ON FINITENESS CONJECTURES FOR MODULAR QUATERNION ALGEBRAS

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Abstract. It is conjectured that there exist finitely many isomorphism classes of simple endomorphism algebras of abelian varieties of GL_2-type over $\mathbb{Q}$ of bounded dimension. We explore this conjecture when particularized to quaternion endomorphism algebras of abelian surfaces by giving a moduli interpretation which translates the question into the diophantine arithmetic of Shimura curves embedded in Hilbert surfaces. We address the resulting problems on these curves by local and global methods, including Chabauty techniques on explicit equations of Shimura curves.

1. Introduction

An abelian variety A defined over $\mathbb{Q}$ is said to be *of* GL_2*-type over* $\mathbb{Q}$ if the endomorphism algebra $\mathrm{End}^0_{\mathbb{Q}}(A) = \mathrm{End}_{\mathbb{Q}}(A) \otimes \mathbb{Q}$ is a (necessarily totally real or complex multiplication) number field of degree $[\mathrm{End}^0_{\mathbb{Q}}(A) : \mathbb{Q}] = \dim A$. An abelian variety A is called *modular over* $\mathbb{Q}$ if it is a quotient of the Jacobian variety $J_1(N)$ of the modular curve $X_1(N)$ defined over $\mathbb{Q}$. If moreover A is simple over $\mathbb{Q}$, its modularity over $\mathbb{Q}$ is equivalent to the existence of an eigenform $f \in S_2(\Gamma_1(N))$ such that A is isogenous over $\mathbb{Q}$ to the abelian variety A_f attached by Shimura to f. As is well-known, all simple modular abelian varieties A over $\mathbb{Q}$ are of GL_2-type over $\mathbb{Q}$ and the generalized Shimura-Taniyama-Weil Conjecture predicts that the converse holds (cf. [29]).

We might ask what simple algebras B arise as the endomorphism algebra $\mathrm{End}^0_L(A)$ of an abelian variety A of GL_2-type over $\mathbb{Q}$ for some field extension $L/\mathbb{Q}$. As modular computations show, it might be predicted that these should not constitute a very large class of algebras (cf. also [9]). This is gathered in the following conjecture.

Conjecture 1.1. *Let $g \geq 1$ be a positive integer. The set $\mathcal{E}_g = \{\mathrm{End}^0_L(A)\}$ of isomorphism classes of endomorphism algebras of abelian varieties A of* GL_2*-type over $\mathbb{Q}$ of dimension g over any field extension $L/\mathbb{Q}$ is a finite set.*

The conjecture holds for $g = 1$: namely,

$$\mathcal{E}_1 = \{\mathbb{Q}\} \cup \{\mathbb{Q}(\sqrt{-d}) : d = 1, 2, 3, 7, 11, 19, 43, 67, 163\}.$$

On the other hand, the case $g \geq 2$ is completely open. It is known that if A is an abelian surface of GL_2-type over $\mathbb{Q}$ then, for any number field L, the endomorphism algebra $\mathrm{End}^0_L(A)$ is either a quadratic field, $\mathrm{M}_2(\mathbb{Q})$, $\mathrm{M}_2(K)$ for an imaginary

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quadratic field K, or a division quaternion algebra over $\mathbb{Q}$ (cf. [28], Proposition 1.1 and Theorem 1.3, [36]).

The aim of this article is to address the question for $g = 2$, and in particular for quaternion endomorphism algebras. We make the following definitions.

Definition 1.2. *Let B_D be a division quaternion algebra over $\mathbb{Q}$ of reduced discriminant D and let $m \neq 1$ be a square-free integer.*

(1) *The pair (D, m) is modular (resp. of GL_2-type) over $\mathbb{Q}$ if there exists an abelian surface A modular (resp. of GL_2-type) over $\mathbb{Q}$ such that $\mathrm{End}^0_{\overline{\mathbb{Q}}}(A) \simeq B_D$ and $\mathrm{End}^0_{\mathbb{Q}}(A) \simeq \mathbb{Q}(\sqrt{m})$.*

(2) *The quaternion algebra B_D is modular (resp. of GL_2-type) over $\mathbb{Q}$ if (D, m) is modular (resp. of GL_2-type) for some square-free integer m.*

As a first observation, we point out that if (D, m) is a pair of GL_2-type, then [12] yields that $m > 1$ and Shimura [36] proved that B_D is *indefinite*, i.e., that $B_D \otimes \mathbb{R} \simeq \mathrm{M}_2(\mathbb{R})$. We state a particular consequence of Conjecture 1.1 separately.

Conjecture 1.3. *The set of pairs (D, m) of GL_2-type over $\mathbb{Q}$ is finite.*

This conjecture implies that there are only finitely many quaternion algebras of GL_2-type over $\mathbb{Q}$ and a fortiori that there are only finitely many modular quaternion algebras over $\mathbb{Q}$ as well. It is also worth noting that, given a fixed quaternion algebra B_D, there are infinitely many real quadratic fields $\mathbb{Q}(\sqrt{m})$ that embed in B_D, since any field $\mathbb{Q}(\sqrt{m})$ with m such that $(\frac{m}{p}) \neq 1$ for all $p|D$ does.

One of the motivations for Conjecture 1.3 is computational. Let us agree to define the *minimal level* of a modular pair (D, m) as the minimal N such that there exists a newform $f \in S_2(\Gamma_0(N))$ with $(B_D, \mathbb{Q}(\sqrt{m})) \simeq (\mathrm{End}^0_{\overline{\mathbb{Q}}}(A_f), \mathrm{End}^0_{\mathbb{Q}}(A_f))$. The computations below are due to Koike and Hasegawa for $N \leq 3000$, and extended to $N \leq 5000$ by Clark and Stein.

Proposition 1.4. [1], [18] *The only modular pairs (D, m) of minimal level $N \leq 5000$ are:*

(D, m)	$(6, 2)$	$(6, 3)$	$(6, 6)$	$(10, 10)$	$(14, 7)$	$(15, 15)$
N	675	1568	243	2700	1568	> 3000

In Theorem 1.9 we show that, despite the finiteness Conjecture 1.3, the above are not the only examples of pairs (D, m) of GL_2-type over $\mathbb{Q}$. According to the generalized Shimura-Taniyama-Weil Conjecture in dimension two, the pairs of Theorem 1.9 below should actually be *modular* pairs.

On the other hand, it is remarkable that not a single example of a quaternion algebra B_D nor a pair (D, m) has ever been excluded from being modular or of GL_2-type over $\mathbb{Q}$. In this work we present the first examples, either obtained by local methods or by methods using global information.

The strategy followed in this paper is to prove that the condition for a pair (D, m) to be of GL_2-type over $\mathbb{Q}$ is equivalent to the existence of a point in a suitable subset of the set of rational points on an Atkin-Lehner quotient of the Shimura curve X_D attached to B_D, which we make fully explicit in Section 4.

The article is organized as follows. In Section 2, we give a moduli interpretation of Conjecture 1.3 which is based on the results in [32], [33]. By using this moduli interpretation, in Section 3 we obtain the following finiteness theorem.

Theorem 1.5. *If the pair (D, m) is of GL_2-type over $\mathbb{Q}$, then $m|D$ and the set of rational points on the quotient of the curve X_D by the Atkin-Lehner involution ω_m contains a non-Heegner point. In particular, for a given quaternion algebra B_D the set of pairs (D, m) of GL_2-type is finite.*

For the proof of Theorem 1.5 we refer to Theorem 3.1. In view of the above result, we wonder which pairs (D, m), $m|D$, are of GL_2-type and the first natural step in our programm is the consideration of the set of rational points on $X_D^{(m)}$ over the local completions of $\mathbb{Q}$.

The study of local points on $X_D^{(m)}$ over $\mathbb{Q}_p$ for primes $p \nmid D$ has been considered in [35] by means of trace formulae of Hecke operators. As we show in Section 3, Proposition 3.3 (i), this serves us to exhibit some isolated pairs $(p \cdot q, q)$ which are not of GL_2-type over $\mathbb{Q}$.

Over the field $\mathbb{R}$ of real numbers and the p-adic fields $\mathbb{Q}_p$ for $p|D$, this question was studied by Ogg in [27]. We obtain from his work and Theorem 1.5 the following result, which establishes that all pairs (D, m) for $m = D/p_1 \cdot \ldots \cdot p_k$, $k \geq 2$, are not of GL_2-type.

A stronger result is given in Proposition 3.3 (ii), (iii) and (iv). For the sake of neatness, we only give here the precise statement in the case that $D = p \cdot q$ is the product of two primes.

Proposition 1.6. *Let (D, m) be a pair of GL_2-type over $\mathbb{Q}$. Then either $m = D$ or $m = D/p$ for some $p|D$.*

Assume that $(D, m) = (p \cdot q, q)$.

- (i) *If p, $q \neq 2$, then $(\frac{q}{p}) = -1$ and either $p \equiv 3$ mod 4 or $p \equiv 5$ mod 12, $q \equiv 3$ mod 4.*
- (ii) *If $p = 2$, then $q \equiv 3$, 5 or 7 mod 8.*
- (iii) *If $q = 2$, then $p \equiv 3$ or 5 mod 8.*

It follows from Proposition 1.6 and the Čebotarev density Theorem that there actually exist infinitely many pairs $(p \cdot q, q)$ which are not of GL_2-type.

However, local methods are insufficient to prove that a given quaternion algebra B_D is not of GL_2-type over $\mathbb{Q}$. More precisely, due to Proposition 3.3 (v), it follows that only global approaches can be used to prove that the pairs (D, D) are not of GL_2-type over $\mathbb{Q}$, because the curves $X_D^{(D)}$ have rational points everywhere locally.

The second step is to study the set of global rational points in $X_D^{(m)}(\mathbb{Q})$ and to compare its cardinality with the number of rational Heegner points on that curve. This procedure allows us to exclude a pair (D, m) from being of GL_2-type over $\mathbb{Q}$ whenever the rational Heegner points exhaust all points in $X_D^{(m)}(\mathbb{Q})$.

By using the explicit equations of Atkin-Lehner quotients of Shimura curves in [16], [22], [23] and Chabauty methods using elliptic curves, we are able to prove that several quaternion algebras B_D and pairs (D, m) are not of GL_2-type over $\mathbb{Q}$, even in cases when the above purely local methods do not apply. These are collected in the following theorem.

Theorem 1.7.

- (i) *The quaternion algebra B_{155} is not of GL_2-type over $\mathbb{Q}$.*

(ii) *The pairs*

$$(D, m) \in \{(15,5), (21,3), (26,2), (33,11), (38,2), (46,23), (58,2), (91,91), (106,53), (118,59), (123,123), (142,2), (158,158), (202,101), (326,326), (446,446)\}$$

are not of GL_2*-type over* $\mathbb{Q}$.

The third step is to characterize the rational non-Heegner points $P \in X_D^{(m)}(\mathbb{Q})$ which can be represented by a polarized abelian surface defined over $\mathbb{Q}$. The solution to this problem depends on the vanishing of an obstruction $\xi_m(P)$ in the 2-torsion of the Brauer group of $\mathbb{Q}$. In Section 4, Theorem 4.3, we explicitly compute $\xi_m(P) \in \mathrm{Br}_2(\mathbb{Q})$ and thus give a necessary and sufficient condition for the existence of such an abelian surface. As an application, we obtain the following.

Theorem 1.8. *The pairs* $(D, m) = (10, 2)$*,* $(15, 3)$*,* $(26, 13)$ *and* $(38, 19)$ *are not of* GL_2*-type over* $\mathbb{Q}$*.*

Theorem 1.9.

(i) *The pairs*

$$(D, m) \in \{(6,2), (6,3), (6,6), (10,5), (10,10), (22,2), (22,11), (22,22), (14,7), (14,14), (15,3), (15,15), (21,21), (33,33), (46,46), (26,26), (38,38), (58,29), (58,58)\}$$

are of GL_2*-type over* $\mathbb{Q}$*.*

(ii) *For the pairs*

$$(D, m) \in \{(6,2), (6,3), (6,6), (10,5), (10,10), (14,14), (15,15), (21,21), (22,2), (22,11), (22,22), (33,33), (46,46)\},$$

there exist infinitely many nonisomorphic abelian surfaces A *defined over* $\mathbb{Q}$ *such that* $\mathrm{End}^0_{\mathbb{Q}}(A) = \mathbb{Q}(\sqrt{m})$ *and* $\mathrm{End}_{\overline{\mathbb{Q}}}(A) = \mathcal{O}_D$ *is a maximal order in* B_D*.*

(iii) *There exist exactly two abelian surfaces* $A/\mathbb{Q}$ *with* $\mathrm{End}^0_{\mathbb{Q}}(A) = \mathbb{Q}(\sqrt{7})$ *and* $\mathrm{End}_{\overline{\mathbb{Q}}}(A) = \mathcal{O}_{14}$ *up to isomorphism over* $\overline{\mathbb{Q}}$*.*

It is remarkable that in the cases $(D, m) = (10, 2)$ and $(15, 3)$, the curve $X_D^{(m)}$ is isomorphic to $\mathbb{P}^1_{\mathbb{Q}}$ and therefore there are infinitely many non-Heegner rational points in $X_D^{(m)}(\mathbb{Q})$. However, it turns out that $\xi_m(P) \neq 1$ for all these points (cf. Section 5) and thus (D, m) is not of GL_2-type over $\mathbb{Q}$.

Remark 1.10. All pairs (D, m) for $D \leq 33$ are covered by the above results.

2. Towers of Shimura curves and Hilbert surfaces

Let $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$ be the algebraic group over $\mathbb{R}$ obtained by restriction of scalars of the multiplicative group. A Shimura datum is a pair (G, X), where G is a connected reductive affine algebraic group over $\mathbb{Q}$ and X is a $G(\mathbb{R})$-conjugacy class in the set of morphisms of algebraic groups $\mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$, as in [25].

Let $\mathbb{A}_f$ denote the ring of finite adeles of $\mathbb{Q}$. For any compact open subgroup U of $G(\mathbb{A}_f)$, let

$$\mathrm{Sh}_U(G, X)(\mathbb{C}) = G(\mathbb{Q}) \setminus (X \times G(\mathbb{A}_f))/U,$$

which has a natural structure of quasi-projective complex algebraic variety, that we may denote by $\mathrm{Sh}_U(G, X)_{\mathbb{C}}$.

Let (G, X) and (G', X') be two Shimura data and let U, U' be compact open subgroups of $G(\mathbb{A}_f)$ and $G'(\mathbb{A}_f)$, respectively. A morphism $f : G \to G'$ of algebraic groups which maps X into X' and U into U' induces a morphism

$$\mathrm{Sh}_f : \mathrm{Sh}_U(G, X)_{\mathbb{C}} \to \mathrm{Sh}_{U'}(G', X')_{\mathbb{C}}$$

of algebraic varieties.

In this section, we consider two particular instances of Shimura varieties: Shimura curves attached to an indefinite quaternion algebra and Hilbert surfaces attached to a real quadratic number field.

2.1. **Shimura curves.** Let B_D be an indefinite quaternion algebra over $\mathbb{Q}$ of reduced discriminant D and fix an isomorphism $\Phi : B_D \otimes \mathbb{R} \xrightarrow{\simeq} \mathrm{M}_2(\mathbb{R})$. Let $\mathcal{O}_D \subset B_D$ be a maximal order and let $G/\mathbb{Z}$ be the group scheme $\mathcal{O}_D^*$. We have that $G(\mathbb{Q}) = B_D^*$ and $G(\mathbb{A}_f) = \prod_p (\mathcal{O}_D)_p^*$, where for any prime number p, we let $\mathcal{O}_p = \mathcal{O} \otimes \mathbb{Z}_p$. Let $X = \mathbb{H}^{\pm}$ be the $\mathrm{GL}_2(\mathbb{R})$-conjugacy class of the map $a+bi \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. As complex analytical spaces, $\mathbb{H}^{\pm}$ is the union of two copies of Poincaré's upper half plane $\mathbb{H}$.

For any compact open subgroup U of $G(\mathbb{A}_f)$, let $X_{U,\mathbb{C}} = \mathrm{Sh}_U(G, X)_{\mathbb{C}}$ be the Shimura curve attached to the Shimura datum (G, X) and U. It is the union of finitely many connected components of the form $\Gamma_i \setminus \mathbb{H}$, where Γ_i are discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$.

Let $P \mapsto \overline{P}$ denote the conjugation map on B_D. Fix a choice of an element $\mu \in B_D^*$ such that $\mu^2 + \delta = 0$ for some $\delta \in \mathbb{Q}^*, \delta > 0$ and let $\varrho_\mu : B_D \to B_D$, $\beta \mapsto \mu^{-1}\bar{\beta}\mu$. For any scheme S over $\mathbb{Q}$, let $\mathcal{F}_{\mathcal{O}_D,\mu}(S)$ be the set of isomorphism classes of $(A, \iota, \mathcal{L}, \nu)$, where A is an abelian scheme over S, $\iota : \mathcal{O}_D \hookrightarrow \mathrm{End}_S(A)$ is a ring monomorphism, $\mathcal{L}$ is a polarization on A such that the Rosati involution $* : \mathrm{End}_S(A) \otimes \mathbb{Q} \to \mathrm{End}_S(A) \otimes \mathbb{Q}$ is ϱ_μ on B_D and ν is an U-level structure on A (cf. [2], p. 128). As is well known, $X_{U,\mathbb{C}}$ coarsely represents the moduli functor $\mathcal{F}_{\mathcal{O}_D,\mu}$.

A point $[A, \iota, \mathcal{L}, \nu] \in X_{U,\mathbb{C}}(\mathbb{C})$ is called a *Heegner point* or a *CM point* if ι is not surjective or, equivalently, if A is isogenous to the square of an elliptic curve with complex multiplication.

The modular interpretation implies that the reflex field of the Shimura datum (G, X) is $\mathbb{Q}$ and that $X_{U,\mathbb{C}}$ admits a canonical model $X_{U,\mathbb{Q}}$ over $\mathbb{Q}$, which is the coarse moduli space for any of the above moduli functors $\mathcal{F}_{\mathcal{O}_D,\mu}$ extended to arbitrary bases over $\mathbb{Q}$ (cf. [2], [25, Section 2]). We remark that the algebraic curve $X_{U,\mathbb{Q}}$ does not depend on the choice of $\mu \in B_D^*$, although its moduli interpretation does.

As a particular case, let $\mathcal{O} \subset \mathcal{O}_D$ be an integral order contained in $\mathcal{O}_D$, and let $\hat{\mathcal{O}}^* = \prod_p \mathcal{O}_p^*$. Let us simply denote by $X_{\mathcal{O},\mathbb{Q}}$ the Shimura curve $X_{\hat{\mathcal{O}}^*,\mathbb{Q}}$. Again, for any fixed $\mu \in B_D^*$, $\mu^2 + \delta = 0$, it admits the following alternative modular interpretation: $X_{\mathcal{O},\mathbb{Q}}$ coarsely represents the functor $\hat{\mathcal{F}}_{\mathcal{O},\mu} :$ Sch/$\mathbb{Q} \to$ Sets , sending a scheme S over $\mathbb{Q}$ to the set of isomorphism classes of triplets $(A, \iota, \mathcal{L})$, where $(A, \mathcal{L})$ is a polarized abelian scheme over S as above and $\iota : B_D \hookrightarrow \mathrm{End}_S(A) \otimes \mathbb{Q}$ is a ring monomorphism such that $\iota(B_D) \cap \mathrm{End}_S(A) = \iota(\mathcal{O})$.

2.2. **Hilbert surfaces.** Let F be a real quadratic extension of $\mathbb{Q}$, let R_F be its ring of integers and let G be the $\mathbb{Z}$-group scheme $\mathrm{Res}_{R_F/\mathbb{Z}}(\mathrm{GL}_2(R_F))$. Since $F \otimes \mathbb{R} \simeq \mathbb{R}^2$, we have $G(\mathbb{R}) = \mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R})$. Let $X = \mathbb{H}^{\pm} \times \mathbb{H}^{\pm}$. For any compact open subgroup U of $G(\mathbb{A}_f)$, let $H_{U,\mathbb{C}} = \mathrm{Sh}_U(G,X)_{\mathbb{C}}$ be the Hilbert surface attached to the Shimura datum (G,X) and U.

The Hilbert surface $H_{U,\mathbb{C}}$ admits, in the same way as $X_{U,\mathbb{C}}$, a canonical model $H_{U,\mathbb{Q}}$ over $\mathbb{Q}$ which is the coarse moduli space of abelian surfaces $(A, j, \mathcal{L}, \nu)$ together with a ring homomorphism $j : R_F \to \mathrm{End}(A)$, a polarization $\mathcal{L}$ on A such that $*_{|j(R_F)}$ is the identity map and a U-level structure. When U is the restriction of scalars of $\mathrm{GL}_2(\hat{R})$ for a given quadratic order $R \subseteq R_F$, we write $H_{R,\mathbb{Q}} := \mathrm{Sh}_U(G,X)_{\mathbb{Q}}$. As in the Shimura curve case, $H_{R,\mathbb{Q}}$ can also be regarded as the coarse moduli space of polarized abelian surfaces with real multiplication by R and no level structure.

2.3. **Forgetful maps.** We consider various forgetful maps between Shimura curves and Hilbert surfaces with level structure.

For any integral quaternion order $\mathcal{O}$ of B_D, let $\hat{\mathcal{O}}^* \subseteq \hat{\mathcal{O}}_D^*$ be the natural inclusion of compact groups. The identity map on the Shimura data $(\mathcal{O}_D^*, \mathbb{H}^{\pm})$ induces a morphism

$$X_{\mathcal{O},\mathbb{Q}} \longrightarrow X_{\mathcal{O}_D,\mathbb{Q}}$$

which can be interpreted in terms of moduli as forgetting the level structure: $[A, \iota, \mathcal{L}, \nu] \mapsto [A, \iota, \mathcal{L}]$.

Similarly, for any quadratic order R of F, there is a natural morphism

$$H_{R,\mathbb{Q}} \longrightarrow H_{R_F,\mathbb{Q}}.$$

Finally, let $R \subset \mathcal{O}$ be a real quadratic order *optimally embedded* in $\mathcal{O}$, which means that $R = F \cap \mathcal{O}$, and fix an element $\mu \in B_D^*$, $\mu^2 + \delta = 0$, $\delta \in \mathbb{Q}^*$, $\delta > 0$ symmetric with respect to R (that is, $\varrho_\mu | R = 1_R$). Regard X_D as representing the moduli functor $\hat{\mathcal{F}}_{\mathcal{O},\mu}$. Attached to the pair (R, μ) there is a distinguished forgetful morphism

$$\begin{array}{cccc} \pi_{(R,\mu)} : & X_{\mathcal{O},\mathbb{Q}} & \longrightarrow & H_{R,\mathbb{Q}} \\ & [A, \iota : \mathcal{O} \to \mathrm{End}(A), \mathcal{L}] & \mapsto & [A, \iota_{|R} : R \to \mathrm{End}(A), \mathcal{L}] \end{array}$$

of Shimura varieties which consists on forgetting the ring endomorphism structure in the moduli interpretation of these varieties.

Let R' be a quadratic order of F optimally embedded in $\mathcal{O}_D$. Writing $R = R' \cap \mathcal{O}$, we obtain the following commutative diagram.

$$\begin{array}{ccc} X_{\mathcal{O},\mathbb{Q}} & \longrightarrow & X_{\mathcal{O}_D,\mathbb{Q}} \\ \pi_{(R,\mu)} \downarrow & & \downarrow \pi_{(R',\mu)} \\ H_{R,\mathbb{Q}} & \longrightarrow & H_{R',\mathbb{Q}} \end{array}$$

The main consequence we wish to derive from the above with regard to the problem posed in Section 1 is the following.

Let A/K be an abelian surface over a number field K with quaternionic multiplication over $\overline{\mathbb{Q}}$ by a quaternion algebra B_D. Let $\mathcal{O} \simeq \mathrm{End}_{\overline{\mathbb{Q}}}(A)$. Assume further that $\mathrm{End}_K(A) \otimes \mathbb{Q}$ is a real quadratic number field F and let $R = \mathrm{End}_K(A)$; as was remarked in Section 1, this is always the case for modular abelian surfaces over $\mathbb{Q}$. By construction, the order R is optimally embedded in $\mathcal{O}$. Since A is projective over K, it admits a (possibly non-principal) polarization $\mathcal{L}$ defined over K. Let $*$ denote the Rosati involution on B_D attached to $\mathcal{L}$. It follows from [31], pp. 6–7, that $* = \varrho_\mu$ for some $\mu \in B_D^*$ with $\mu^2 + \delta = 0$ for some $\delta \in \mathbb{Q}^*$, $\delta > 0$. By choosing an explicit isomorphism $\iota : \mathcal{O} \xrightarrow{\sim} \mathrm{End}_{\overline{\mathbb{Q}}}(A)$, the triplet $(A, \iota, \mathcal{L})$ produces a point P

in $X_{\mathcal{O},\mathbb{Q}}(\overline{\mathbb{Q}})$, when we regard the Shimura curve as coarsely representing the functor $\hat{\mathcal{F}}_{\mathcal{O},\mu}$.

Moreover, we have $\iota_{|R} : R \simeq \mathrm{End}_K(A)$. ¿From the fact that $\mathcal{L}$ is defined over K, it follows that $*_{|R}$ is an anti-involution on R. Since R is totally real, it follows that $*_{|R}$ is the identity. Hence, the point P is mapped to a point $P_R \in H_{R,\mathbb{Q}}(K)$ by the forgetful map $\pi_{(R,\mu)} : X_{\mathcal{O},\mathbb{Q}} \to H_{R,\mathbb{Q}}$.

Let $\mathcal{O}_D$ be a maximal order in B_D containing $\mathcal{O}$ and let $R' = F \cap \mathcal{O}_D$, where we regard $F = R \otimes \mathbb{Q}$ as naturally embedded in B_D. By the above commutative diagram of Shimura varieties, we obtain a point $P_{R'} \in H_{R',\mathbb{Q}}(K)$ which lies in the image of the forgetful map $X_{\mathcal{O}_D,\mathbb{Q}} \to H_{R',\mathbb{Q}}$.

The above discussion yields the following proposition.

Proposition 2.1. *Let B_D be an indefinite division quaternion algebra over $\mathbb{Q}$, let $\mathcal{O}_D$ be a maximal order and let $F = \mathbb{Q}(\sqrt{m})$ for some square-free integer $m > 1$.*

Assume that, for any order R of F, optimally embedded in $\mathcal{O}_D$, and $\mu \in B_D^*$ symmetric with respect to R, the set of rational points of $\pi_{(R,\mu)}(X_{\mathcal{O}_D,\mathbb{Q}})$ in the Hilbert surface $H_{R,\mathbb{Q}}$ consists entirely of Heegner points. Then, the pair (D, m) is not of GL_2-type over $\mathbb{Q}$.

If this holds for (D, m) for all square-free integers $m > 1$, then B_D is not of GL_2-type over $\mathbb{Q}$.

3. Atkin-Lehner quotients of Shimura curves

Fix a maximal order $\mathcal{O}_D$ in an indefinite division quaternion algebra B_D and let us simply denote $X_D = X_{\mathcal{O}_D,\mathbb{Q}}$. The curve X_D is equipped with a group of automorphisms $W \subseteq \mathrm{Aut}(X_D)$ which is called the *Atkin-Lehner group.* As an abstract group, $W = \mathrm{Norm}_{B_D^*}(\mathcal{O}_D)/(\mathbb{Q}^* \cdot \mathcal{O}_D^*)$ and we have $W \simeq (\mathbb{Z}/2\mathbb{Z})^{2r}$, where $2r = \#\{p \text{ prime} : p|D\}$ is the number of ramified primes of D. A full set of representatives of W is $\{\omega_m : m|D, m > 0\}$, where $\omega_m \in \mathcal{O}_D$, $\mathrm{n}(\omega_m) = m$. As elements of W, these satisfy $\omega_m^2 = 1$ and $\omega_m \cdot \omega_n = \omega_{mn}$ for any two coprime divisors m, $n|D$ (cf. [30]). Recall that we write $X_D^{(m)}$ for the quotient curve $X_D/\langle\omega_m\rangle$.

The following result relates the condition for a quaternion algebra B_D to be of GL_2-type to the diophantine arithmetic of the Atkin-Lehner quotients of X_D.

Theorem 3.1. *Let $m > 1$ be a square-free integer. Assume that the pair (D, m) is of GL_2-type over $\mathbb{Q}$. Then,*

(i) *$m|D$ and all prime divisors $p|\frac{D}{m}$ do not split in $\mathbb{Q}(\sqrt{m})$.*
(ii) *$X_D^{(m)}(\mathbb{Q})$ contains a non-Heegner point.*

Proof. Assume that (D, m) is of GL_2-type over $\mathbb{Q}$. By Proposition 2.1, there exists an order R of $F = \mathbb{Q}(\sqrt{m})$ optimally embedded in $\mathcal{O}_D$ and $\mu \in B_D^*$ symmetric with respect to R such that the set of rational points of $\pi_{(R,\mu)}(X_{\mathcal{O}_D,\mathbb{Q}})$ in the Hilbert surface $H_{R,\mathbb{Q}}$ contains a non-Heegner point.

It was shown in [32], Theorem 4.4, [33], Section 6, that if $m|D$ then there is a birational equivalence

$$\pi_{(R,\mu)}(X_{\mathcal{O}_D,\mathbb{Q}}) \overset{\sim}{\dashrightarrow} X_D^{(m)}$$

and if $m \nmid D$, there is a birational equivalence

$$\pi_{(R,\mu)}(X_{\mathcal{O}_D,\mathbb{Q}}) \overset{\sim}{\dashrightarrow} X_D.$$

These birational morphisms are defined over $\mathbb{Q}$ and become isomorphisms when restricted to the set of non-Heegner points.

Since $X_D(\mathbb{R}) = \emptyset$ (by [38]), it follows that $m|D$ and $X_D^{(m)}(\mathbb{Q})$ contains a non-Heegner point. Moreover, since F must embed in B, it follows from the theory of optimal embeddings that all primes $p|D$ do not split in F. □

As an immediate consequence of (i), we obtain the following corollary.

Corollary 3.2. *Given a discriminant D of a division quaternion algebra over $\mathbb{Q}$, the set of modular pairs (D, m) is finite.*

In view of Theorem 3.1, the diophantine properties of these curves are crucial for the understanding of Conjecture 1.3. We first study under what circumstances the curve $X_D^{(m)}$ has no points over some completion of $\mathbb{Q}$.

Part (i) of the proposition below follows from [35]. Parts (ii), (iii) and (iv) follow from [27] and [35]. Part (v) has been shown in [9] by using supersingular abelian surfaces and also in [35] by means of trace formulae of Hecke operators.

Proposition 3.3. *Let $X_D^{(m)}$ be as above and let ℓ and p_i denote a prime number.*

(i) *Let $\ell \nmid D$. If there exists no imaginary quadratic field K that splits B_D and contains an integral element of norm ℓ or ℓm, then $X_D^{(m)}(\mathbb{Q}_\ell) = \emptyset$.*

(ii) *If $m = D/(p_1 \cdot \ldots \cdot p_k)$, $k \geq 2$, then $X_D^{(m)}(\mathbb{Q}_v) = \emptyset$ for some place $v|D \cdot \infty$.*

(iii) *If $m = D/p$ for p an odd prime satisfying one the following conditions:*
- $(\frac{m}{p}) = 1$
- $\exists \ell | m$: *(a)* $(\frac{-p}{\ell}) = 1$ *or* $(\frac{-m}{p}) \neq 1$ *and (b)* $\ell \equiv 1$ *or* $p \equiv 3$ *mod* 4
- $\exists \ell | m$: *(a)* $(\frac{-m/\ell}{p}) = 1$ *and (b)* $m \neq \ell$ *or* $p \equiv 1$ *mod* 12
- D *is even and* $\exists \ell | \frac{D}{2}$: $\ell \equiv 1$ *mod* 4 *and* $(\frac{-m/2}{\ell}) = 1$,

then $X_D^{(m)}(\mathbb{Q}_v) = \emptyset$ for some place $v|D \cdot \infty$.

(iv) *If $m = D/2$ satisfies one the following conditions:*
- $(\frac{m}{2}) = 1$
- $\exists \ell | \frac{D}{2}$: $\ell \equiv 1$ *mod* 8
- $\exists \ell | \frac{D}{2}$: $(\frac{-m/\ell}{2}) = 1$,

then $X_D^{(m)}(\mathbb{Q}_v) = \emptyset$ for some place $v|D \cdot \infty$.

(v) *$X_D^{(D)}(\mathbb{Q}_v) \neq \emptyset$ for all places v of $\mathbb{Q}$.*

When $D = p \cdot q$ is the product of two primes, the congruence conditions of parts (iii) and (iv) above simplify notably and they were stated in Proposition 1.6 for the convenience of the reader. As a consequence of part (v), we conclude that only global methods may enable us to prove that any pair (D, D) is not of GL_2-type.

Finally we remark that part (i) of the above result allows us to produce examples of pairs like $(159, 3), (215, 43), (591, 3)$ and $(1247, 43)$ which are not of GL_2-type over $\mathbb{Q}$ and are not covered by the statements (ii)-(v).

4. A descent theorem on rational models of abelian surfaces with quaternionic multiplication

Let X_D be the Shimura curve of discriminant D and let the curve $X_D^{(m)}$ be as in the previous section. Let R denote the ring of integers of $\mathbb{Q}(\sqrt{m})$. For any choice of $d > 0$ such that $B_D \simeq (\frac{-Dd,m}{\mathbb{Q}})$, we may regard $X_D^{(m)}$ as parameterizing abelian surfaces $(A, \iota : R \hookrightarrow \mathrm{End}(A), \mathcal{L})$ with real multiplication by R and a polarization $\mathcal{L}$ of degree d symmetric with respect to ι. A point $P = [A, \iota : R \hookrightarrow \mathrm{End}(A), \mathcal{L}] \in X_D^{(m)}$ is rational over a field K if and only if the field of moduli of $(A, \iota, \mathcal{L})$ is K. However, $(A, \iota, \mathcal{L})$ may or may not admit a rational model over K.

The obstruction $\xi_m(P)$ for a point $P \in X_D^{(m)}(K)$ to be represented by a triplet $(A, \iota : R \hookrightarrow \mathrm{End}(A), \mathcal{L})$ defined over K lies in the two torsion part of the Brauer group $\mathrm{Br}(K) = H^2(G_K, \bar{K}^*)$, where G_K stands for the absolute Galois group over K. According to this, the set of rational points on $X_D^{(m)}$ over $\mathbb{Q}$ may be described as

$$X_D^{(m)}(\mathbb{Q}) = \bigcup_{\xi \in \mathrm{Br}_2(\mathbb{Q})} X_D^{(m)}(\mathbb{Q})_\xi,$$

where we let $X_D^{(m)}(\mathbb{Q})_\xi = \{P \in X_D^{(m)}(\mathbb{Q}) : \xi_m(P) = \xi\}$.

Remark 4.1. The obstruction $\xi_m(P)$ of a point $P = [A, \iota : R \hookrightarrow \mathrm{End}(A), \mathcal{L}] \in X_D^{(m)}(\mathbb{Q})$ only depends on the polarized abelian surface $(A, \mathcal{L})$. Indeed, assume that $(A, \mathcal{L})$ is defined over $\mathbb{Q}$. Let $\tilde{P} = [A, \iota : \mathcal{O}_D \hookrightarrow \mathrm{End}(A), \mathcal{L}] \in X_D(K)$ be the preimage of P in X_D for a certain imaginary quadratic field K. Let $\sigma \in G_{\mathbb{Q}}$ denote the complex conjugation. Since $\omega_m(\tilde{P}) = [A, \omega_m^{-1}\iota\omega_m, \mathcal{L}]$, we have that

$$\mathrm{End}^0_{\mathbb{Q}}(A) = \{\alpha \in \mathrm{End}^0_K(A) : \alpha^\sigma = \alpha\} = \{\alpha \in \mathrm{End}^0_K(A) : \omega_m^{-1}\alpha\omega_m = \alpha\} \simeq \mathbb{Q}(\sqrt{m})\,.$$

This implies that $\iota : R \hookrightarrow \mathrm{End}_{\mathbb{Q}}(A)$.

Let $X_D^{(m)}(\mathbb{Q})_h$ denote the set of rational Heegner points on $X_D^{(m)}$ and let us denote by $X_D^{(m)}(\mathbb{Q})_{nh} = X_D^{(m)}(\mathbb{Q}) \backslash X_D^{(m)}(\mathbb{Q})_h$ the set of rational non-Heegner points.

Definition 4.2. *We define the set of descent points on $X_D^{(m)}$ to be*

$$X_D^{(m)}(\mathbb{Q})_d = X_D^{(m)}(\mathbb{Q})_{\xi=1} \cap X_D^{(m)}(\mathbb{Q})_{nh}.$$

Set

$$r_m := \# X_D^{(m)}(\mathbb{Q})\,, \quad rh_m := \# X_D^{(m)}(\mathbb{Q})_h\,, \quad rd_m := \# X_D^{(m)}(\mathbb{Q})_d\,.$$

Note that r_m and rd_m may be $+\infty$. It follows from the above discussion that the pair (D, m) is of GL_2-type over $\mathbb{Q}$ if and only if $X_D^{(m)}(\mathbb{Q})_d \neq \emptyset$.

In [26], Murabayashi proved a descent result for principally polarized abelian surfaces with quaternionic multiplication under certain hypotheses. We give an alternative proof of his result that allows us to generalize it to arbitrarily polarized abelian surfaces and which is unconditionally valid.

Theorem 4.3. *Let $P \in X_D(K)$ be a non-Heegner point over an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-\delta})$ such that $\pi(P) \in X_D^{(m)}(\mathbb{Q})$ and write*

$$\xi_m(P) := (\frac{-\delta, m}{\mathbb{Q}}) \otimes B_D.$$

If $\xi_m(P)$ vanishes, then there is a polarized abelian surface $(A, \mathcal{L})$ defined over $\mathbb{Q}$ together with an isomorphism $\iota : \mathcal{O}_D \xrightarrow{\simeq} \mathrm{End}_K(A)$ such that $P = [A, \iota, \mathcal{L}]$. In that case, we have $\mathrm{End}^0_{\mathbb{Q}}(A) = \mathbb{Q}(\sqrt{m})$.

Proof. If there exists a polarized abelian variety $(A, \mathcal{L})$ defined over $\mathbb{Q}$ such that $P = [A, \iota, \mathcal{L}]$, then we know (cf. [29], [26] when $\deg(\mathcal{L}) = 1$) that $B_D \simeq (\frac{-\delta, m}{\mathbb{Q}})$ and hence $\xi_m(P) = \mathrm{M}_2(\mathbb{Q}) = [1] \in \mathrm{Br}(\mathbb{Q})$.

Let us assume that $B_D \simeq (\frac{-\delta, m}{\mathbb{Q}})$. Choose $\mu, \omega \in \mathcal{O}_D$, $\mu^2 = -\delta$, $\omega^2 = m$, $\mu\omega = -\omega\mu$ and let $R = \mathbb{Q}(\omega) \cap \mathcal{O}_D$.

Since $K = \mathbb{Q}(\sqrt{-\delta})$ embeds in B_D, it follows from [20], Ch. 2, that the point P may be represented by the $\overline{\mathbb{Q}}$-isomorphism class of a polarized simple abelian surface with quaternionic multiplication $(A_0, \iota_0, \mathcal{L}_0)$ completely defined over K and such that the Rosati involution that $\mathcal{L}_0$ induces on B_D is ϱ_μ. We stress that $\mathcal{L}_0$ will not always be principal.

Since ϱ_μ is symmetric with respect to R, the forgetful morphism $\pi_{(R,\mu)} : X_D \to H_{R,\mathbb{Q}}$ is the composition of the projection $\pi : X_D \to X_D^{(m)}$ and an immersion of $X_D^{(m)}$ into $H_{R,\mathbb{Q}}$, at least when we restrict these morphisms to the respective dense subsets of non-Heegner points. Hence, because $\pi(P) \in X_D^{(m)}(\mathbb{Q})$ is a non-Heegner point, this amounts to saying that the field of moduli of the triplet $(A_0, \iota_{0|_R}, \mathcal{L}_0)$ is $\mathbb{Q}$.

Let $\sigma \in G_{\mathbb{Q}} \backslash G_K$. Then there exists an isomorphism $\nu : A_0 \to A_0^\sigma$ such that $\nu^*(\mathcal{L}_0^\sigma) = \mathcal{L}_0$ and $\nu \cdot \omega^{-1} \cdot \alpha \cdot \omega = \alpha^\sigma \cdot \nu$ for all endomorphisms $\alpha \in B_D = \mathrm{End}_K^0(A)$. In particular,

$$\nu \cdot \omega = \omega^\sigma \cdot \nu\,, \quad \nu \cdot \mu = -\mu^\sigma \cdot \nu\,.$$

We split the proof into two parts.

Step 1: We show that ν may be assumed to be defined over K.

To prove this claim, note that $\mathrm{Aut}_K(A_0, \mathcal{L}_0) = \{\pm 1\}$ (cf.[26] or [31], Theorem 2.2) and let $\rho_\nu : G_K \to \mathrm{Aut}_K(A_0, \mathcal{L}_0) = \{\pm 1\}$ be the group homomorphism defined by $\rho_\nu(\tau) = \nu^{-1} \cdot \nu^\tau$.

Suppose that ν was not defined over K, that is $\rho_\nu(G_K) = \{\pm 1\}$. Let L/K be the quadratic extension such that $G_L = \ker \rho_\nu$. Since L is the minimal field of definition of all homomorphisms in $\mathrm{Hom}(A_0, A_0^\sigma)$ and $\mathrm{Hom}(A_0^\sigma, A_0)$, we deduce that $L/\mathbb{Q}$ is a Galois extension. Since K is imaginary, $L/\mathbb{Q}$ can not be cyclic and there exists a square-free integer $d > 1$ such that $L = K(\sqrt{d})$.

It follows from the Skolem-Noether Theorem (cf. [40], p. 6) that we may choose basis of $V_K = H^0(A_0, \Omega^1_{A_0/K})$ such that the matrix expressions of ω^* and μ^* acting on V_K are

$$M_m = \begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}, \quad M_\delta = \begin{pmatrix} \sqrt{-\delta} & 0 \\ 0 & -\sqrt{-\delta} \end{pmatrix},$$

respectively.

Let $N \in \mathrm{GL}_2(K(\sqrt{d}))$ be the matrix expression of $\nu \in \mathrm{Hom}(A_0, A_0^\sigma)$ with respect to this basis of V_K and its Galois conjugate of V_K^σ. Then N satisfies

$$N^\tau = -N\,, \quad M_m \cdot N = N \cdot M_m^\sigma = N \cdot M_m\,, \quad M_\delta \cdot N = -N \cdot M_\delta^\sigma = N \cdot M_\delta\,,$$

for $\tau \in G_K \backslash G_L$. Hence, $N = \sqrt{d} \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix}$, $\beta \in K$. Fix σ in $G_{\mathbb{Q}(\sqrt{d})}$, $\sigma \notin G_K$. We have $\nu^\sigma \cdot \nu \in \mathrm{Aut}(A_0, \mathcal{L}_0) = \{\pm 1\}$, thus $N \cdot N^\sigma = \pm\,\mathrm{id}$ and $\beta \cdot \beta^\sigma = 1/d$. Hence, the normal closure F of $K(\sqrt{\beta})/\mathbb{Q}$ is dihedral containing $K(\sqrt{d})$ and $F/\mathbb{Q}(\sqrt{-d \cdot \delta})$ is cyclic. Let $\rho_\beta : G_K \to \{\pm 1\}$ be the surjective morphism such that $\ker \rho_\beta = G_{K(\sqrt{\beta})}$. Attached to the cocycle $\rho_\beta \in H^1(G_K, \{\pm 1\})$ there is an abelian surface A_1 defined over K together with an isomorphism $\lambda : A_0 \to A_1$ such that $\lambda^\tau = \lambda \cdot \rho_\beta(\tau)$. We claim that $\phi = \lambda^\sigma \cdot \nu \cdot \lambda^{-1} : A_1 \to A_1^\sigma$ is defined over K. Indeed, for any $\tau \in G_K$,

$$\phi^\tau = (\lambda^{\sigma\cdot\tau\cdot\sigma^{-1}})^\sigma \cdot \nu^\tau \cdot (\lambda^{-1})^\tau = \rho_\beta(\sigma\cdot\tau\cdot\sigma^{-1}\cdot\tau^{-1}) \cdot \rho_\nu(\tau) \cdot \phi\,.$$

Since $\sigma \cdot \tau \cdot \sigma^{-1} \cdot \tau^{-1} \in G_{K(\sqrt{\beta})}$ if and only if $\tau \in G_{K(\sqrt{d})}$, we obtain that $\phi^\tau = \phi$.

Moreover, all endomorphisms of A_1 are of the form $\lambda\cdot\varphi\cdot\lambda^{-1}$ with φ in $\mathrm{End}_K(A_0)$. These are all defined over K because $(\lambda\cdot\varphi\cdot\lambda^{-1})^\tau = \delta_\beta(\tau\cdot\tau^{-1})\lambda\cdot\varphi\cdot\lambda^{-1} = \lambda\cdot\varphi\cdot\lambda^{-1}$.

We therefore assume that ν is defined over K.

Step 2: We show that A_0 admits a model over $\mathbb{Q}$ with all its endomorphisms defined over K.

We do so by applying Weil's criterion (cf. [41]). Since $\nu^\sigma \cdot \nu \in \mathrm{Aut}(A_0, \mathcal{L}_0)$, we have $\nu^\sigma \cdot \nu = \epsilon\,\mathrm{id}$ with $\epsilon \in \{\pm 1\}$. Using the same basis of $H^0(A_0, \Omega^1_{A_0/K})$ and $H^0(A_0^\sigma, \Omega^1_{A_0^\sigma/K})$ as above, the matrix expression $N \in \mathrm{GL}_2(K)$ of ν is such that $M_m \cdot N = N \cdot M_m^\sigma = N \cdot M_m\,,\quad M_\delta \cdot N = -N \cdot M_\delta^\sigma = N \cdot M_\delta\,.$ It follows that

$$N = \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix}, \quad \beta \in K\,.$$

Hence, $\beta \cdot \beta^\sigma = \epsilon$. Since K is imaginary, $\epsilon = 1$. Weil's criterion [41] applies to ensure the existence of an abelian surface A defined over $\mathbb{Q}$ and isomorphic over K to A_0. Since A is isomorphic over K to A_0, we obtain that $\mathrm{End}_K^0(A) = B_D$. The equality $\mathrm{End}_{\mathbb{Q}}^0(A) = \mathbb{Q}(\sqrt{m})$ follows from Remark 4.1. □

Let (D,m) be a pair of GL_2-type over $\mathbb{Q}$. One may wonder how many abelian surfaces $A/\mathbb{Q}$ exist up to isomorphism such that $\mathrm{End}_{\mathbb{Q}}^0(A) = \mathbb{Q}(\sqrt{m})$ and $\mathrm{End}_{\overline{\mathbb{Q}}}(A) = \mathcal{O}_D$. We make this precise in what follows.

Definition 4.4. *Let $\mathcal{Q}_{(D,m)}(\mathbb{Q})$ denote the set of $\overline{\mathbb{Q}}$-isomorphism classes of abelian surfaces A defined over $\mathbb{Q}$ such that $\mathrm{End}_{\overline{\mathbb{Q}}}(A) = \mathcal{O}_D$ is a maximal order in B_D and $\mathrm{End}_{\mathbb{Q}}^0(A) = \mathbb{Q}(\sqrt{m})$.*

For a pair (D,m) of GL_2-type over $\mathbb{Q}$, regard the Atkin-Lehner group $W_m = W/\langle\omega_m\rangle$ as a subgroup of $\mathrm{Aut}_{\mathbb{Q}}(X_D^{(m)})$ which freely acts on $X_D^{(m)}(\mathbb{Q})_d$.

Theorem 4.5. *Let (D,m) be a pair of GL_2-type over $\mathbb{Q}$. There is a canonical one-to-one correspondence*

$$\mathcal{Q}_{(D,m)}(\mathbb{Q}) \longleftrightarrow W_m \backslash X_D^{(m)}(\mathbb{Q})_d$$

and hence, if $D = p_1 \cdot \ldots \cdot p_{2r}$, then

$$|\mathcal{Q}_{(D,m)}(\mathbb{Q})| = \frac{r d_m}{2^{2r-1}}\,.$$

Proof. Let $[A] \in \mathcal{Q}_{(D,m)}(\mathbb{Q})$ represented by an abelian surface A defined over $\mathbb{Q}$. Let $d \geq 1$ be the minimal integer such that $B_D \simeq (\frac{-Dd,m}{\mathbb{Q}})$. We know from Proposition 4.6 that there exists at least one polarization $\mathcal{L}$ on A of degree d defined over $\mathbb{Q}$. Fix an isomorphism $\iota : \mathcal{O}_D \xrightarrow{\simeq} \mathrm{End}_{\overline{\mathbb{Q}}}(A)$ that restricts to $\iota \otimes \mathbb{Q}_{|\mathbb{Q}(\sqrt{m})} : \mathbb{Q}(\sqrt{m}) \xrightarrow{\simeq} \mathrm{End}_{\mathbb{Q}}^0(A)$. By [11], Theorem 3.4, we have that $\mathrm{End}_{\overline{\mathbb{Q}}}(A) = \mathrm{End}_K(A) = \mathcal{O}_D$ for some imaginary quadratic field K.

Let $* : B_D \to B_D$ be the Rosati involution induced by $\mathcal{L}$. It follows from [20], Ch. 1, [31], pp. 6–7, that $\beta^* = \varrho_\mu(\beta) = \mu^{-1}\bar{\beta}\mu$ for some $\mu \in \mathcal{O}_D$, $\mu^2 + Dd =$

0. If we regard the Shimura curve X_D as coarsely representing $\hat{\mathcal{F}}_{\mathcal{O},\mu}$, the triplet $[A,\iota,\mathcal{L}]$ produces a point P in $X_D(K)$. By [32], Theorem 4.4, [33], Section 6, the point P projects onto a rational point $\pi(P)=[A,\iota_{|\mathbb{Q}(\sqrt{m})},\mathcal{L}]\in X_D^{(m)}(\mathbb{Q})$. By construction we actually have $\pi(P)\in X_D^{(m)}(\mathbb{Q})_d$. Since, as shown in [20], Ch. 1, [32], Section 3.1, the group W_m acts on $\pi(P)=[A,\iota_{|\mathbb{Q}(\sqrt{m})},\mathcal{L}]$ by fixing the isomorphism class of A and switching ι and $\mathcal{L}$, we deduce that A produces a well-defined point in $W_m\backslash X_D^{(m)}(\mathbb{Q})_d$. Constructing the inverse map from $W_m\backslash X_D^{(m)}(\mathbb{Q})_d$ onto $\mathcal{Q}_{(D,m)}(\mathbb{Q})$ is now obvious. □

As a refinement of the above considerations, we wonder for which pairs (D,m) there exists a curve $C/\mathbb{Q}$ of genus 2 such that the Jacobian $J(C)$ has multiplication by $\mathbb{Q}(\sqrt{m})$ over $\mathbb{Q}$ and quaternionic multiplication by $\mathcal{O}_D$ over $\overline{\mathbb{Q}}$.

Proposition 4.6. [11] *Let $A/\mathbb{Q}$ be an abelian surface such that* $\mathrm{End}^0_{\mathbb{Q}}(A)=\mathbb{Q}(\sqrt{m})$, *$m>1$ square-free and* $\mathrm{End}_{\overline{\mathbb{Q}}}(A)=\mathcal{O}_D$. *Then $m|D$ and A admits a polarization* $\mathcal{L}\in H^0(\mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q}),\mathrm{NS}(A_{\overline{\mathbb{Q}}}))$ *of degree $d>0$ if and only if* $B=(\frac{-Dd,m}{\mathbb{Q}})$.

Let $P=(A,\iota,\mathcal{L})\in X_D(K)$ be a non-Heegner point such that $\pi(P)\in X_D^{(m)}(\mathbb{Q})$. We know by [20], Ch. 1 and [31], Corollary 6.3, that there is a principal polarization $\mathcal{L}_0$ on A defined over K and hence there exists a genus 2 curve C_0 defined over K whose canonically polarized Jacobian variety is isomorphic to $(A,\mathcal{L}_0)$ over K. As a corollary of Theorem 4.3 and Proposition 4.6, we obtain the following. We keep the same notation as above.

Corollary 4.7. *There exists a curve $C/\mathbb{Q}$ defined over $\mathbb{Q}$ such that $(A,\mathcal{L}_0)\simeq(\mathrm{Jac}(C),\Theta_C)$ over $\mathbb{Q}$ if and only if*

$$B_D=\left(\frac{-\delta,m}{\mathbb{Q}}\right)=\left(\frac{-D,m}{\mathbb{Q}}\right).$$

Next, we illustrate the above results with several examples.

Example 4.8. In [19], Hashimoto and Tsunogai provided a family of curves of genus 2 whose Jacobians have quaternionic multiplication by $\mathcal{O}_6$. These families specialize to infinitely many curves defined over $\mathbb{Q}$. However, one can not expect that to be always possible for a discriminant D even when there is an Atkin-Lehner quotient $X_D^{(m)}\simeq\mathbb{P}^1_{\mathbb{Q}}$. As we pointed out in Section 1, computations due to Hasegawa [18] exhibit B_{14} as a modular quaternion algebra. This is indeed possible because $X_{14}^{(14)}\simeq\mathbb{P}^1_{\mathbb{Q}}$ but there does not exist a curve $C/\mathbb{Q}$ of genus 2 whose Jacobian $J(C)$ is of GL_2-type over $\mathbb{Q}$ and has quaternionic multiplication by $\mathcal{O}_{14}$ over $\overline{\mathbb{Q}}$, because $B_{14}\not\simeq(\frac{-14,2}{\mathbb{Q}})$, $(\frac{-14,7}{\mathbb{Q}})$ nor $(\frac{-14,14}{\mathbb{Q}})$.

Example 4.9. An affine equation of the Shimura curve X_6 is $x^2+y^2+3=0$ and the action of w_6 on this model is $(x,y)\mapsto(-x,y)$. We have $X_6^{(6)}\simeq\mathbb{P}^1_{\mathbb{Q}}$ and there exist infinitely many points on $X_6(K)$, $K=\mathbb{Q}(\sqrt{-21})$, mapping to a rational point on $X_6^{(6)}$. Since K splits B_6, it follows from [20], Ch. 2, and [33], Section 6, that there exist simple principally polarized abelian surfaces $(A,\mathcal{L}_0)/K$ which are isomorphic to their Galois conjugate abelian surface $(A^\sigma,\mathcal{L}_0^\sigma)$. However, they do not admit a rational model over $\mathbb{Q}$, since $(\frac{-21,6}{\mathbb{Q}})\not\simeq B_6$. Thus, there do not exist curves C defined over $\mathbb{Q}$ such that $(A,\mathcal{L}_0)\simeq(J(C),\Theta_C)$ for these abelian surfaces.

Example 4.10. Let f be the newform of $S_2(\Gamma_0(243))$ with q-expansion

$$f = q + \sqrt{6}\, q^2 + 4\, q^4 + \dots .$$

The modular abelian surface A_f obtained as an optimal quotient of the Jacobian of $X_0(43)$ satisfies that $\operatorname{End}_{\mathbb{Q}(\sqrt{-3})}(A_f)$ is a maximal order of the quaternion algebra B_6. We know by [31], Theorem 7.1, that there is a single class of principal polarizations $\mathcal{L}_0$ on A_f up to $\overline{\mathbb{Q}}$-isomorphism defined over $K = \mathbb{Q}(\sqrt{-3})$. Hence $A_f \otimes K$ is the Jacobian of a curve $C/\mathbb{Q}(\sqrt{-3})$. Since $\mathcal{L}_0$ is isomorphic to its Galois conjugate $\mathcal{L}_0^\sigma$, it follows that C_0 is isomorphic to C_0^σ but, although its Jacobian $\operatorname{Jac}(C_0) = A_f \otimes K$ admits a projective model over $\mathbb{Q}$, the curve C_0 can not be defined over $\mathbb{Q}$ because $B_6 \not\simeq (\frac{-6,6}{\mathbb{Q}})$. In fact, by using similar methods to [17], we obtain the following equation for C_0:

$$y^2 = (2+2\sqrt{-3})x^6 + 12(-3+\sqrt{-3})x^5 - 12(3+7\sqrt{-3})x^4 + 4(69+7\sqrt{-3})x^3 + \\ 8(-11+7\sqrt{-3})x^2 - 18(1+5\sqrt{-3}) + 12(2+\sqrt{-3}).$$

It can be checked that its Igusa invariants are rational and there is a morphism $\nu : C_0 \to C_0^\sigma$ defined over K such that $\nu^\sigma \cdot \nu$ is the hyperelliptic involution.

5. Rational points on quotient Shimura curves $X_D^{(m)}$ of genus ≤ 1

The set of rational Heegner points $X_D^{(m)}(\mathbb{Q})_h$ on an Atkin-Lehner quotient of a Shimura curve is finite and its cardinality rh_m can be computed by using the following formula, which stems from the work of Jordan on complex multiplication (cf. [20], Ch. 3, [35]).

Proposition 5.1. *Let $D = p_1 \cdot \ldots \cdot p_{2r}$ and let $m|D$. For $i = 1$ or 2, let us denote by $\mathcal{R}_i$ the set of orders of imaginary quadratic fields whose class number is i. For any $R \in \mathcal{R}_1$, set $p_R = 2$ when $\operatorname{disc}(R) = -2^k$ and p_R to be the single odd prime dividing $\operatorname{disc}(R)$, otherwise. The number rh_m is given by the following formula:*

$$rh_m = \begin{cases} \begin{array}{r} 2^{2r-1}\#\{R \in \mathcal{R}_1 : \left(\frac{R}{p_i}\right) = -1 \text{ for all } p_i|D\} + \\ 2^{2r-2}\#\{R \in \mathcal{R}_1 : p_R|D\,, \left(\frac{R}{p_i}\right) = -1 \text{ for all } p_i|\frac{D}{p_R}\} + \\ \#\{R \in \mathcal{R}_2 : 2D = -\operatorname{disc}(R)\} \end{array} & , \text{if } m = D, \\ \begin{array}{r} 2^{r-2}\#\{R \in \mathcal{R}_1 : p_R = \frac{D}{m}\} + \#\{R \in \mathcal{R}_2 : -\frac{D}{\operatorname{disc}(R)} \in \mathbb{Q}^{*2}\} + \\ \frac{(-1)^p+1}{2}\#\{R \in \mathcal{R}_2 : 2D = -\operatorname{disc}(R)\} \end{array} & , \text{if } m = \frac{D}{p}, \\ 0 & \text{otherwise}. \end{cases}$$

where $(\frac{R}{p})$ stands for the Eichler symbol.

We refer to [40], p. 43 for the definition of the Eichler symbol. In any case, $0 \leq rd_m \leq r_m - rh_m$. The condition $r_m = rh_m$ implies $rd_m = 0$ and allows us to claim the non existence of an abelian surface $A/\mathbb{Q}$ with $\operatorname{End}^0_{\mathbb{Q}} A = \mathbb{Q}(\sqrt{m})$ and $\operatorname{End}^0_{\overline{\mathbb{Q}}} A = B_D$. Nevertheless, proving the existence of such an abelian surface, i. e. $rd_m > 0$, requires the knowledge of an equation for X_D and the action of ω_m on it.

There are exactly twelve Shimura curves X_D of genus $g \leq 2$. For all of them, $D = p \cdot q$ with p, q primes and affine equations for these curves are known (cf. [16], [22], [23]) except for $D = 2 \cdot 17$ ($g = 1$). For these equations, the Atkin-Lehner involutions $\omega_p, \omega_q, \omega_{p.q}$ act on the curve, sending (x, y) to $(-x, y)$, $(x, -y)$

and $(-x, y)$ in some suitable order. The next table shows equations, genera and the actions of ω_p and ω_q for these curves.

$D = p \cdot q$	g	X_D	$\omega_p(x, y)$	$\omega_q(x, y)$
$2 \cdot 3$	0	$x^2 + y^2 + 3 = 0$	$(-x, -y)$	$(\ \ x, -y)$
$2 \cdot 5$	0	$x^2 + y^2 + 2 = 0$	$(\ \ x, -y)$	$(-x, -y)$
$2 \cdot 11$	0	$x^2 + y^2 + 11 = 0$	$(-x, -y)$	$(\ \ x, -y)$
$2 \cdot 7$	1	$(x^2 - 13)^2 + 7^3 + 2y^2 = 0$	$(-x, \ \ y)$	$(-x, -y)$
$3 \cdot 5$	1	$(x^2 + 3^5)(x^2 + 3) + 3y^2 = 0$	$(-x, \ \ y)$	$(-x, -y)$
$3 \cdot 7$	1	$x^4 - 658x^2 + 7^6 + 7y^2 = 0$	$(-x, -y)$	$(-x, \ \ y)$
$3 \cdot 11$	1	$x^4 + 30x^2 + 3^8 + 3y^2 = 0$	$(-x, \ \ y)$	$(-x, -y)$
$2 \cdot 23$	1	$(x^2 - 45)^2 + 23 + 2y^2 = 0$	$(-x, \ \ y)$	$(-x, -y)$
$2 \cdot 13$	2	$y^2 = -2x^6 + 19x^4 - 24x^2 - 169$	$(-x, -y)$	$(-x, \ \ y)$
$2 \cdot 19$	2	$y^2 = -16x^6 - 59x^4 - 82x^2 - 19$	$(-x, -y)$	$(-x, \ \ y)$
$2 \cdot 29$	2	$2y^2 = -x^6 - 39x^4 - 431x^2 - 841$	$(-x, -y)$	$(\ \ x, -y)$

Table 1. Equations and Atkin-Lehner involutions on Shimura curves

Unfortunately, the construction of the above equations does not allow us to distinguish the rational Heegner points among the rational points on the curves $X_D^{(m)}$, unless they are fixed by some Atkin-Lehner involution (however, for $D = 6$ and 10, cf. [10]). This forces the proof of next theorem to be more elaborate.

Theorem 5.2. *For the eleven values of D as above, the triplets (r_m, rh_m, rd_m) take the following values:*

$D = p \cdot q$	(r_p, rh_p, rd_p)	(r_q, rh_q, rd_q)	(r_D, rh_D, rd_D)
$2 \cdot 3$	$(\infty, 1, \infty)$	$(\infty, 1, \infty)$	$(\infty, 8, \infty)$
$2 \cdot 5$	$(\infty, 2, 0)$	$(\infty, 2, \infty)$	$(\infty, 11, \infty)$
$2 \cdot 7$	$(0, 0, 0)$	$(6, 2, 4)$	$(\infty, 8, \infty)$
$2 \cdot 11$	$(\infty, 2, \infty)$	$(\infty, 2, \infty)$	$(\infty, 8, \infty)$
$2 \cdot 13$	$(1, 1, 0)$	$(3, 1, 0)$	$(\infty, 10, > 0)$
$2 \cdot 19$	$(1, 1, 0)$	$(3, 1, 0)$	$(\infty, 8, > 0)$
$2 \cdot 23$	$(0, 0, 0)$	$(2, 2, 0)$	$(\infty, 8, \infty)$
$2 \cdot 29$	$(1, 1, 0)$	$(\infty, 2, > 0)$	$(\infty, 13, > 0)$
$3 \cdot 5$	$(\infty, 2, 0)$	$(4, 4, 0)$	$(\infty, 10, \infty)$
$3 \cdot 7$	$(2, 2, 0)$	$(0, 0, 0)$	$(\infty, 10, \infty)$
$3 \cdot 11$	$(0, 0, 0)$	$(2, 2, 0)$	$(\infty, 10, \infty)$

Table 2. Rational, Heegner and descent points

Proof. We split the proof in three parts according to the genus g of X_D.

Case $g = 0$. In all these cases the equation is $x^2 + y^2 = -d$, for some prime $d|D$. For each pair (D, m), the points on $X_D(\sqrt{-\delta})$ of the form $(a, b\sqrt{-\delta})$, $(b\sqrt{-\delta}, a)$ or $(a\sqrt{-\delta}, b\sqrt{-\delta})$, with $a, b \in \mathbb{Q}$ and a square-free integer $\delta \geq 1$, are the only affine points on $X_D(\mathbb{Q}(\sqrt{-\delta}))$ which may provide rational points on $X_D^{(m)}(\mathbb{Q})$, depending on whether ω_m maps (x, y) to either $(x, -y)$, $(-x, y)$ or $(-x, -y)$. Let $(\cdot, \cdot)$ denote the global Hilbert symbol over $\mathbb{Q}$. When $\omega_m(x, y) = (x, -y)$ or $(-x, y)$, such rational points exist if and only if $(\delta, -d) = 1$. Similarly, when $\omega_m(x, y) = (-x, -y)$, there exist points on $X_D(\mathbb{Q}(\sqrt{-\delta}))$ which project onto a rational point on $X_D^{(m)}(\mathbb{Q})$ if and only if $(-1, d\cdot\delta) = 1$. It is easy to check that, for all pairs $(D,m) \neq (10,2)$, these conditions and the descent condition of Theorem 4.3 have infinitely many solutions for δ. If we let l be a prime number, we may take δ as follows.

(D,m)	δ	Conditions on l
$(6,2)$	$3\,l$	$l \equiv 1 \pmod 8$
$(6,3),(6,6)$	l	$l \equiv 1 \pmod{24}$
$(10,5)$	$2\,l$	$l \equiv 1 \pmod{20}$
$(10,10)$	$2\,l$	$l \equiv 1 \pmod{40}$
$(22,2)$	$11\,l$	$l \equiv 1 \pmod 8$
$(22,11),(22,22)$	l	$l \equiv 1 \pmod{88}$

For the pair $(10,2)$, we have $d=2$ and the condition $(\delta,-2)=1$ implies $5 \nmid \delta$ and thus $(\frac{-\delta,2}{\mathbb{Q}}) \not\simeq B_{10}$, since $(-\delta,2)_5 = 1$. Moreover, the two points $(1 : \pm\sqrt{-1} : 0)$ at infinity on the curve X_{10} produce a rational point on $X_{10}^{(2)}$ which is Heegner because its preimages are fixed points by the Atkin-Lehner involution ω_5. We conclude that for $D=10$, $rd_2 = 0$ despite $r_2 = \infty$.

Case $g=1$. The genus of $X_D^{(m)}$ is zero except for the cases in which ω_m maps (x,y) to $(-x,-y)$. The latter holds for the pairs $(D,m) = (14,7), (15,5), (21,3), (33,11)$ and $(46,23)$, and in these cases $g(X_D^{(m)}) = 1$. To be more precise, the curves $X_D^{(m)}$ are elliptic curves over $\mathbb{Q}$ and there is a single isogeny class of conductor D in each case. Their Mordell-Weil rank over $\mathbb{Q}$ is 0 and the orders of the group of rational torsion points on them are 6, 4, 2, 2 and 2, respectively. Only for the pair $(D,m) = (14,7)$ do we have $r_m > rh_m$. But in this case, the two rational Heegner points can be recognized from the affine equation of the curve $X_{14}^{(7)}$ because they are fixed points by some Atkin-Lehner involution. It then turns out that the four rational non-Heegner points on $X_{14}^{(7)}$ are the projections of the points $(\pm 8\sqrt{-1}, \pm 56\sqrt{-1}), (\pm 2\sqrt{-2}, \pm 14\sqrt{-2}) \in X_{14}(\overline{\mathbb{Q}})$. Since $\delta = 1, 2$ satisfy the descent condition, we have $rd_7 = 4$.

When ω_m acts as $(x,y) \mapsto (-x,y)$, we have $r_m = 0$ except for $(D,m) = (15,3)$. The affine equation $(X+3^5)(X+3) + 3y^2 = 0$ for $X_{15}^{(3)}$ shows that there are no rational points at infinity on this model. Moreover, it turns out that for all $(X,y) \in X_{15}^{(3)}(\mathbb{Q})$, the 5-adic valuation of the X-coordinate is $v_5(X) = 0$. Since $\delta \equiv -X \pmod{\mathbb{Q}^2}$, we have $5 \nmid \delta$. Thus, $(\frac{-\delta,3}{\mathbb{Q}}) \not\simeq B_{15}$ because $(-\delta,3)_5 = 1$. It follows that $rd_3 = 0$.

For the case $\omega_m : (x,y) \mapsto (x,-y)$, we have $m = D$ and the curve $X_D^{(D)}$ admits an affine model of the form $f(x) + dY = 0$, where $f(x) \in \mathbb{Z}[x]$ is monic of degree 4 and d is a prime dividing D. Moreover, $(\frac{-d,D}{\mathbb{Q}}) \simeq B_D$. A point $(x_0, Y_0) \in X_D^{(D)}(\mathbb{Q})$ satisfies the descent condition if and only if $(f(x_0), D) = 1$, that is, $f(x_0) = u_0 - Dv_0^2$ for some $u_0, v_0 \in \mathbb{Q}$. Hence, the descent condition for (x_0, Y_0) turns out to be equivalent to the existence of a rational point on the algebraic surface $S_D : f(x) = u^2 - Dv^2$ with $x = x_0$. For $D = 14, 15, 21, 33$ and 46 we have the following rational points on S_D: $(x_0, u_0, v_0) = (0, 64, 16), (1, 44, 8), (1, 944, 192), (8, 145, 16)$ and $(4, 408, 60)$, respectively. The elliptic curves $E_D : f(x) = u^2 - Dv_0^2$ have at the least three rational points: two rational points at infinity and the affine point (x_0, u_0). It can be easily checked that, for the above values of D, the rational torsion subgroup $E_{D,\mathrm{tors}}(\mathbb{Q})$ of E_D is of order 2. We conclude that the Mordell-Weil rank of $E_D(\mathbb{Q})$ is greater to 0 and thus $rd_D = \infty$.

Case $g = 2$. The three curves X_D are bielliptic. In Cremona notation, the three elliptic quotients $X_D/\langle\omega_2\rangle$ are $26B2$, $38B2$, $58B2$, while $X_{26}/\langle\omega_{13}\rangle$, $X_{38}/\langle\omega_{19}\rangle$, $X_{58}/\langle\omega_{58}\rangle$ are $26A1$, $38A1$ and $58A1$, respectively.

For the three curves $X_D^{(2)}$, we have $rh_2 = r_2$ and hence the pairs $(D, 2)$ are not of GL_2-type over $\mathbb{Q}$. For $(26, 13)$ and $(38, 19)$, the single rational Heegner point corresponds to the projection of the two points at infinity because they are fixed points by ω_2. The preimages of the other two rational points are $(\pm\sqrt{-5}, \pm 26)$ and $(\pm\sqrt{-5}/2, \pm 19/4)$ for $D = 26$ and 38 respectively. In both cases $(-5, q)_5 = -1$ and hence $rd_q = 0$. For the remaining cases, we have $r_m = \infty$ and we can easily find $rh_m + 1$ rational points on $X_D^{(m)}$ satisfying the descent condition. □

When the genus of X_D is larger than 2, there exist exactly 21 elliptic Atkin-Lehner quotients $X_D^{(m)}$ (cf. [30]), but only three of them have rank zero. Namely, these are $X_{106}^{(53)}$, $X_{118}^{(59)}$ and $X_{202}^{(101)}$, which correspond to the elliptic curves $106D1$, $118D1$ and $202A_1$ respectively. In all these cases the number of rational points is $r_m = 1$. Thus $rh_m = 1$ since $r_m - rh_m$ is even whenever $r_m < \infty$. In particular, we get $rd_m = 0$.

Combining the above with Theorems 4.5 and 5.2, we obtain Theorems 1.8 and 1.9 and part (ii) of Theorem 1.7 for all pairs but a few ones which deserve more attention: namely, those (D, m) such that $X_D^{(m)}$ is a curve of genus 2. Indeed, computing the full list of rational points on these curves is a harder task that we address in the next section.

6. Covering techniques on bielliptic Shimura curves $X_D^{(m)}$ of genus 2

It was shown in [16] that there exist exactly ten Shimura curve quotients $X_D^{(m)}$ which are bielliptic of genus 2. Applying Proposition 5.1, the triplets (D, m, rh_m) are $(91, 91, 10)$, $(123, 123, 10)$, $(141, 141, 10)$, $(142, 2, 0)$, $(142, 142, 10)$, $(155, 155, 10)$, $(158, 158, 10)$, $(254, 254, 8)$, $(326, 326, 4)$ and $(446, 446, 6)$.

In this section we study the set of rational points on these curves. We first list the $\mathbb{Q}$-rational points on each $X_D^{(m)}$ which are easily found by a short search. Table 3 lists some small rational points on the bielliptic curves $X_D^{m})$ of genus 2.

In this section, we show that Table 3 lists all rational points for each $X_D^{(m)}$. The case $D = 142$, $m = 2$ is straightforward: there are no points in $X_{142}^{(2)}(\mathbb{R})$ from which it follows that there are none in $X_{142}^{(2)}(\mathbb{Q})$. For the other values of D, m, each $X_D^{(m)}$ has points everywhere locally and so cannot be resolved in this way. We first recall the techniques from [3],[6],[13],[14],[15], which we summarize here in a simplified form adapted to the curves $X_D^{(m)}$. The fact that each $X_D^{(m)}$ is bielliptic allows a specialized version ([13]) of the same ideas of [7]; similar methods are available for arbitrary hyperelliptic curves, as described in [5] and [6]. Each of the curves $X_D^{(m)}$ is of genus 2 and of the form

$$X_D^{(m)} : \; Y^2 = f_3X^6 + f_2X^4 + f_1X^2 + f_0, \text{ with } f_i \in \mathbb{Z}.$$

Any such curve $X_D^{(m)}$ has a map $(X, Y) \mapsto (X^2, Y)$ from $X_D^{(m)}$ to the elliptic curve $Y^2 = f_3w^3 + f_2w^2 + f_1w + f_0$, and map $(X, Y) \mapsto (1/X^2, Y/X^3)$ from $X_D^{(m)}$ to

D	m	$X_D^{(m)}$	$X_D^{(m)}(\mathbb{Q})$
91	D	$Y^2 = -X^6 + 19X^4 - 3X^2 + 1$	$(0, \pm 1), (\pm 1, \pm 4), (\pm 3, \pm 28)$
123	D	$Y^2 = -9X^6 + 19X^4 + 5X^2 + 1$	$(0, \pm 1), (\pm 1, \pm 4), (\pm 1/3, \pm 4/3)$
141	D	$Y^2 = 27X^6 - 5X^4 - 7X^2 + 1$	$(\pm 1, \pm 4), (\pm 1/3, \pm 4/9)$, $(0, \pm 1), (\pm 11/13, \pm 4012/2197)$
142	2	$Y^2 = -16X^6 - 87X^4 - 146X^2 - 71$	$\emptyset$
142	D	$Y^2 = 16X^6 + 9X^4 - 10X^2 + 1$	$\pm\infty, (0, \pm 1), (\pm 1, \pm 4), (\pm 1/3, \pm 4/27)$
155	D	$Y^2 = 25X^6 - 19X^4 + 11X^2 - 1$	$\pm\infty, (\pm 1, \pm 4), (\pm 1/3, \pm 4/27)$
158	D	$Y^2 = -8X^6 + 9X^4 + 14X^2 + 1$	$(\pm 1, \pm 4), (0, \pm 1), (\pm 1/3, \pm 44/27)$
254	D	$Y^2 = 8X^6 + 25X^4 - 18X^2 + 1$	$(0, \pm 1), (\pm 1, \pm 2), (\pm 2, \pm 29)$
326	D	$Y^2 = X^6 + 10X^4 - 63X^2 + 4$	$\pm\infty, (0, \pm 2)$
446	D	$Y^2 = -16X^6 - 7X^4 + 38X^2 + 1$	$(0, \pm 1), (\pm 1, \pm 4)$

Table 3. Known Rational Points on the bielliptic $X_D^{(m)}$ of genus 2

the elliptic curve $Z^2 = f_0x^3 + f_1x^2 + f_2x + f_3$. The Jacobian of $X_D^{(m)}$ is $\mathbb{Q}$-isogenous to the product of these elliptic curves over $\mathbb{Q}$ which, in all of these examples, each have rank 1 (and no nontrivial torsion). It follows that $\mathrm{Jac}(X_D^{(m)})(\mathbb{Q})$ has rank 2, and so Chabauty techniques [8] cannot be used, since they only apply when the rank of the Mordell-Weil group of the Jacobian is strictly less than the genus of the curve. It is therefore necessary to imitate the technique in [13], which we briefly summarize here in a simplified form suited to these examples. We first fix one of the above two elliptic curves – it does not matter which one; we shall use the latter elliptic curve, since the resulting models will typically be slightly simpler. Define $E_D^{(m)}, (x_0, Z_0), \phi, t$ as follows.

$$E_D^{(m)} : \; Z^2 = f_0x^3 + f_1x^2 + f_2x + f_3, \tag{1}$$

$$(x_0, Z_0) \text{ generates } E_D^{(m)}(\mathbb{Q}), \tag{2}$$

$$\phi : \; X_D^{(m)} \longrightarrow E_D^{(m)} : (X, Y) \mapsto (1/X^2, Y/X^3), \tag{3}$$

$$t := \text{ root of } f_0x^3 + f_1x^2 + f_2x + f_3, \tag{4}$$

so that $E_D^{(m)}(\mathbb{Q})/2E_D^{(m)}(\mathbb{Q}) = \{\infty, (x_0, Z_0)\}$. Suppose that $(X, Y) \in X_D^{(m)}(\mathbb{Q})$. Then, applying ϕ, we let $x = 1/X^2$ and $Z = Y/X^3$ so that $(x, Z) \in E_D^{(m)}(\mathbb{Q})$. We recall the injective homomorphism $\mu : E_D^{(m)}(\mathbb{Q})/2E_D^{(m)}(\mathbb{Q}) \to \mathbb{Q}(t)^*/(\mathbb{Q}(t)^*)^2$ defined by $\mu(\infty) = 1$ and $\mu((x, Z)) = f_0(x - t)$ from Chapter X of [39]. It follows that $\mu((x, Z))$ equals either 1 or $f_0(x_0 - t)$ in $\mathbb{Q}(t)^*/(\mathbb{Q}(t)^*)^2$. Hence, either $f_0Z^2/(x - t)$ or $(x_0 - t)Z^2/(x - t)$ is a square. We can eliminate Z^2 using (1) and simplification yields:

$$\begin{array}{rl} \text{either} & f_0(f_0x^2 + (f_0t + f_1)x + (t^2f_0 + tf_1 + f_2)) \in (\mathbb{Q}(t)^*)^2 \\ \text{or} & (x_0 - t)(f_0x^2 + (f_0t + f_1)x + (t^2f_0 + tf_1 + f_2)) \in (\mathbb{Q}(t)^*)^2. \end{array}$$

Note that we do not really need (x_0, Z_0); we only need the square class of $x_0 - t$. This can already be determined from the 2-Selmer group of $E_D^{(m)}$, without computing an explicit generator of the Mordell-Weil group. In our examples, however, the curve $E_D^{(m)}$ has small coefficients and finding an actual generator is little more work.

Since $x = 1/X^2$ is a square itself, we can multiply either quantity with x without changing its square class. Hence, we have shown that if $(X, Y) \in X_D^{(m)}(\mathbb{Q})$ then there exists $y \in \mathbb{Q}(t)$ such that $(x, y) = (1/X^2, y)$ is a $\mathbb{Q}(t)$-rational point on one of the curves

$$F_D^{(m)} : y^2 = f_0 x\big(f_0 x^2 + (f_0 t + f_1)x + (t^2 f_0 + t f_1 + f_2)\big), \tag{5}$$

$$G_D^{(m)} : y^2 = (x_0 - t)x\big(f_0 x^2 + (f_0 t + f_1)x + (t^2 f_0 + t f_1 + f_2)\big). \tag{6}$$

This gives a strategy for trying to prove that we have found all of $X_D^{(m)}(\mathbb{Q})$; it is sufficient find all $(x, y) \in F_D^{(m)}(\mathbb{Q}(t))$ such that $x \in \mathbb{Q}$ and similarly for $G_D^{(m)}$. This can be attempted using the techniques in [3],[6],[13],[14],[15], which apply local techniques to bound the number of such points. Since these articles already contain several worked examples of this type, we merely provide a brief sketch of one case, to give an idea of the general strategy, and to allow the reader to interpret the tabular summary given later. This will be followed by a description of any unusual features of difficult special cases. The full details of the computations are available at:

`ftp://ftp.liv.ac.uk/pub/genus2/shimura/` or
`http://www.cecm.sfu.ca/~nbruin/shimura/`

Consider $X_{142}^{(142)} : Y^2 = 16X^6 + 9X^4 - 10X^2 + 1$, where we wish to show that the only $\mathbb{Q}$-rational points have X-coordinate $\infty, 0, 1, \pm 1/3$ (we use ∞, depending on context, as the notation both for the point at infinity and its X-coordinate). Then $E_{142}^{(142)}$ of (1) is the elliptic curve $V^2 = x^3 - 10x^2 + 9x + 16$ over $\mathbb{Q}$, which has rank 1, with generator $(x_0, V_0) = (1, 4)$. Under $(X, Y) \mapsto (1/X^2, Y/X^3)$, the known points in $X_{142}^{(142)}(\mathbb{Q})$ map to $(0, \pm 4)$, ∞, $(1, \pm 4)$ and $(9, \pm 4) = 2(1, \pm 4)$ in $E_{142}^{(142)}(\mathbb{Q})$. Letting t be the cubic number satisfying $t^3 - 10t^2 + 9t + 16$, the curves (5),(6) become

$$\begin{aligned} F_{142}^{(142)} : y^2 &= x^3 + (t - 10)x^2 + (t^2 - 10t + 9)x, \\ G_{142}^{(142)} : y^2 &= (1 - t)\big(x^3 + (t - 10)x^2 + (t^2 - 10t + 9)x\big). \end{aligned}$$

Our known points in $E_{142}^{(142)}(\mathbb{Q})$ induce $\infty, (0, 0), (9, \pm(3t^2 - 15t)/4) \in F_{142}^{(142)}(\mathbb{Q}(t))$ and $\infty, (0, 0), (1, \pm 4) \in G_{142}^{(142)}(\mathbb{Q}(t))$. It is sufficient to show that there are no other points (x, y) in $F_{142}^{(142)}(\mathbb{Q}(t))$ or $G_{142}^{(142)}(\mathbb{Q}(t))$ for which $x \in \mathbb{Q}$. Note that, in each case $\infty, (0, 0)$ give the entire torsion group, and so we have a point of infinite order. Furthermore, a standard complete 2-descent or 2-isogeny descent, as recently implemented by N. Bruin in Magma [24] (or for an older version, see [4]), gives a Selmer bound of 1 on the rank, and so both $F_{142}^{(142)}(\mathbb{Q}(t))$ or $G_{142}^{(142)}(\mathbb{Q}(t))$ have rank 1.

Consider, for example, the second curve G_{142}. One can easily check with finite field arguments, that $R = (0, 0) + 4(1, 4) \in G_{142}(\mathbb{Q}(t))$ is in the kernel of reduction modulo 3, which is inert in $\mathbb{Q}(t)/\mathbb{Q}$ (so that $\langle R \rangle$ is of finite index in $G_{142}(\mathbb{Q}(t))$), and can check that $\langle R \rangle$, $(0, 0) + \langle R \rangle$, $(1, 4) + \langle R \rangle$ and $(1, -4) + \langle R \rangle$ and the only 4 cosets containing possible $(x, y) \in G_{142}^{(142)}(\mathbb{Q}(t))$ for which $x \in \mathbb{Q}$ (note that, although $(0, 0), (1, 4)$ do not generate $G_{142}(\mathbb{Q}(t))$, we do have that $(0, 0), (1 - t, t^2 - t - 4)$ generate $G_{142}(\mathbb{Q}(t))$ and that $(1, 4) = (0, 0) + 2(1 - t, t^2 - t - 4)$, so that $(0, 0), (1, 4)$ can be treated as if they are generators for the purposes of our 3-adic argument). This means that we only need to consider $(x, y) = nR, (0, 0) + nR, (1, 4) + nR, (1, -4) + nR$, for $n \in \mathbb{Z}_3$, and it is sufficient in each case to show that $n = 0$ is the only case where $x \in \mathbb{Q}$. Using the formal exp and log functions in the 3-adic formal group [13], we can express nR as $\exp(n \cdot \log(R))$ and deduce that

$$1/(x\text{-coordinate of } nR) \equiv 8 \cdot 3^2 n^2 + (2 \cdot 3^2 n^2 + 3^3 n^4)t + (3^2 n^2 + 3^3 n^4)t^2 \pmod{3^4},$$

where each coefficient of a power of t is a power series in n defined over $\mathbb{Z}_3$ whose coefficients converge to 0. If $x \in \mathbb{Q}$ then the coefficients of t and t^2 must be 0. Taking the coefficient of t, we have a power series n for which $n = 0$ is a known double root, and for which the coefficient of n^2 has 3-adic absolute value strictly greater than all subsequent coefficients of powers of n. It follows that $n = 0$ is the only solution. A 3-adic analysis of $(0,0) + nR, (1,4) + nR, (1,-4) + nR$ also shows that these can only have $\mathbb{Q}$-rational x-coordinate when $n = 0$. We know that the only $(x,y) \in G_{142}^{(142)}(\mathbb{Q}(t))$ with $x \in \mathbb{Q}$ are the points with $x = \infty, 0, 1$. Similarly, a 5-adic argument shows that the only $(x,y) \in F_{142}^{(142)}(\mathbb{Q}(t))$ with $x \in \mathbb{Q}$ are the points with $x = \infty, 0, 9$, as required.

These methods can be applied when the ranks of $F_D^{(m)}(\mathbb{Q}(t))$ and $G_D^{(m)}(\mathbb{Q}(t))$ are less than the degree of $\mathbb{Q}(t)$, that is, less than 3. Fortune is in our favour, since the ranks for these examples indeed all turn out to be $0, 1$ or 2. Note that, in the above rank 1 example, there was information to spare, since either the coefficient of t or that of t^2 could be used to bound the number of solutions. For the rank 2 cases, one can still obtain a bound, but the information from both power series must be used.

The following table summarizes the computations.

$D = m$	$F_D^{(m)}$ and $G_D^{(m)}$		r	$x \in \mathbb{Q}$	p
91	$F_D^{(m)} : y^2 =$	$x^3 + (t-3)x^2 + (t^2 - 3t + 19)x$	1	$\infty, 0, \frac{1}{9}$	5
91	$G_D^{(m)} : y^2 =$	$(1-t)\big(x^3 + (t-3)x^2 + (t^2 - 3t + 19)x\big)$	1	$\infty, 0, 1, 4$	5
123	$F_D^{(m)} : y^2 =$	$x^3 + (t+5)x^2 + (t^2 + 5t + 19)x$	1	$\infty, 0, 9$	5
123	$G_D^{(m)} : y^2 =$	$(1-t)\big(x^3 + (t+5)x^2 + (t^2 + 5t + 19)x\big)$	1	$\infty, 0, 1$	7
141	$F_D^{(m)} : y^2 =$	$x^3 + (t-7)x^2 + (t^2 - 7t - 5)x$	1	$\infty, 0, 9$	7
141	$G_D^{(m)} : y^2 =$	$(1-t)\big(x^3 + (t-7)x^2 + (t^2 - 7t - 5)x\big)$	2	$\infty, 0, 1, \frac{169}{121}$	7
142	$F_D^{(m)} : y^2 =$	$x^3 + (t-10)x^2 + (t^2 - 10t + 9)x$	1	$\infty, 0, 9$	5
142	$G_D^{(m)} : y^2 =$	$(1-t)\big(x^3 + (t-10)x^2 + (t^2 - 10t + 9)x\big)$	1	$\infty, 0, 1$	3
155	$F_D^{(m)} : y^2 =$	$x^3 + (t-11)x^2 + (t^2 - 11t + 19)x$	1	$\infty, 0, 9$	7
155	$G_D^{(m)} : y^2 =$	$(t-1)\big(x^3 + (t-11)x^2 + (t^2 - 11t + 19)x\big)$	2	$\infty, 0, 1$	3
158	$F_D^{(m)} : y^2 =$	$x^3 + (t+14)x^2 + (t^2 + 14t + 9)x$	2	$\infty, 0, 1, 9$	5
158	$G_D^{(m)} : y^2 =$	$(-3-t)\big(x^3 + (t+14)x^2 + (t^2 + 14t + 9)x\big)$	0	$\infty, 0$	–
254	$F_D^{(m)} : y^2 =$	$x^3 + (t-18)x^2 + (t^2 - 18t + 25)x$	0	$\infty, 0$	–
254	$G_D^{(m)} : y^2 =$	$(1-t)\big(x^3 + (t-18)x^2 + (t^2 - 18t + 25)x\big)$	2	$\infty, 0, 1, \frac{1}{4}$	5
326	$F_D^{(m)} : y^2 =$	$4\big(4x^3 + (4t-63)x^2 + (4t^2 - 63t + 10)x\big)$	1	$\infty, 0$	5
326	$G_D^{(m)} : y^2 =$	$-t\big(4x^3 + (4t-63)x^2 + (4t^2 - 63t + 10)x\big)$	0	$\infty, 0$	–
446	$F_D^{(m)} : y^2 =$	$x^3 + (t+38)x^2 + (t^2 + 38t - 7)x$	1	$\infty, 0, 1$	3
446	$G_D^{(m)} : y^2 =$	$(5-t)\big(x^3 + (t+38)x^2 + (t^2 + 38t - 7)x\big)$	0	$\infty, 0$	–

Table 4. Summary of computations

The second column gives the models for the curves $F_D^{(m)}$ and $G_D^{(m)}$, and the third column gives the rank over $\mathbb{Q}(t)$, where the cubic number t is as defined in (4). In all cases, the torsion over $\mathbb{Q}(t)$ consists only of ∞ and $(0,0)$. The fourth column gives the list of all $x \in \mathbb{Q}$ which are x-coordinates of a point (x,y) on the curve and defined over $\mathbb{Q}(t)$; the final column gives a prime p such that a p-adic argument proves that no other such $x \in \mathbb{Q}$ are possible. Of course, the rank 0 cases are trivial and so no such prime is required.

The computations referenced above verify the following theorem.

Theorem 6.1. *The curves $X_D^{(m)}$ have no $\mathbb{Q}$-rational points apart from those given in Table 3.*

We conclude with mention of a few special features of the computations. Recall that in the sketched worked example for the case $D = m = 142$, it turned out that the Selmer bound from a complete 2 descent was the same as the rank. However, for the six cases $F_{91}^{(91)}, G_{91}^{(91)}, G_{123}^{(123)}, F_{155}^{(155)}, F_{254}^{(254)}, G_{326}^{(326)}$, this bound is two greater than the actual rank. In order to find a sharp bound, one can perform a complete 2-descent on the 2-isogenous curve. It follows that each of these cases has nontrivial members of the 2-part of the Shafarevich-Tate group over $\mathbb{Q}(t)$.

In the other direction, there are two curves $G_{155}^{(155)}$ and $F_{326}^{(326)}$, where the group generated by images of the known points in $X_D^{(m)}(\mathbb{Q})$ is less than the actual rank, so one must search for the missing independent points of infinite order. For example, the 2-Selmer bound on the rank of $G_{155}^{(155)}(\mathbb{Q}(t))$ is 2, and the images of the known points in $X_{155}^{(155)}(\mathbb{Q})$ give only $\infty, (0,0), (1,4)$, of which only $(1,4)$ is of infinite order, so that we are missing an independent point of infinite order. In this case, a naive short search discovers the required point $((t^2+10t+25)/4, 13t^2-11t+10)$. The 2-Selmer bound on the rank of $F_{326}^{(326)}(\mathbb{Q}(t))$ is 1, and the images of the known points in $X_{326}^{(326)}(\mathbb{Q})$ do not give any points of infinite order, and in fact the required point is

$$(x,y) = \left(\tfrac{63540t^2-1005167t+228495}{2888}, \tfrac{903341332t^2-1429154471t+325168047}{109744}\right).$$

This could not be found by a naive search, and we needed to use the improved search techniques described in the appendix of [6], and recently implemented by N. Bruin into Magma [24].

Theorem 6.1 combined with Proposition 1.6 allows us to complete the proof of Theorem 1.7. From Theorem 6.1 and an analogue of Theorem 4.5 we can also derive the following result.

Corollary 6.2. *For each of the pairs*

$$(D,m) = (141,141), (142,142), (254,254)$$

we know exactly the number of $\overline{\mathbb{Q}}$-isomorphism classes of abelian surfaces $A/\overline{\mathbb{Q}}$ that admit an embedding $\iota : \mathbb{Q}(\sqrt{m}) \hookrightarrow \mathrm{End}_{\overline{\mathbb{Q}}}(A))$ whose field of moduli is $\mathbb{Q}$ and such that $\mathrm{End}_{\overline{\mathbb{Q}}}(A) \simeq \mathcal{O}_D$. The numbers are 2, 1 and 1, respectively.

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