightarrow{\phi} F \longrightarrow 0.$$

   (a) Show that for a prime ideal $\mathfrak{p}$ of A, we have a short exact sequence of $A_{\mathfrak{p}}$-modules
$$0 \longrightarrow N_{\mathfrak{p}} \longrightarrow F_{\mathfrak{p}} \xrightarrow{\phi_{\mathfrak{p}}} F_{\mathfrak{p}} \longrightarrow 0.$$

   (b) Let P be the maximal ideal of $A_{\mathfrak{p}}$ and $k = A_{\mathfrak{p}}/P$ be its residue field. Show that if we tensor the short exact sequence in (a) with k, we still have a short exact sequence
$$0 \longrightarrow k \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}} \longrightarrow k \otimes_{A_{\mathfrak{p}}} F_{\mathfrak{p}} \xrightarrow{1\otimes\phi_{\mathfrak{p}}} k \otimes_{A_{\mathfrak{p}}} F_{\mathfrak{p}} \longrightarrow 0.$$

   (c) Use the short exact sequence in (b) to say that $k \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}} = 0$ and then use Nakayama's lemma to show that $N_{\mathfrak{p}} = 0$. Conclude that the map ϕ is an isomorphism and that every set of n generators of F is a basis of F.