

# FORMALITY OF THE FRAMED LITTLE 2-DISCS OPERAD AND SEMIDIRECT PRODUCTS

JEFFREY GIANIRACUSA AND PAOLO SALVATORE

ABSTRACT. We prove that the operad of framed little 2-discs is formal. Tamarkin and Kontsevich each proved that the unframed 2-discs operad is formal. The unframed 2-discs is an operad in the category of  $S^1$ -spaces, and the framed 2-discs operad can be constructed from the unframed 2-discs by forming the operadic semidirect product with the circle group. The idea of our proof is to show that Kontsevich's chain of quasi-isomorphisms is compatible with the circle actions and so one can essentially take the operadic semidirect product with the homology of  $S^1$  everywhere to obtain a chain of quasi-isomorphisms between the homology and the chains of the framed 2-discs.

## 1. INTRODUCTION

We begin by recalling two closely related operads. First, let  $D_2$  denote the *little 2-discs operad* of Boardman and Vogt. In arity  $n$  it is the space of embeddings of the union of  $n$  discs into a standard disc, where each disc is embedded by a map which is a translation composed with a dilation. At the level of spaces, group complete algebras over this operad are 2-fold loop spaces, and at the level of homology an algebra over  $H_*(D_2)$  is precisely a Gerstenhaber algebra.

A variant of the  $D_2$  operad is the *framed little 2-discs* operad, denoted  $fD_2$ , introduced by Getzler [2]. Here the little discs are allowed to be embedded by a composition of a dilation, rotation, and translation. The (unframed) little 2-discs operad  $D_2$  is an operad in the category of  $S^1$ -spaces, where the circle acts by conjugation. Markl and Salvatore-Wahl [9] presented the framed little 2-discs operad as the semidirect product of the circle group  $S^1$  with  $D_2$ . In particular,  $fD_2(n) = D_2(n) \times (S^1)^n$ . Getzler observed that algebras over the homology operad  $H_*(fD_2)$  are precisely Batalin-Vilkovisky algebras, and at the space level Salvatore-Wahl proved that a group complete algebra over  $fD_2$  is a 2-fold loop space on a based space with a circle action.

The operad  $D_2$  is homotopy equivalent to the Fulton-MacPherson operad  $FM = FM_2$  [8] (we drop the subscript since we will only be discussing 2-discs in this note); the space  $FM(n)$  is a compactification of the configuration space of  $n$  ordered distinct points in the plane modulo translations and positive dilations. As with  $D_2$ , the circle acts on  $FM$  by rotations. The semidirect product construction for this action gives the *framed Fulton-MacPherson operad*  $fFM = fFM_2$ , which is homotopy equivalent to  $fD_2$ , and such that  $fFM(n) = FM(n) \times (S^1)^n$ . Both  $FM$  and  $fFM$  are operads of semi-algebraic sets.

Tamarkin [11] and Kontsevich [6, 5] proved the following formality theorem.

**Theorem 1.1.** *The operad  $C_*(FM)$  of chains on  $FM$  with real coefficients is quasi-isomorphic to its homology operad  $H_*(FM)$ , the Gerstenhaber operad.*

(One can also use singular or semi-algebraic chains in the statement; we will return to this point later.) Kontsevich's proof seems more geometric and has the advantage of extending to a proof of

---

*Date:* 23 November, 2009.

*2000 Mathematics Subject Classification.* Primary: 18D50; Secondary: 55P48, 81Q30, 81T45.

*Key words and phrases.* semidirect product operad, framed little discs, operad formality, graph complex.

formality for the little  $k$ -discs for all  $k \geq 2$ ; this proof has been explained in greater detail by Lambrechts and Volic [7].

In general, formality of an operad is a powerful property with many theoretical and computational applications. The above operad formality theorem plays an important role in Tamarkin's proof [12] of Kontsevich's deformation quantization theorem. Our purpose in writing this note is to show that Kontsevich's proof of formality of the operad  $FM$  can be adapted to show the formality of the operad  $fFM$ . Our main result is:

**Theorem 1.2.** *The operad  $C_*(fFM)$  of chains on  $fFM$  with real coefficients is quasi-isomorphic to its homology operad  $H_*(fFM) = BV$ , the Batalin-Vilkovisky operad.*

An independent proof of this formality, built from Tamarkin's method rather than Kontsevich's, is due to Severa [10]. One interesting application of this formality result is given in [1], where it is used to construct homotopy BV algebra structures on objects such as the chains on double loop spaces.

Online  $fD_2$ , the operad  $fFM$  in fact has the structure of a *cyclic operad*, and so it is natural to ask if the formality can be made compatible with the cyclic structure. After the present work was completed we found a proof [3] of the stronger result that  $fFM$  is formal as a cyclic operad, although that proof is significantly more involved and required the introduction of a new type of graph complex.

Kontsevich's proof showed formality of the little  $k$ -discs operad for *all*  $k$ , and so it is reasonable to ask if the framed  $k$ -discs operads are all formal as well (as operads, or better yet as cyclic operads). We plan to address this question in future work. The proofs given in this paper, [10], and [3] address only the case  $k = 2$ . These arguments do not work for  $k > 2$  for various reasons. In the Tamarkin formality argument it is essential that the operad spaces are  $K(\pi, 1)$ s, and this is no longer true for  $k > 2$ . The argument in this paper does not immediately extend to higher  $k$  because one would have to replace the group  $S^1 = SO(2)$  with  $SO(k)$  and find a quasi-isomorphism  $H^*SO(k) \rightarrow \Omega^*SO(k)$  that is compatible with Kontsevich's integration map; we do not know if this is possible. There are similar obstacles to adapting the argument in [3] to higher  $k$ .

**1.1. Outline of the proof.** First recall the outline of Kontsevich's proof of Theorem 1.1. It goes by constructing a certain DG-algebra  $G(n)$  of graphs together with a quasi-isomorphism  $I : G(n) \rightarrow \Omega^*(FM(n))$  to the DG-algebra of semi-algebraic forms, and a projection  $G(n) \rightarrow H^*(G(n)) = H^*(FM(n))$  that is also a quasi-isomorphism. Both of these quasi-isomorphisms are essentially morphisms of DGA cooperads (this is not quite true — the subtleties here are discussed nicely in [7]). By dualizing one can obtain from this a chain of quasi-isomorphisms giving formality of  $FM$ .

What we show in this note is that Kontsevich's formality proof is in compatible in a precise sense with the circle action. The circle action on  $FM$  makes  $H^*(FM)$  into a cooperad in  $H := H^*(S^1)$ -comodules. Kontsevich's DGA cooperad of admissible graphs  $G$  has a differential given by contracting edges; we define a degree  $-1$  derivation  $\Delta$  on  $G(n)$  given by deleting edges. This derivation defines a  $H$ -comodule structure on  $G(n)$ ; we check that that this comodule structure is compatible with the cooperad structure and that the projection  $G(n) \rightarrow H^*(FM(n))$  is a morphism of  $H$ -comodules. Using the quasi-isomorphism  $H^*(S^1) \rightarrow \Omega^*(S^1)$ , this morally allows us to form a diagram of quasi-isomorphisms of semidirect product cooperads

$$\Omega^*(FM \rtimes S^1) \leftarrow G \rtimes H \rightarrow H^*(FM) \rtimes H \cong H^*(FM \rtimes S^1).$$

However, the proof is not quite so simple because the functor of semi-algebraic forms is contravariant monoidal and so  $\Omega^*(FM \rtimes S^1)$  is not a cooperad on the nose. Nevertheless, this issue can be overcome easily, exactly as discussed in [7] in the unframed case.

2. A DEGREE  $-1$  DERIVATION ON ADMISSIBLE GRAPHS

Consider the circle action  $\rho_n : S^1 \times FM(n) \rightarrow FM(n)$ , and let  $H$  denote the coalgebra  $H^*(S^1) = \mathbb{R}[d\theta]$ . The circle action induces an  $H$ -comodule structure on  $H^*(FM(n))$ ; the coaction is given by the formula

$$\rho_n^*(x) = [d\theta] \otimes \Delta(x) + 1 \otimes x$$

where

$$\Delta : H^*(FM(n)) \rightarrow H^{*-1}(FM(n))$$

is a degree  $-1$  derivation. Clearly the  $H$ -comodule structure and the derivation  $\Delta$  determine each other.

We shall now lift the  $H$ -comodule structure to Kontsevich's admissible graph complex  $G(n)$  by lifting the derivation  $\Delta$ . Recall that  $G(n)$  is the complex of real vector spaces spanned by admissible graphs on  $n$  external vertices [5]. The grading is defined by

$$(\# \text{ edges}) - 2(\# \text{ internal vertices}).$$

Graphs are *admissible* if they are at least trivalent at each internal vertex and satisfy a few additional conditions. Each graph is equipped with a total ordering of its edges, and a permutation of the edges acts on the corresponding generator of  $G(n)$  by its sign. Given a graph  $g$  with edges  $e_1, \dots, e_k$ , the differential is defined by

$$dg = \sum_i (-1)^i g/e_i$$

where  $g/e_i$  is the graph obtained from  $g$  by collapsing the edge  $e_i$ . Any non-admissible terms occurring in the sum are set to zero. Recall that the complex  $G(n)$  has a graded commutative algebra structure given by disjoint union of internal vertices and the union of edges. The Kontsevich integral defines a morphism of differential graded algebras  $I : G(n) \rightarrow \Omega^*(FM(n))$  (the target is the algebra of semi-algebraic forms defined in [4]) and this is a quasi-isomorphism.

**Definition 2.1.** A linear operator  $\Delta : G(n) \rightarrow G(n)$  of degree  $-1$  is defined as follows. Given a graph  $g \in G(n)$  with ordered set of edges  $e_1, \dots, e_k$ ,

$$\Delta(g) = \sum_i (-1)^{i+1} (g - e_i)$$

where  $g - e_i$  is the graph obtained by deleting the edge  $e_i$  from  $g$  (without identifying the endpoints together). If a summand is a non-admissible graph then we set it to zero.

**Proposition 2.2.** The operator  $\Delta$  satisfies:

- (1)  $\Delta^2 = 0$ ;
- (2) it is a derivation of the algebra  $G(n)$ ;
- (3) it graded commutes with the differential  $d$  of  $G(n)$ , i.e.  $d\Delta = -\Delta d$ .

Hence the rule

$$g \mapsto 1 \otimes g + [d\theta] \otimes \Delta(g)$$

gives  $G(n)$  the structure of a DG-comodule over the coalgebra  $H := H^*(S^1)$ .

Let  $\theta_{ij} : FM(n) \rightarrow S^1$  be the map measuring the angle of the line from the  $i$ -th to the  $j$ -th point with the first coordinate axis. The algebra  $G(n)$  is freely generated by *indecomposable* graphs, those that do not get disconnected by removing a small neighbourhood of the set of external vertices. If  $g$  is indecomposable with no internal vertices then it has only an edge between some vertices  $i$  and  $j$ , and we denote it  $g = \alpha_{ij}$ . Kontsevich considers the algebra map

$$q_n : G(n) \rightarrow H^*(G(n)) = H^*(FM(n))$$

sending all graphs with internal vertices to 0, and such that

$$q_n(\alpha_{ij}) = \theta_{ij}^*(d\theta) := d\theta_{ij}.$$

**Proposition 2.3** (Kontsevich, Lambrechts-Volic). *The collection of maps  $\{q_n\}$  assemble to a quasi-isomorphism of DG-cooperads  $q : G \rightarrow H^*(FM)$ .*

**Proposition 2.4.** *The projection  $q_n : G(n) \rightarrow H^*(FM(n))$  is a map of  $H$ -comodules, i.e.  $q \circ \Delta = \Delta \circ q$ .*

*Proof.* For any graph  $g$ , the summands of  $\Delta(g)$  have the same number of internal vertices as  $g$ . Therefore if  $g$  is indecomposable with some internal vertices then  $q(\Delta(g)) = \Delta(q(g)) = 0$ . If  $g$  is indecomposable with no internal vertices, then  $g = \alpha_{ij}$  for some  $i, j$ , and  $\Delta(g) = 1$  (the unit of the algebra  $G(n)$  is the graph on  $n$  external vertices with no edges). Then  $q(g) = [d\theta_{ij}]$ . Since the map  $\theta_{ij}$  is  $S^1$ -equivariant, we have that

$$\Delta([d\theta_{ij}]) = \theta_{ij}^*(\Delta([d\theta])) = \theta_{ij}^*(1) = 1$$

and so  $\Delta(q(g)) = 1 = q(\Delta(g))$ .  $\square$

### 3. COMPATIBILITY OF $\Delta$ WITH THE KONTSEVICH INTEGRAL AND THE COOPERAD STRUCTURES

We show in the next lemma that the operator  $\Delta$  is compatible with the integration map  $I : G(n) \rightarrow \Omega^*(FM(n))$ .

**Lemma 3.1.** *For  $g \in G(n)$ ,*

$$\rho_n^*(I(g)) = d\theta \times I(\Delta(g)) + 1 \times I(g) \in \Omega^*(S^1 \times FM(n)).$$

*Proof.* If  $g = \alpha_{ij}$ , then  $I(g) = d\theta_{ij}$ ,  $\Delta(g) = 1$ , and  $\rho_n^*(d\theta_{ij}) = d\theta \times 1 + 1 \times d\theta_{ij}$ . If  $g$  is a graph with no internal vertices, then it is a product of forms  $d\theta_{ij}$  and the result follows by multiplying the corresponding expressions, since  $\rho_n^*, I$  are algebra maps and  $\Delta$  is a derivation. If  $g$  has  $k$  internal vertices then  $I(g) = p_*(I(h))$ , for some  $h \in G(n+k)$ , where  $p_*$  denotes the push-forward along the semi-algebraic bundle projection  $p : F(n+k) \rightarrow F(n)$ . It follows from the definition of  $I$  that  $p_*(I(\Delta(h))) = I(\Delta(g))$ . The diagram

$$\begin{array}{ccc} S^1 \times F(n+k) & \xrightarrow{\rho_{n+k}} & F(n+k) \\ S^1 \times p \downarrow & & \downarrow p \\ S^1 \times F(n) & \xrightarrow{\rho_n} & F(n) \end{array}$$

is a pullback of semi-algebraic sets. By Proposition 8.13 in [4]

$$(\rho_n)^* \circ p_* = (S^1 \times p)_* \circ \rho_{n+k}^*.$$

Since  $h$  has no internal vertices

$$\rho_{n+k}^*(I(h)) = d\theta \times I(\Delta(h)) + 1 \times I(h)$$

and so

$$\begin{aligned} \rho_n^*(I(g)) &= \rho_n^*(p_*(I(h))) = (S^1 \times p)_*(\rho_{n+k}^*(I(h))) \\ &= d\theta \times p_*(I(\Delta(h))) + 1 \times p_*(I(h)) \\ &= d\theta \times I(\Delta(g)) + 1 \times I(g). \end{aligned} \quad \square$$

We show next that the operator  $\Delta$  is compatible with the cooperad structure of  $G$  constructed by Kontsevich. The tensor product of  $H$ -comodules is a  $H$ -comodule, such that  $\Delta$  on the tensor product is defined by the Leibniz rule.

**Proposition 3.2.** *The cooperad structure map*

$$\circ_i : G(m+n-1) \rightarrow G(m) \otimes G(n)$$

(with  $1 \leq i \leq m$ ) commutes with  $\Delta$ ; i.e. it is a map of  $H$ -comodules.

*Proof.* Given a graph  $g \in G(m+n-1)$ ,

$$\circ_i(g) = \sum_j (-1)^{s(j)} g'_j \otimes g''_j,$$

where  $j$  ranges over partitions of the set  $V$  of internal vertices into two sets  $V'_j$  and  $V''_j$ . Then  $g''_j \in G(n)$  is the full subgraph of  $g$  containing the external vertices  $\{i, \dots, i+n-1\}$  (relabelled) and the internal vertices in  $V''_j$ . The graph  $g'_j$  is obtained from  $g$  by collapsing  $g''_j$  to a single external vertex, and relabelling external vertices. The sign  $s(j)$  is the sign of the permutation moving the edges from the ordering of  $g$  to the ordering of  $g'_j$  followed by the ordering of  $g''_j$ . If such graphs have repeated edges or are not admissible then they are identified to zero. From the definition one sees that

$$\circ_i(\Delta(g)) = \sum_j (-1)^{s(j)} (\Delta(g'_j) \otimes g''_j + (-1)^{|g'_j|} g'_j \otimes \Delta(g''_j)) = \Delta(\circ_i(g)). \quad \square$$

#### 4. FROM $H$ -COMODULES TO SEMIDIRECT PRODUCT COOPERADS

By Proposition 3.2 above one can form the semidirect product cooperad  $G \rtimes H$  with  $(G \rtimes H)(n) = G(n) \otimes H^{\otimes n}$ , by extending to the differential graded setting the construction in section 4 of [9], and dualizing it. Explicitly the cooperad structure maps

$$\circ_i : (G \rtimes H)(m+n-1) \rightarrow (G \rtimes H)(m) \otimes (G \rtimes H)(n)$$

are the algebra maps defined by sending

$$\begin{aligned} d\theta_k &\mapsto d\theta'_i + d\theta''_{k-i+1} \text{ for } i \leq k \leq n+i-1, \\ d\theta_k &\mapsto d\theta'_k \text{ for } k < i, \\ d\theta_k &\mapsto d\theta''_{k-i+1} \text{ for } k \geq n+i, \end{aligned}$$

and for  $g \in G(m+n-1)$ ,

$$g \mapsto \sum_j (-1)^{s(j)} (g'_j \otimes d\theta'_i \otimes \Delta(g''_j) + g'_j \otimes g''_j).$$

By Proposition 2.4 and Proposition 2.3 the collection  $q$  induces a quasi-isomorphism of cooperads

$$G \rtimes H \rightarrow H^*(FM) \rtimes H = H^*(fFM).$$

The semi-algebraic differential forms on  $FM$  do not exactly constitute a cooperad because semi-algebraic forms is a contravariant monoidal functor. The cross product of forms  $\Omega^*(FM(m)) \otimes \Omega^*(FM(n)) \rightarrow \Omega^*(FM(m) \times FM(n))$  (which is a quasi-isomorphism) and the operad composition  $\circ_i : FM(m) \times FM(n) \rightarrow FM(m+n-1)$  induce a zigzag

$$\Omega^*(FM(m+n-1)) \rightarrow \Omega^*(FM(m) \times FM(n)) \leftarrow \Omega^*(FM(m)) \otimes \Omega^*(FM(n)).$$

Nevertheless, there is a compatibility rule between operadic composition in  $G$  and  $FM$ .

**Lemma 4.1** (Lemma 8.19 of [7]). *The pullback along the operad composition map  $\circ_i^{FM} : FM(m) \times FM(n) \rightarrow FM(m+n-1)$ , for  $g \in G(m+n-1)$ , gives*

$$(\circ_i^{FM})^*(I(g)) = \sum_j (-1)^{s(j)} I(g'_j) \times I(g''_j)$$

where  $\circ_i^G(g) = \sum_j (-1)^{s(j)} g'_j \otimes g''_j$ .

We state next an analogous compatibility condition for the *framed* case. There are quasi-isomorphisms

$$\Omega^*(FM(n)) \otimes H^{\otimes n} \rightarrow \Omega^*(FM(n)) \otimes \Omega^*(S^1)^{\otimes n} \rightarrow \Omega^*(FM(n) \times (S^1)^n) = \Omega^*(fFM(n)).$$

The first map sends fundamental classes of circles to volume forms, and the second map is the cross product of forms. The composition with the Kontsevich integral gives a quasi-isomorphism of algebras

$$I' : G(n) \otimes H^{\otimes n} = (G \rtimes H)(n) \rightarrow \Omega^*(fFM(n)).$$

**Lemma 4.2.** *The diagram*

$$\begin{array}{ccc} (G \rtimes H)(m+n-1) & \xrightarrow{\circ_i^{G \rtimes H}} & (G \rtimes H)(m) \otimes (G \rtimes H)(n) \\ \downarrow I' & & \downarrow I' \otimes I' \\ \Omega^*(fFM(m+n-1)) & \xrightarrow{(\circ_i^{fFM})^*} & \Omega^*(fFM(m) \times fFM(n)) \longleftarrow \Omega^*(fFM(m)) \otimes \Omega^*(fFM(n)) \end{array}$$

*commutes.*

*Proof.* By definition of semidirect product the composition in  $fFM$  is

$$(x, z_1, \dots, z_m) \circ_i^{fFM} (y, w_1, \dots, w_n) = (x \circ_i^{FM} \rho_m(z_i, y), z_1, \dots, z_{i-1}, z_i w_1, \dots, z_i w_n, z_{i+1}, \dots, z_m).$$

The lemma follows from this, Lemma 3.1 and Lemma 4.1.  $\square$

We proceed similarly as in section 10 of [7] observing that integration on semi-algebraic chains of forms associated to graphs defines pairings

$$C_*(fFM(n)) \otimes (G \rtimes H)(n) \rightarrow \mathbb{R}$$

sending  $c \otimes g \mapsto \int_c I(g)$ , and their adjoints give a quasi-isomorphism of operads

$$C_*(fFM) \rightarrow (G \rtimes H)^*.$$

This together with the fact that  $q^* : H_*(fFM) \rightarrow (G \rtimes H)^*$  is a quasi-isomorphism of operads establishes theorem 1.2.

#### REFERENCES

- [1] I. Gálvez-Carrillo, A. Tonks, and B. Vallette, Homotopy Batalin-Vilkovisky algebras. arXiv:0907.2246.
- [2] E. Getzler, Batalin-Vilkovisky algebras and two-dimensional topological field theories. *Comm. Math. Phys.* 159 (1994), no. 2, 265–285.
- [3] J. Giansiracusa and P. Salvatore, cyclic formality of the framed 2-discs operad and genus zero stable curves, arXiv:0911.4430
- [4] R. Hardt, P. Lambrechts, V. Turchin and I. Volic, Real homotopy theory of semi-algebraic sets, arXiv:0806.0476
- [5] M. Kontsevich, Operads and motives in deformation quantization. *Lett. Math. Phys.* 48 (1999) 35–72.
- [6] M. Kontsevich, Deformation Quantization of Poisson Manifolds. *Lett. Math. Phys.* 66 (2003) 157–216.
- [7] P. Lambrechts and I. Volic, Formality of the little N-discs operad, arXiv:0808.0457
- [8] P. Salvatore, Configuration spaces with summable labels. *Cohomological methods in homotopy theory* 375–395, *Progr. Math.*, 196, Birkhäuser, Basel, 2001.
- [9] P. Salvatore and N. Wahl, Framed discs operads and Batalin-Vilkovisky algebras, *Q.J.Math.* 54 (2003), 213–231.
- [10] P. Severa, Formality of the chain operad of framed little disks, arXiv:math/0902.3576
- [11] D. Tamarkin, Formality of chain operad of little discs, *Lett. Math. Phys.* 66 (2003), n.1-2, 65–72
- [12] D. Tamarkin, Another proof of M. Kontsevich formality theorem, arXiv:math/9803025

MATHEMATICAL INSTITUTE, UNIVERSITY OF OXFORD, 24 - 29 ST. GILES, OXFORD, OX1 3LB, UNITED KINGDOM

*E-mail address:* giansira@maths.ox.ac.uk

DIPARTIMENTO DI MATEMATICA, UNIVERSITA' DI ROMA "TOR VERGATA", VIA DELLA RICERCA SCIENTIFICA, 00133 ROMA, ITALY

*E-mail address:* salvator@mat.uniroma2.it