BASICS OF ALGEBRAIC GEOMETRY: A QUICK REVISION

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Abstract. These notes are preliminary to a lecture course on Affine Geometry. Their aim is to refresh selected standard basics on affine and projective varieties. Some pieces of the text are borrowed in different sources, see the list of references. Some facts known in a quite general setting are exposed in rather simplified versions. Certain results valid in any dimension are discussed just for surfaces. While using transcendental tools, we restrict to the complex number field $\mathbb{C}$ even for facts available over more general fields. Certain important topics remain behind the scene, e.g., abstract algebraic varieties, coherent sheaves, sheaf cohomology, schemes, Albanese map, canonical ring, classification of singularities, etc. The reader will not find mentioning of the Hodge theory, Kähler differentials, Chern classes, projective modules, stacks, birational rigidity, flatness, etc. We address to the literature for a more extended and profound treatment, proofs, and solutions of exercises.

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Chapter I. AFFINE VARIETIES

1. Ideals

Throughout these notes k stands for an algebraically closed field of characteristic zero. Let A be an integral domain over k, that is, a commutative algebra over k with a unit element and without zero divisors (elements $a, b \in A$ are called *zero divisors* if $a \cdot b = 0$, whereas $a \neq 0$ and $b \neq 0$). In particular, A has no nilpotent element, i.e., $(a \in A, a^n = 0$ for some $n \in \mathbb{N}) \Rightarrow (a = 0)$. Recall that A can be embedded into a unique smallest field, called the *fraction field of* A, $\operatorname{Frac} A := \left\{\frac{f}{g} \,|\, g \neq 0, f, g \in A\right\}$.

Definitions 1.1. A subalgebra $I \subset A$ is called an *ideal* if I is absorbing, that is, $i \cdot a \in I \ \forall i \in I, \forall a \in A$. The *radical* of I is the ideal

$$\sqrt{I} = \{a \in A \,|\, a^n \in I, \ \exists n \in \mathbb{N}\}\,.$$

If $\sqrt{I} = I$, then I is called a *radical ideal*. An ideal $I \subset A$ is called a *prime ideal* if $(ab \in I) \Rightarrow (a \in I$ or $b \in I)$. Any prime ideal is a radical one. A proper ideal $I \subset A$ is called *maximal* if $(I \subset J) \Rightarrow (J = I)$ for any proper ideal $J \subset A$. As usual, we let $(b_1, \ldots, b_n)$ denote the ideal generated by $b_1, \ldots, b_n \in A$, that is, the minimal ideal which contains these elements.

Notation 1.2. We let $\mathbb{A}^n = \mathbb{A}^n_k$ denote the affine n-space over k, and $\mathcal{O}(\mathbb{A}^n) = k[X_1, \ldots, X_n]$ the polynomial algebra over k in n variables.

Remark 1.3. Usually, $\mathcal{O}_X$ stands for the *structure sheaf* of an algebraic variety X, while the space of global sections of this sheaf is denoted by $\mathcal{O}_X(X)$, or $\Gamma(X, \mathcal{O}_X)$, or else by $H^0(X, \mathcal{O}_X)$. Simplifying the notation we write $\mathcal{O}(X)$ for the algebra $\mathcal{O}_X(X)$ of global regular functions on X, that is, the *structure ring* or *coordinate ring* of X.

Example 1.4. If $Y \subset \mathbb{A}^n$ then $\mathbb{I}(Y) := \{p \in \mathcal{O}(\mathbb{A}^n) \,|\, p|_Y = 0\}$ is a radical ideal in $\mathcal{O}(\mathbb{A}^n)$. It is maximal if and only if Y is a singleton.

Definition 1.5. A module M over a commutative algebra A is said to be *of finite type* if

$$\exists b_1, \ldots, b_k \in M \,|\, \forall b \in M, \ \ b = \sum_{i=1}^{k} a_i b_i \ \text{ for some } \ a_1, \ldots, a_k \in A\,.$$

1.6 (Hilbert Basis Theorem). *Any ideal $I \subset \mathcal{O}(\mathbb{A}^n)$ is an $\mathcal{O}(\mathbb{A}^n)$-module of finite type, i.e., $I = (b_1, \ldots, b_k)$ for some $b_1, \ldots, b_k \in I$.*

Definition 1.7. A ring R is called *Noetherian* if any ideal $I \subset R$ is finitely generated.

Example 1.8. According to the Hilbert Basis Theorem, the polynomial ring $\mathcal{O}(\mathbb{A}^n)$ is Noetherian.

Proposition 1.9. *In a Noetherian ring R, any radical ideal $I \subset R$ has a unique decomposition (up to order) into prime ideals containing it. In other words, $I = P_1 \cap \ldots \cap P_k$ for some prime ideals $P_i \subset R$, $i = 1, \ldots, k$.*

2. Affine varieties

Definitions 2.1.

- A map $f \colon \mathbb{A}^n \to \mathbb{A}^m$, $f = (f_1, \ldots, f_m)$, is called *regular* or *morphism* if $f_i \in \mathcal{O}(\mathbb{A}^n)$ $\forall i = 1, \ldots, m$.
- If f is a morphism, then $X = f^{-1}(0)$ is called a *Zariski closed subset* (or an *affine algebraic set*[1]) of $\mathbb{A}^n$.
- The complement $\mathbb{A}^n \setminus X$ is called a *Zariski open subset* of $\mathbb{A}^n$.

Example 2.2. For any ideal $I = (f_1, \ldots, f_m) \subset \mathcal{O}(\mathbb{A}^n)$ the zero set of I,

$$\mathbb{V}(I) := \{x \in \mathbb{A}^n \mid f_i(x) = 0, i = 1, \ldots, m\} = \{x \in \mathbb{A}^n \mid f(x) = 0 \ \forall f \in I\},$$

is a Zariski closed subset of $\mathbb{A}^n$.

2.3 (Hilbert Nullstellensatz). $\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}$.

Corollary 2.4. *There is a one-to-one correspondence*

$$\{\text{radical ideals } \ I \subset \mathcal{O}(\mathbb{A}^n)\} \ \longleftrightarrow \ \{\text{Zariski closed sets } \ X \subset \mathbb{A}^n\}$$

given by $X \mapsto \mathbb{I}(X)$ and $I \mapsto \mathbb{V}(I)$.

Proposition 2.5. *The Zariski open subsets of $\mathbb{A}^n$ form a topology on $\mathbb{A}^n$ called the* Zariski topology. *In particular, the union and the intersection of two Zariski open subsets is again a Zariski open subset.*

Definitions 2.6.

- An *affine variety* X is a Zariski closed subset in $\mathbb{A}^n$ such that $\mathbb{I}(X)$ is a prime ideal.
- • A Zariski closed subset X in $\mathbb{A}^n$ is called *irreducible* if the equality $X = X_1 \cup X_2$, where X_1, X_2 are Zariski closed subsets of X, implies that either $X = X_1$, or $X = X_2$.
- The *Zariski topology* on X is the one induced by the Zariski topology on $\mathbb{A}^n$.

Exercises 2.7. Let $k = \mathbb{C}$, and let $X \subset \mathbb{A}^n$ be an affine variety. Show that

- $\mathcal{O}(X) := \mathcal{O}(\mathbb{A}^n)/\mathbb{I}(X)$ is a finitely generated integral domain, consisting of the traces on X of polynomials in n variables;
- any maximal ideal I of $\mathcal{O}(X)$ is of the form $I = \mathbb{I}(x) \cap \mathcal{O}(X)$ for a point $x \in X$;

[1] It became quite common to reserve the expression *affine variety* for irreducible Zariski closed subsets of $\mathbb{A}^n$. We adopt this terminology, although it creates some inconveniences, especially when speaking about curves.

- any nonempty Zariski open subset $U \subset X$ is dense in X, and the intersection of two such subsets is as well;
- the Zariski topology of X is not separated, unless X is a singleton;
- $X \subset \mathbb{A}^n_{\mathbb{C}} = \mathbb{R}^{2n}$ is compact in the Hausdorff topology if and only if X is a singleton. *Hint:* use the maximum modulus principle for holomorphic functions;
- any Zariski closed subset $X' \subset \mathbb{A}^n$ admits a unique decomposition $X' = X_1 \cup \ldots \cup X_k$ into irreducible Zariski closed subsets (called the *irreducible components of* X') such that $(X_i \subset X_j) \Rightarrow (i = j)$;
- the Cartesian product $Z = X \times Y$ of two affine varieties is an affine variety, and $\mathcal{O}(Z) \cong \mathcal{O}(X) \otimes \mathcal{O}(Y)$ (here "$\cong$" means an isomorphism of algebras);
- $\mathrm{SL}(n, \mathbb{C})$ is an affine variety.

Definition 2.8. An *affine domain* is a finitely generated integral domain (over k).

2.9. There is a one-to-one correspondence

$$\{\text{affine varieties}\} \quad \longleftrightarrow \quad \{\text{affine domains}\}$$

given by $X \mapsto \mathbb{A} = \mathcal{O}(X)$ and $A \mapsto \operatorname{Specm} A$, where $\operatorname{Specm} A$ stands for the set of maximal ideals of A equipped with a structure of an affine variety e.g. as follows. If A is generated by $a_1, \ldots, a_n \in A$, then $X = \operatorname{Specm} A := \mathbb{V}(J(A)) \subset \mathbb{A}^n$, where

$$J(A) = \{p \in \mathcal{O}(\mathbb{A}^n) \,|\, p(a_1, \ldots, a_n) = 0\}$$

is the *ideal of relations* of A. In other words, $A = \mathcal{O}(X) = \mathcal{O}(\mathbb{A}^n)/J(A)$.

3. Dimension

Definitions 3.1.

– Given an affine variety X, the *rational function field* of X is the field of fractions $k(X) = \operatorname{Frac} \mathcal{O}(X)$ of the *structure algebra* $\mathcal{O}(X)$.
– The elements $f_1, \ldots, f_d \in k(X)$ are said to be *algebraically independent* if

$$(p(f_1, \ldots, f_d) = 0 \;\text{ for }\; p \in k[X_1, \ldots, X_d]) \Rightarrow (p = 0)\,.$$

– The *dimension* $\dim X$ is the transcendence degree $\operatorname{tr.deg} k(X)$ of the field extension $k \subset k(X)$, that is, the maximal number of algebraically independent elements $f_1, \ldots, f_d$ of $k(X)$.

Example 3.2. An affine variety X is called a *curve* if $\dim X = 1$, a *surface* if $\dim X = 2$, and a *d-fold* if $\dim X = d \geq 3$. A plane curve $C = \{(x, y) \in \mathbb{A}^2 \,|\, p(x, y) = 0\}$, where $p \in k[X, Y]$, is irreducible if the polynomial p is. For instance, the affine cubic $C = \{x(xy - 1) = 0\}$ is reducible (that is, non-irreducible) and has two irreducible components.

4. Morphisms

Definitions 4.1. If $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ are affine varieties and $F\colon \mathbb{A}^n \to \mathbb{A}^m$ is a morphism (see Def. 2.1) such that $F(X) \subset Y$, then the map $f = F|_X\colon X \to Y$ is called a *morphism* from X to Y. The induced pull-back homomorphism $f^*\colon \mathcal{O}(Y) \to \mathcal{O}(X)$, $g \mapsto g \circ f$, is called a *comorphism*. A morphism $X \to Y$ which admits an inverse morphism $Y \to X$ is called an *isomorphism*. The varieties X and Y are said to be *isomorphic* (notation: $X \cong Y$) if there exists an isomorphism $X \to Y$. An isomorphism $X \to X$ is called an *automorphism*. The automorphisms $X \to X$ form a group denoted $\operatorname{Aut} X$ and called the *automorphism group* of X.

Exercises 4.2. Show that

- any homomorphism $\mathcal{O}(Y) \to \mathcal{O}(X)$ is a comorphism;
- the affine line $\mathbb{A}^1$ and the hyperbola $H = \{(x, y) \in \mathbb{A}^2 \mid xy - 1 = 0\}$ are not isomorphic;
- $\mathrm{GL}(n, \mathbb{C})$ is an affine variety. *Hint*: it can be given as

$$\mathrm{GL}(n, \mathbb{C}) = \{(A, t) \in M(n, \mathbb{C}) \times \mathbb{A}^1 \cong \mathbb{A}^{n^2+1} \mid t \cdot \det A = 1\},$$

  where $M(n, \mathbb{C})$ is the vector space of square matrices of order n over $\mathbb{C}$;
- $\operatorname{Aut} \mathbb{A}^1 = \operatorname{Aff} \mathbb{A}^1$, and $\operatorname{Aut} \mathbb{A}^1_* \cong \mathbb{G}_m \rtimes (\mathbb{Z}/2\mathbb{Z})$, where $\mathbb{A}^1_* := \mathbb{A}^1 \setminus \{0\}$ and $\mathbb{G}_m$ stands for the multiplicative group of the base field k.

Definitions 4.3.

– A morphism $f\colon X \to Y$ is called *birational* if the restriction $f|_U\colon U \to V$ to suitable Zariski open dense affine subsets $U \subset X$ and $V \subset Y$ is an isomorphism.
– A morphism $f\colon X \to Y$ is called *finite* if $f^*\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ is an embedding which makes $\mathcal{O}(X)$ an $f^*\mathcal{O}(Y)$-module of finite type.

Exercises 4.4. Let $k = \mathbb{C}$. Show that

- a birational morphism $f\colon X \to Y$ induces a pull-back isomorphism of rational function fields $f^*\colon k(Y) \xrightarrow{\cong} k(X)$;
- any non-constant morphism $\mathbb{A}^1 \to \mathbb{A}^1$ and $\mathbb{A}^1_* \to \mathbb{A}^1_*$ is finite, and any morphism $\mathbb{A}^1 \to \mathbb{A}^1_*$ is constant;
- any finite morphism $f\colon X \to Y$ is quasi-finite, that is, any fiber $f^{-1}(y)$ of f is finite. In fact, the cardinality of $f^{-1}(y)$ does not exceed the rank of $\mathcal{O}(X)$ as an $f^*\mathcal{O}(Y)$-module (called the *degree* of f), and coincides with the degree $\deg f$ for a generic point $y \in Y$; deduce that $\dim X \le \dim Y$;
- if $\dim X = \dim Y$, then any finite morphism $X \to Y$ is a closed map, in particular, is surjective;
- give an example of a non-surjective finite morphism.

Proposition 4.5 (Noether normalization). *Any affine variety $X \subset \mathbb{A}^n$ of dimension d admits a finite morphism $X \to \mathbb{A}^d$, which is the restriction to X of a linear map $\mathbb{A}^n \to \mathbb{A}^d$.*

Definition 4.6. A morphism $f\colon X \to Y$ of affine varieties is called *dominant* if $f(X)$ is Zariski dense in Y.

Exercise 4.7. Consider the morphism $f : \mathbb{A}^2 \to \mathbb{A}^2$ given by $f(x, y) = (x, xy)$. Is it finite? Is it dominant? Is it open? Is it closed? Describe the fibers of f.

Proposition 4.8. *Let $f\colon X \to Y$ be a dominant morphism of affine varieties. Then for any point $y \in f(X)$, any irreducible component Z of $f^{-1}(x)$ is an affine variety of dimension $\dim Z \geq \dim X - \dim Y$, with the equality sign for the points y in a Zariski dense open subset of Y.*

Exercise 4.9 (Krull Theorem). Let X be an affine variety, and let $f \in \mathcal{O}(X)$ be a nonconstant regular function on X. Show that any irreducible component of $f^{-1}(0)$ has dimension $\dim X - 1$.

Exercise 4.10. Show that for any automorphism $F = (f_1, \ldots, f_n)\colon \mathbb{A}^n \to \mathbb{A}^n$ the Jacobian determinant $\det(\operatorname{Jac} F)$ is a nonzero constant.

Remark 4.11 (Jacobian Conjecture). The famous Jacobian Conjecture claims that the converse is also true, that is, if a morphism $F\colon \mathbb{A}^n \to \mathbb{A}^n$ has a constant nonzero Jacobian determinant, then F is an automorphism. The conjecture is open for any $n \geq 2$, see, e.g., [4], [36].

5. Normal varieties

Definition 5.1. Let A be a domain over k with the fraction field $\operatorname{Frac} A$. The *integral closure* $\bar{A}$ of A is the domain

$$\bar{A} = \{f \in \operatorname{Frac} A \,|\, p(f) = 0 \text{ for some } p(T) = T^n + a_1 T^{n-1} + \ldots + a_n \in A[T]\}\,.$$

If $\bar{A} = A$ then A is called *integrally closed.*

Definitions 5.2. An affine variety X is called *normal* if $\mathcal{O}(X)$ is integrally closed in $\operatorname{Frac} \mathcal{O}(X)$. For an affine domain A, the integral closure $\bar{A}$ is again an affine domain, and the affine variety $X_{\text{norm}} := \operatorname{Specm} \bar{A}$ is called the *normalization* of $X = \operatorname{Specm} A$. The identical embedding $A \hookrightarrow \bar{A}$ leads to the *normalization morphism* $\nu\colon X_{\text{norm}} \to X$, where ν is finite and birational.

Proposition 5.3. *Let $f : X \to Y$ be a morphism of affine varieties. If Y is normal, then there exists a unique lift of f to a morphism $\widetilde{f}\colon X_{\text{norm}} \to Y$ such that the following* *diagram is commutative:*

$$
\begin{tikzcd}
 & X_{\mathrm{norm}} \arrow[dl, "\nu"'] \arrow[dr, "\widetilde{f}"] & \\
X \arrow[rr, "f"] & & Y
\end{tikzcd}
\tag{1}
$$

6. Singularities

Definitions 6.1. Let $X \subset \mathbb{A}^n$ be an affine variety of dimension $d > 0$ given by a system of equations

$$f_1 = \ldots = f_m = 0, \ \text{ where } \ f_i \in k[X_1, \ldots, X_n], \ \ i = 1, \ldots, m\,.$$

– A point $x \in X$ is called *simple* (or *smooth*, or *regular*, or *non-singular*) if

$$r := \mathrm{rk}(\mathrm{Jac}\,(f_1, \ldots, f_m)(x)) = n - d =: \mathrm{codim}_{\,\mathbb{A}^n} X\,,$$

and *singular* if $r < n - d$.

– The *Zariski tangent space* $T_x X$ of X at x is the kernel $\ker dF(x)$ of the differential $dF(x)\colon \mathbb{A}^n \to \mathbb{A}^m$ at x of the morphism $F = (f_1, \ldots, f_m)\colon \mathbb{A}^n \to \mathbb{A}^m$. (Recall that the matrix of $dF(x)$ is the Jacobian matrix $\mathrm{Jac}\,(f_1, \ldots, f_m)(x)$.)

– Let $X \subset \mathbb{A}^2$ be an affine curve over $\mathbb{C}$ passing through the origin. The local ring $\mathbb{C}\{x, y\}$ of convergent power series in two variables being factorial, there is a unique factorization of the defining polynomial $p \in \mathbb{C}[x, y]$ of C into irreducible factors in $\mathbb{C}\{x, y\}$. These factors define *local analytic branches* of C at the origin. If p is irreducible in $\mathbb{C}\{x, y\}$, then one says that $(C, 0)$ is *unibranch*.

– X is called *smooth* (or *regular*, or *non-singular*) if any point $x \in X$ is.

Exercises 6.2. Let $k = \mathbb{C}$. Show that

- if $x \in X$ is a simple point, then $\dim T_x X = d$, otherwise $\dim T_x X > d$;
- a point $x \in X$ is smooth if and only if X is a smooth submanifold of $\mathbb{A}^n = \mathbb{R}^{2n}$ in a neighborhood of x in X. *Hint*: apply the Implicit Function Theorem;
- letting $\mathcal{O}(X, x) \subset \mathrm{Frac}\,\mathcal{O}(X)$ be the local ring of all rational functions on X regular at x, and $\mathfrak{m}_x$ be the maximal ideal of functions in $\mathcal{O}(X, x)$ vanishing in x, one has $T_x X \cong (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$.

Remark 6.3. There are several other equivalent definitions of a smooth point; see, e.g., [9], [22].

Proposition 6.4. *The set* $\mathrm{sing}\, X$ *of all singular points of an affine variety* X *is a proper Zariski closed subset of* X*. Hence its complement* $\mathrm{reg}\, X = X \setminus \mathrm{sing}\, X$ *is a Zariski dense open subset of* X*.*

Exercises 6.5. Let $k = \mathbb{C}$. Show that

- the *Neil parabola* (or a *cuspidal cubic*) $\Gamma_{2,3} = \{(x, y) \in \mathbb{A}^2 \mid x^3 - y^2 = 0\}$ has a unique singular point, namely, the origin $0 \in \Gamma_{2,3}$ (such a singular point is called an *ordinary cusp*);
- $(\Gamma_{2,3}, 0)$ is unibranch;
- $(\Gamma_{2,3})_{\text{norm}} \cong \mathbb{A}^1$. *Hint*: the morphism

$$\nu \colon \mathbb{A}^1 \to \Gamma_{2,3}, \quad T \mapsto (T^2, T^3)$$

  is birational, because the pull-back comorphism $\nu^* \colon \mathcal{O}(\Gamma_{2,3}) \to \mathcal{O}(\mathbb{A}^1) = \mathbb{C}[t]$ identifies $\mathcal{O}(\Gamma_{2,3})$ with the subalgebra $\mathbb{C}[T^2, T^3] \subset \mathbb{C}[T]$, where both rings have $\mathbb{C}(T)$ as their fields of fractions;
- the nodal cubic $C = \{(x, y) \in \mathbb{A}^2 \mid x^2 - y^2(y - 1) = 0\}$ has a unique singular point $0 \in C$ (the origin) with two smooth *local branches* at 0 which meet transversally at 0 (such a singular point is called a *node*, or an *ordinary double point*);
- $C_{\text{norm}} \cong \mathbb{A}^1$;
- any normal affine curve is smooth.
- Consider the Veronese cone $V_d = \mathbb{A}^2/\mu_d = \operatorname{Specm} \mathcal{O}(\mathbb{A}^2)^{\mu_d}$, $d > 1$, where

$$\mu_d = \{(x, y) \mapsto (\zeta x, \zeta y) \mid \zeta^d = 1\},$$

  and $\mathcal{O}(\mathbb{A}^2)^{\mu_d} \subset \mathcal{O}(\mathbb{A}^2)$ is the subalgebra of μ_d-invariants. Show that V_d is a normal affine surface with a unique singular point $\bar{0} \in V_d$ (the image of the origin; such an isolated singular point $(V_d, \bar{0})$ is called a *quotient singularity*).

Chapter II. PROJECTIVE VARIETIES

7. Homogeneous ideals

Definition 7.1. The *n-dimensional projective space* $\mathbb{P}^n = \mathbb{P}^n(k)$ over k is the set of equivalence classes of $(n+1)$-tuples of elements of k, not all zero, with respect to the equivalence relation

$$(a_0, \dots, a_n) \sim (b_0, \dots, b_n) \Leftrightarrow \exists \lambda \in k^* \,|\, b_i = \lambda a_i \ \ \forall i = 0, \dots, n\,.$$

An element $p = (p_0 : \cdots : p_n) \in \mathbb{P}^n$ is called a *point*, p_i's are *homogeneous coordinates* of p. A zero set in $\mathbb{P}^n$ of a homogeneous polynomial $f \in k[x_0, ..., x_n]$ of degree d is well defined, since $f(\lambda p_0, \dots, \lambda p_n) = \lambda^d f(p_0, \dots, p_n)$.

Definition 7.2. The ideal $I \subset k[x_0, \dots, x_n]$ is called *homogeneous*, if I is generated by homogeneous polynomials.

Exercise 7.3. Let $k[x_0, \dots, x_n]_d \subset k[x_0, \dots, x_n]$ be the subspace of all homogeneous polynomials in $n+1$ variables of degree d. Show that an ideal $I \subset k[x_0, \dots, x_n]$ is homogeneous if and only if it is *graded*, that is,

$$I = \bigoplus_{i=0}^{\infty} I_d. \quad \text{where} \quad I_d = I \cap k[x_0, \dots, x_n]_d\,.$$

Definition 7.4. A subset $Z \subset \mathbb{P}^n$ is called *Zariski closed* (or a *projective algebraic set*) if Z is the zero set of a homogeneous ideal $I \subset k[x_0, \dots, x_n]$.

Remark 7.5. This same homogeneous ideal I defines an affine Zariski closed set $\widehat{X} = \mathbb{V}(I) \subset \mathbb{A}^{n+1}$ called the *affine cone* over X. The cone $\widehat{X}$ is saturated by the lines in $\widehat{X}$ passing through the origin $0 \in \mathbb{A}^{n+1}$.

Exercises 7.6. Show that

- the sum, product and intersection of homogeneous ideals is again a homogeneous ideal, and the same for the radical of an ideal;
- if a homogeneous ideal I is not prime, then there are homogeneous polynomials f, g such that $fg \in I$ but $f, g \notin I$;
- the Zariski topology is well defined on $\mathbb{P}^n$;
- any Zariski closed set in $\mathbb{P}^n$ admits a unique decomposition into irreducible components.

Exercise 7.7. Let $k = \mathbb{C}$. Show that any Zariski closed subset X in $\mathbb{P}^n$ is compact in the Hausdorff topology. *Hint*: show first that this is true for $X = \mathbb{P}^n$.

Remark 7.8. *Abstract algebraic geometry* studies *abstract algebraic varieties* obtained by gluing together affine charts via local isomorphisms. An analog of *compact* or *projective variety* in abstract algebraic geometry is *proper* or *complete variety.*

Definitions 7.9.

– A *projective variety* $X \subset \mathbb{P}^n$ is a Zariski closed set in $\mathbb{P}^n$ defined by a homogeneous prime ideal.
– X is called *linearly degenerate* if $X \subset H$, where $H \cong \mathbb{P}^{n-1}$ is a linear hyperplane in $\mathbb{P}^n$. Replacing the ambient projective space $\mathbb{P}^n$ by the linear span $\langle X \rangle \cong \mathbb{P}^k$ of X, where $k \leq n$, one may assume that X is linearly non-degenerate.

Exercise 7.10. Show that any projective variety is irreducible.

8. Morphisms of projective varieties

The affine space $\mathbb{A}^n$ embeds into the projective space $\mathbb{P}^n$ e.g. as follows:

$$\mathbb{A}^n \hookrightarrow \mathbb{P}^n, \quad (p_1, \ldots, p_n) \mapsto (1 : p_1 : \cdots : p_n) .$$

On the other hand, $\mathbb{P}^n$ can be covered by $n+1$ affine charts, $\mathbb{P}^n = U_0 \cup \cdots \cup U_n$, where $U_i = \{p = (p_0 : \cdots : p_n) \in \mathbb{P}^n \,|\, p_i \neq 0\} \cong \mathbb{A}^n$.

Exercise 8.1. Given a projective variety $X \subset \mathbb{P}^n$, show that X is covered by the affine varieties $X_i = X \cap U_i$, $i = 0, \ldots, n$.

Definitions 8.2. Let $X \subset \mathbb{P}^n$ be a linearly non-degenerate projective variety.
– We let $\dim X = \max_{0 \leq i \leq n}\{\dim X_i\}$, where the X_i are as in 8.1.
– A point $x \in X$ is called *singular* (resp., *simple* or *regular*) if x is a singular (resp., simple) point of X_i for some i.
– X is called *normal* if the X_i are normal for all $i = 0, \ldots, n$.
– A map $f\colon X \to Y$ of projective varieties $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ is called *regular* or *morphism* if $f^{-1}(Y_j)$ is a Zariski open set in X for any $j \in \{0, \ldots, m\}$, and the restriction $f|_{X_i \cap f^{-1}(Y_j)}\colon X_i \cap f^{-1}(Y_j) \to Y_j$ is a morphism for any $i \in \{0, \ldots, n\}$ and $j \in \{0, \ldots, m\}$.
– A morphism $f\colon X \to Y$ of projective varieties is called
  – *dominant* if $f(X)$ contains a Zariski open dense subset of Y;
  – *birational* if it is generically one-to-one, that is, restricts to a bijection of some Zariski open dense subsets;
  – *finite* if f is quasi-finite, that is, if $f^{-1}(y)$ is a finite set for any $y \in Y$;
  – *isomorphism* if it admits an inverse morphism;
  – *embedding* if f is an isomorphism of X onto the image $f(X) \subset Y$.
– A function f on X is called *rational* (resp., *regular*) if $f|_{X_i}$ is a rational (resp., regular) function for any $i = 0, \ldots, n$. The rational functions on X form a field denoted $k(X)$ [2]

[2] Attention: sometimes $k(X)$ is also applied for the Kodaira dimension of X. In these notes we use $\kappa(X)$ for the Kodaira dimension.

Proposition 8.3. *The set* $\operatorname{sing} X$ *of singular points of a projective variety* X *is a proper Zariski closed subset of* X*. If* X *is normal, then every irreducible component of* $\operatorname{sing} X$ *has codimension at least 2 in* X*.*

Theorem 8.4 (Noether normalization). *For a projective variety* $X \subset \mathbb{P}^n$ *of dimension* d *the following hold.*

(a) *There is a finite morphism* $X \to \mathbb{P}^d$*, which is the restriction to* X *of a linear projection* $\mathbb{P}^n \dashrightarrow \mathbb{P}^d$ *with center at a linear subspace* $\mathbb{P}^k \subset \mathbb{P}^n$ *disjoint with* X*, where* $k + d = n - 1$*.*

(b) *There exists a (unique) normal projective variety* X_{norm} *and a finite birational morphism* $\nu\colon X_{\mathrm{norm}} \to X$ *such that any morphism* $f : X \to Y$*, where* Y *is normal, admits a unique lift to a morphism* $\bar{f}\colon X_{\mathrm{norm}} \to Y$ *closing the commutative diagram (1).*

Exercises 8.5. Let X, Y be projective varieties over $\mathbb{C}$. Show that

- any normal projective curve is smooth;
- any morphism $f\colon X \to Y$ is a proper and closed map;
- the image $f(X) \subset Y$ is a projective subvariety of Y;
- any dominant (resp., birational) morphism $X \to Y$ is surjective, and any regular function on X is constant;
- the projection $X \times Z \to Z$, where Z is a complex affine variety, is proper and closed in the Zariski topology;
- $\mathbb{C}(X) = \operatorname{Frac} \mathcal{O}(U)$ for any Zariski open dense affine subset $U \subset X$;
- a morphism $f\colon X \to Y$ is birational if and only if the pull-back comorphism $f^*\colon \mathbb{C}(Y) \to \mathbb{C}(X)$ is an isomorphism (cf. 4.4);
- the Cartesian product $X \times Y$ is a projective variety. *Hint*: use the (linearly non-degenerate) Segre embedding

$$\mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^{nm+n+m}, \quad ((a_0 : \cdots : a_n),\ (b_0 : \cdots : b_m)) \mapsto (\cdots : a_i b_j : \cdots)\,;$$

- the image of the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$,

$$((a_0 : a_1),\ (b_0 : b_1)) \mapsto (x_0 : x_1 : x_2 : x_3) = (a_0 b_0 : a_0 b_1 : a_1 b_0 : a_1 b_1)\,,$$

  is the smooth quadric $Q = \{x_0 x_3 - x_1 x_2 = 0\} \subset \mathbb{P}^3$;
- a point P on the plane projective curve

$$C = \{(x : y : z) \in \mathbb{P}^2 \mid f(x, y, z) = 0\}\,,$$

  where $f \in \mathbb{C}[x, y, z] \setminus \{0\}$ is an irreducible homogeneous polynomial, is singular if and only if $\partial f/\partial x(P) = \partial f/\partial y(P) = \partial f/\partial z(P) = 0$;
- for a simple point $P \in C$, show that the tangent line to C at P has the homogeneous equation

$$x \partial f/\partial x(P) + y \partial f/\partial y(P) + z \partial f/\partial z(P) = 0\,;$$

- the plane *Weierstrass cubic*

$$y^2z - (x^3 + g_2xz^2 + g_3z^3) = 0, \quad \text{where} \quad g_2, g_3 \in \mathbb{C}\,, \tag{2}$$

  is nonsingular if and only if the polynomial $p(x) = x^3+g_2x+g_3$ has no multiple root;
- find the line tangent to this curve at the point $P = (0 : 0 : 1) \in \mathbb{P}^2$;
- give an example of two isomorphic projective varieties with non-isomorphic affine cones. *Hint*: use the *Veronese embedding* $v_d\colon \mathbb{P}^n \hookrightarrow \mathbb{P}^{N_d}$ given by the degree d monomials in $n+1$ variables. For instance,

$$v_2\colon \mathbb{P}^2 \hookrightarrow \mathbb{P}^5, \quad (x_0 : x_1 : x_2) \mapsto (x_0^2 : x_1^2 : x_2^2 : x_0x_1 : x_0x_2 : x_1x_2)\,;$$

- any morphism $F\colon \mathbb{P}^n \to \mathbb{P}^m$ can be given as $F = (f_0 : \ldots : f_m)$, where the f_i are homogeneous forms in $n+1$ variables of the same degree d, such that $f_0(x) = \ldots = f_m(x) = 0$ only if $x = 0$;
- the automorphism group $\operatorname{Aut}\mathbb{P}^n$ is the linear group $\mathbb{P}\mathrm{GL}\,(n+1, k) = \mathrm{GL}\,(n+1, k)/\mathbb{T}$, where $\mathbb{T} \cong \mathbb{G}_m$ is the subgroup of scalar matrices.

Theorem 8.6. Let $f\colon X \to Y$ be a birational morphism of normal projective varieties over $\mathbb{C}$.

(a) **(Zariski Main Theorem)** *For any point $y \in Y$ the fiber $f^{-1}(y)$ is connected. In particular,* $\dim f^{-1}(y) > 0$ *unless $f^{-1}(y)$ is a singleton.*

(b) **(Purity Theorem)** *The* exceptional locus of *f,*

$$E = E(f) := \bigcup_{\dim f^{-1}(y)>0} f^{-1}(y)\,,$$

is Zariski closed in X, and if $y_0 \in f(E)$ is a smooth point of Y, then E has codimension 1 in X near the points of the fiber $f^{-1}(y_0)$. If Y is smooth, then E is a projective hypersurface.

Exercises 8.7.

- Deduce that any birational morphism between smooth projective curves is an isomorphism.
- Let C be the nodal cubic in $\mathbb{P}^2$ given by equation $x^2z - y^2(y-z) = 0$ (cf. 6.5). Verify that the normalization morphism $\nu\colon \mathbb{P}^1 \to C$ is birational, and that the fiber $\nu^{-1}(P)$ over the point $P = (0 : 0 : 1) \in C$ consists of two points, hence is disconnected. Conclude that the normality assumption in the Zariski Main Theorem is important.

Theorem 8.8 (Stein factorization). *Let $f\colon X \to Y$ be a morphism of projective varieties over $\mathbb{C}$. Then there exist a projective variety $\widetilde{Y}$ and a commutative diagram*

$$\begin{tikzcd} & X \arrow[dl, "\widetilde{f}"'] \arrow[dr, "f"] & \\ \widetilde{Y} \arrow[rr, "g"] & & Y \end{tikzcd} \tag{3}$$

where $\widetilde{f}\colon\ X \to \widetilde{Y}$ is a morphism with connected fibers, and $g\colon \widetilde{Y} \to Y$ is a finite morphism.

Remark 8.9. An analogous fact holds also for morphisms of affine varieties.

9. Projective closure of an affine variety

Definition 9.1. If $F \in k[x_0, \dots, x_n]$ is homogeneous of degree d, we de-homogenize (with respect to x_0) F by setting $f(x_1, \dots, x_n) = F(1, x_1, \dots, x_n)$.

Exercises 9.2. Let $X \subset \mathbb{A}^n$ be an affine variety, $\bar{X} \subset \mathbb{P}^n$ its projective (Zariski) closure under the embedding $\mathbb{A}^n \subset \mathbb{P}^n$. Show that

- the ideal $\mathbb{I}(\widehat{\bar{X}}) \subset k[x_0. \dots, x_n]$ of the normal cone $\widehat{\bar{X}} \subset \mathbb{A}^{n+1}$ over $\bar{X}$ is homogeneous, generated by the homogenization of the elements of $\mathbb{I}(X)$;
- for $k = \mathbb{C}$, $\bar{X}$ coincides with the closure of X in $\mathbb{P}^n$ with respect to the usual Hausdorff topology of $\mathbb{P}^n$;
- $\bar{X} = X \cup \partial X$, where $\partial X = H_0 \cap \bar{X}$ is the hyperplane section of $\bar{X}$ by the hyperplane at infinity $H_0 = \{x_0 = 0\}$;
- the boundary ∂X is nonempty unless X is a singleton. *Hint*: consider the 'hyperbola' $H = \mathbb{V}(ft - 1) \subset \mathbb{A}^1 \times X$, where $f \in \mathcal{O}(X)$ is a nonconstant regular function. Observe that the image of the projection of H to $\mathbb{A}^1$ is $\mathbb{A}^1_*$. Conclude;
- $\dim Z = \dim \bar{X} - 1 = \dim X - 1$ for every irreducible component Z of ∂X. *Hint*: use Krull Theorem 4.9;
- for $k = \mathbb{C}$, the boundary ∂X of X is connected provided that $\dim X \geq 2$. *Hint*: use the Lefschetz Hyperplane Section Theorem, see [26, Ch. I, §7]. (Note that $\mathbb{A}^1_* = \mathbb{A}^1 \setminus \{0\}$ admits a projective completion $\mathbb{A}^1_* \hookrightarrow \mathbb{P}^1$ with a disconnected boundary. Hence the assumption $\dim X \geq 2$ is important.);
- the variety $\mathbb{A}^2_{\mathbb{C}} \setminus \{0\}$ is not affine.

10. Divisors and the Picard group

Definitions 10.1. Let X be a normal affine or projective variety.

– A *prime divisor* of X is an irreducible closed subvariety of codimension 1.

– A *Weil divisor* D on X is a finite formal $\mathbb{Z}$-linear combination of prime divisors, i.e., $D = \sum_i n_i D_i$, where $n_i \in \mathbb{Z}$ and the D_i are prime divisors in X. We let $\operatorname{supp} D = \sum_{n_i > 0} D_i$. Note that $\operatorname{supp} D = \emptyset$ if and only if $D = 0$.
– We let $\operatorname{Div} X$ (or also $\mathrm{WDiv} X$) be the free Abelian group (a lattice) generated by all prime divisors on X.
– A divisor $D = \sum_i n_i D_i$ of X is called *effective* if $n_i \geq 0\ \forall i$, and *reduced* if $n_i = 1\ \forall i$. The effective divisors form a convex cone $\operatorname{Eff} X$ in the lattice $\operatorname{Div} X$.
– A *principal divisor* D on X is the divisor of a rational function $f \in k(X)$, that is, $D = \operatorname{div} f := \operatorname{div}_0 f - \operatorname{div}_\infty f$, where $\operatorname{div}_0 f = \sum_i m_i Z_i$ and $\operatorname{div}_\infty f = \sum_j n_j Y_j$ are effective divisors such that f vanishes to order m_i along Z_i, $1/f$ vanishes to order n_j along Y_j, and f has no zero or pole outside $\operatorname{supp} D$.
– We let $\operatorname{Princ} X$ be the subgroup of principal divisors in the group $\operatorname{Div} X$.
– The quotient $\operatorname{Cl} X = \operatorname{Div} X / \operatorname{Princ} X$ is called the *divisor class group*.
– Two divisors $D, D' \in \operatorname{Div} X$ are called *linearly equivalent*, written $D \sim D'$, if $D - D' = \operatorname{div} f$ is a principal divisor, where $f \in k(X)$. The class of D in $\operatorname{Cl} X$ consists of all divisors D' on X linearly equivalent to D.
– A *Cartier divisor* is a locally principal divisor. Such a divisor D can be given by a data $\{U_i, f_i\}$ so that $D|_{U_i} = \operatorname{div} f_i$, where:
  (1) $\{U_i\}$ is a Zariski open cover of X;
  (2) $f_i \in k(X)^*$ are nonzero rational functions on X;
  (3) $f_i/f_j \in \mathcal{O}(U_i \cap U_j)$ are invertible regular functions on $U_i \cap U_j$. (The data $(\{U_i\},\ f_i/f_j)$ form a Čech cocycle on X, see Definition 13.1.)
– The Cartier divisors form a subgroup $\operatorname{CDiv} X \subset \operatorname{Div} X$.
– The *Picard group* $\operatorname{Pic} X$ is the quotient $\operatorname{Pic} X = \operatorname{CDiv} X / \operatorname{Princ} X \subset \operatorname{Cl} X$.
– Let $\varphi\colon X \to Y$ be a dominant morphism of affine (resp., projective) varieties. If $D \in \operatorname{CDiv} Y$ is given by the data (U_i, f_i), then the *pull-back* $\varphi^* D \in \operatorname{CDiv} X$ is given by the data $(\varphi^{-1}(U_i), \varphi^*(f_i))$.

Exercises 10.2. Show that

- two collections $\{U_i, f_i\}$ and $\{V_j, g_j\}$ define the same Cartier divisor on X if and only if the functions $f_i g_j^{-1} \in \mathcal{O}(U_i \cap V_j)$ are invertible for all i, j;
- if X is smooth, then any Weil divisor on X is Cartier, and so, $\operatorname{Pic} X = \operatorname{Cl} X$;
- the line $\mathbb{V}(x, y)$ on the quadric cone $X = \mathbb{V}(xy - z^2)$ in $\mathbb{A}^3$ is a Weil divisor, which is not Cartier. Deduce that $\operatorname{Cl} X \neq (0)$, and compute this group;
- for the parabola $C = \mathbb{V}(y - x^2) \subset \mathbb{A}^2$ we have $\operatorname{Cl} C = (0)$;
- for the elliptic cubic $E = \mathbb{V}(y^2 - x(x^2 - 1)) \subset \mathbb{A}^2$ every divisor D is linearly equivalent to 0 or to P for a suitable point $P \in E$;
- for a normal affine variety X the algebra $\mathcal{O}(X)$ is *factorial*, or a UFD (a *unique factorization domain*), if and only if $\operatorname{Cl} X = (0)$;
- the Picard group $\operatorname{Pic} \mathbb{P}^n \cong \mathbb{Z}$ is freely generated by the class of a hyperplane denoted by $\mathcal{O}(1) = \mathcal{O}_{\mathbb{P}^n}(1)$. Thus, $\operatorname{Pic} \mathbb{P}^n = \{\mathcal{O}(k)\}_{k \in \mathbb{Z}}$, where $\mathcal{O}(k) = \mathcal{O}(1)^{\otimes k}$;

- any morphism $f\colon X \to Y$ of affine (resp., projective) varieties induces a group homomorphism $f^*\colon \operatorname{Pic} Y \to \operatorname{Pic} X$. *Hint*: use the following lemma.

Lemma 10.3 (Moving Lemma). *For any Cartier divisor D on Y and any point $y \in \operatorname{supp} D$ there exists a Cartier divisor $D' \sim D$ on Y such that $y \notin \operatorname{supp} D'$.*

11. Intersections

There exist several alternative approaches to the intersection theory on algebraic varieties; see [11]. We prefer the homological one. In this section X is a *smooth* projective variety over $\mathbb{C}$ of dimension n (thus, X is a smooth manifold of real dimension $2n$).

Definitions 11.1 (Intersection pairing). For a k-dimensional closed subvariety Y of X we let $[Y]$ be the class of Y in $H_{2k}(X, \mathbb{Z})$.

- The intersection of homology classes defines a bilinear map $H_{2k}(X,\mathbb{Z}) \times H_{2l}(X,\mathbb{Z}) \to H_{2(k+l-n)}(X,\mathbb{Z})$.
- In particular, we have a bilinear *intersection pairing* $H_{2k}(X,\mathbb{Z}) \times H_{2(n-k)}(X,\mathbb{Z}) \to \mathbb{Z} = H_0(X,\mathbb{Z})$.
- For an $(n-k)$-dimensional closed subvariety Z of X we define the *intersection index* $Y \cdot Z$ via $Y \cdot Z := [Y] \cdot [Z] \in \mathbb{Z} = H_0(X,\mathbb{Z})$.
- If $n = 2k$, then the *self-intersection index* $Y^2 = Y \cdot Y \in \mathbb{Z}$ is well defined.
- The natural homomorphism $\operatorname{Pic} X \to H_{2n-2}(X,\mathbb{Z})$ allows to define an n-polylinear form $(\operatorname{Pic} X)^n \to \mathbb{Z}$, $(D_1, \ldots, D_n) \mapsto D_1 \cdot \ldots \cdot D_n \in \mathbb{Z}$.
- The *degree* of a k-dimensional subvariety $Y \subset \mathbb{P}^n$ is $\deg Y = Y \cdot H^k$, where $H \subset \mathbb{P}^n$ is a hyperplane.

Definition 11.2 (Multiplicity of a point on a divisor). Let D be a prime (Cartier) divisor on X, let $P \in \operatorname{supp} D$, and let $f \in \mathcal{O}(U)$ be a regular function that defines D in a neighborhood U of P, that is, $\mathbb{I}(D \cap U) = (f)$. The *multiplicity* $\operatorname{mult}_P(D)$ of P in D (or of D in P) is the positive integer k such that $f \in \mathfrak{m}_P^k \setminus \mathfrak{m}_P^{k+1}$, where $\mathfrak{m}_P$ stands for the maximal ideal of the local ring $\mathcal{O}(X, P)$.

Definition 11.3 (Local intersection index). Let now X be a smooth projective surface, let C_1 and C_2 be two distinct curves on X, and let P be an isolated point of the intersection $C_1 \cap C_2$. One can find an affine neighborhood U of P in X and regular functions f_1 and f_2 on U such that $\mathbb{I}(C_i \cap U) = (f_i)$, $i = 1, 2$. Regarding the f_i as elements of the local ring $\mathcal{O}(X, P)$, the radical $\sqrt{(f_1, f_2)}$ coincides with the maximal ideal $\mathfrak{m}_P \subset \mathcal{O}(X, P)$. It follows that the quotient $\mathcal{O}(X, P)/(f_1, f_2)$ is a finite-dimensional vector space. The *local intersection multiplicity*, or the *local intersection index* $(C_1 \cdot C_2)_P$ of P in $C_1 \cap C_2$ is $\dim \mathcal{O}(X, P)/(f_1, f_2)$. For example, if C_1 and C_2 cross transversely at P (that is, f_1 and f_2 form a system of local parameters at P), then $(f_1, f_2) = \mathfrak{m}_P$, and so, $(C_1 \cdot C_2)_P = 1$.

Exercises 11.4.

- Assume that Y and Z as in 11.1 of complementary dimensions k and $n-k$ meet transversally and at smooth points of both Y and Z only. Show that $Y \cdot Z$ counts the number of intersection points $\mathrm{card}(Y \cap Z)$. Conclude that $Y \cdot Z \geq 0$. *Hint*: use the fact that any complex manifold, viewed as a smooth real manifold, is orientable.
- Show that $\deg Y$ for $Y \subset \mathbb{P}^n$ counts the number of intersection points of Y with a generic $(n-k)$-dimensional linear subspace of $\mathbb{P}^n$. *Hint*: use the fact that $H_{2k}(\mathbb{P}^n) = \langle L_k \rangle \cong \mathbb{Z}$ for any $k = 0, \ldots, n$, where $L_k \cong \mathbb{P}^k$ is a k-dimensional linear subspace in $\mathbb{P}^n$.
- For a divisor $D = \sum_i n_i D_i$ on $\mathbb{P}^n$ we let $\deg D = \sum_i n_i \deg D_i$. Show that $\deg D = 0$ for any principal divisor D on $\mathbb{P}^n$. Deduce that deg defines an isomorphism $\deg\colon \operatorname{Pic} \mathbb{P}^n \xrightarrow{\cong} \mathbb{Z}$.
- For a divisor $D = \sum_i n_i p_i$ on a smooth projective curve Γ we let $\deg D = \sum_i n_i$. Show that $\deg D = 0$ if D is principal.
- Let D be a divisor on X and C be a curve on X such that $C \not\subset \operatorname{supp} D$. Show that $C \cdot D = \deg \nu^* D$, where $\nu\colon C_{\mathrm{norm}} \to C$ is the normalization morphism. Deduce that if D' is a prime divisor on X such that $C \cdot D' < 0$, then $C \subset D'$.
- (**Bézout Theorem**) Show that for two curves C_1 and C_2 in $\mathbb{P}^2$ of degrees d_1 and d_2, respectively, one has $C_1 \cdot C_2 = d_1 d_2$. Extend this formula to reducible curves. Assuming that $C_1 \neq C_2$ (or C_1 and C_2 has no common component, in the reducible case), show that C_1 and C_2 intersect in exactly $d_1 d_2$ points, counted with multiplicities.
- Let $\dim X = 2$, and let $P \in X$ be an isolated intersection point of two curves C_1 and C_2 on X. Show that $(C_1 \cdot C_2)_P \geq m_1 m_2$, where $m_i = \operatorname{mult}_P C_i$.
- In the same setup, verify the formula $C_1 \cdot C_2 = \sum_{P \in C_1 \cap C_2} (C_1 \cdot C_2)_P$.
- Given a curve $C \subset \mathbb{P}^2$ and a point $P \in C$, show that $\operatorname{mult}_P C = \min_l \{(C \cdot l)_P\}$, where l is a line in $\mathbb{P}^2$ through P, and that there exist lines $l' \subset \mathbb{P}^2$ (called the *tangent lines to C at P*) such that $(C \cdot l')_P > \operatorname{mult}_P C$, unless C is a line.
- Let X be an n-dimensional smooth variety, and let $P \in D_1 \cap \ldots \cap D_n$ be an isolated intersection point of n prime divisors $D_1, \ldots, D_n$ in X. Define the *local intersection index* $(D_1 \cdot \ldots \cdot D_n)_P$. Show that $(D_1 \cdot \ldots \cdot D_n)_P = 1$ if and only if the local defining equations of the D_i form a local coordinate system in a neighborhood of P.

Definitions 11.5.

– In the latter setup one says that P is a *normal crossing singularity* of the reduced divisor $D := \sum_{i=1}^n D_i$.

– A reduced divisor $D = \sum_i D_i$ is called a *simple normal crossing divisor* (an *SNC divisor*, for short) if all the components D_i are smooth and D has only normal crossing singularities.

Definitions 11.6 (Numerical equivalence). Let X be a smooth projective variety.

– Two divisors D_1 and D_2 on X are called *numerically equivalent* (denoted $D_1 \equiv D_2$) if for any curve C on X one has $D_1 \cdot C = D_2 \cdot C$.
– The classes of numerically equivalent divisors on X form an Abelian group called the *Neron-Severi group* of X and denoted by $\mathrm{NS}(X)$. Due to the Neron-Severi Theorem, this group is a lattice, that is, a finitely generated free Abelian group.

Exercise 11.7. Verify that two linearly equivalent divisors are numerically equivalent.

12. Linear systems

In this section we let X be a projective or affine normal variety and $D = \sum_i n_i D_i$ be a divisor on X.

Definitions 12.1.

– The *complete linear system of (effective) divisors* $|D|$ associated with D is the set of all effective divisors D' on X linearly equivalent to D: $D' \sim D$.
– The *Riemann–Roch space* $\mathcal{L}(D)$ associated to D is

$$\mathcal{L}(D) = \{f \in k(X) \mid D + \operatorname{div} f \geq 0\} \cup \{0\} \subset k(X)\,.$$

It consists of all rational functions f on X such that
(1) f has no poles except possibly along D_i if $n_i > 0$ (order of pole up to n_i), and
(2) f must have zeros along D_i if $n_i < 0$ (order of zero at least $-n_i$).

Exercise 12.2.

- Show that $\mathcal{L}(0) = \mathcal{O}(X)$, and that $\mathcal{L}(D)$ is an $\mathcal{O}(X)$-submodule of $k(X)$.
- Let $\mathbb{P}V$ stands for the projective space associated to a vector space V. Show that the correspondence $f \mapsto \operatorname{div}(f) + D$ yields a surjection $\varphi\colon \mathbb{P}\mathcal{L}(D) \to |D|$, which is a bijection if X is projective.
- Assume that $D' \sim D$. Show that there is a natural isomorphism of vector spaces $\mathcal{L}(D) \cong \mathcal{L}(D')$.
- Describe the complete linear system $|2h|$, where $h \cong \mathbb{P}^1$ is a projective line in $\mathbb{P}^2$. Find the dimension $\dim |2h|$.

Theorem 12.3 (Cartan-Serre). *For any normal projective variety X and any divisor D on X one has $\dim \mathcal{L}(D) < \infty$.*

Definitions 12.4.

– Fix a linear subspace $V \subset \mathcal{L}(D)$, and let $\mathbb{P}V$ be the projectivization of V. The image $L := \varphi(\mathbb{P}V) \subset |D|$ under the surjection $\varphi\colon \mathbb{P}\mathcal{L}(D) \to |D|$ is called a *linear system* on X. It is called a *(linear) pencil* if $\dim L = 1$.
– The base locus $\operatorname{Bs} V$ of a linear system L on X is the set

$$\operatorname{Bs} L = \{x \in X \mid x \in \operatorname{supp} D' \ \ \forall D' \in L\}\,.$$

– A linear system L is *base point free* if $\operatorname{Bs} L = \emptyset$.
– A *fixed component* F of a linear system L is a prime divisor appearing in the support of every $D' \in L$ (that is, the fixed components of L are the prime divisors contained in the base locus $\operatorname{Bs} L$).

Theorem 12.5 (**Bertini**). *A generic member of a linear system L on X is reduced, irreducible, and smooth away from the base locus* $\operatorname{Bs} L$.

Theorem 12.6 (**Zariski**). *Let X be a smooth projective surface, and let D be an effective divisor on X. If the complete linear system $|D|$ does not have fixed components, then for $m \gg 1$ the complete linear system $|mD|$ is base point free.*

Remark 12.7. This theorem does not hold any longer in higher dimentions. Indeed, Zariski ([13], [38]) contructed an example of a smooth projective threefold X and a prime divisor D on X such that

– $U := X \setminus \operatorname{supp} D$ is affine;
– $\operatorname{Bs} |D|$ is a curve;
– $\operatorname{Bs} |mD| \neq \emptyset \ \forall m \in \mathbb{N}$.

Exercises 12.8.

- Verify that the set of lines through a given point $P \in \mathbb{P}^2$ is a linear system of projective dimension 1 with base locus $\{P\}$ and without fixed components. Is it complete?
- What can you say about the linear system of conics in $\mathbb{P}^2$ containing a given line h?
- Show that the hyperplane sections of a projective variety $X \subset \mathbb{P}^n$ form a base point free linear system of effective divisors on X. Show that a generic member of this system is a smooth and reduced divisor on X provided that X is normal.

13. Vector bundles

Definitions 13.1. Let X be an affine or projective variety.

– A *vector bundle ξ of rank r over X* is a data $((U_i)_i, (\varphi_{ij})_{i,j})$, where $(U_i)_i$ is an affine cover of X, that is, a cover of X by Zariski open affine subsets, and the $\varphi_{ij} \colon U_i \cap U_j \to \mathrm{GL}(r, k)$ are morphisms satisfying the conditions of a *Čech cocycle*, that is., $\varphi_{ji} = \varphi_{ij}^{-1}$ and $\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki} = 1$ for all possible choices of indices i, j, k.
– The *total space* of ξ is an *abstract algebraic variety* E obtained by gluing the affine charts $U_i \times \mathbb{A}^r$ and $U_i \times \mathbb{A}^r$ over the intersections $U_i \cap U_j$ via isomorphisms

$$\Phi_{ij} \colon (U_i \cap U_j) \times \mathbb{A}^r \to (U_j \cap U_i) \times \mathbb{A}^r, \quad (x, v) \mapsto (x, \varphi_{ij}(x)(v))\,.$$

– The isomorphisms Φ_{ij} respect the first projections. Hence there is a surjective morphism $\pi \colon E \to X$ such that for any $x \in X$ the fiber $\pi^{-1}(x)$ carries a

structure of a vector space isomorphic to $\mathbb{A}^r$. Locally over U_i, π coincides with the first projection of the direct product $U_i \times \mathbb{A}^r \to U_i$.
– Given two vector bundles $\xi_i = (\pi_i \colon E_i \to X_i)$, $i = 1, 2$, of the same rank r, one says that ξ_1 and ξ_2 are *isomorphic* if there is a commutative diagram

$$\begin{tikzcd} E_1 \arrow[r, "\Phi"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\ X_1 \arrow[r, "\varphi"] & X_2 \end{tikzcd}$$

where Φ and φ are biregular isomorphisms of algebraic varieties, and for each $x \in X_1$, the restriction $\Phi|_{\pi_1^{-1}(x)} \colon \pi_1^{-1}(x) \to \pi_2^{-1}(\varphi(x))$ is an isomorphism of vector spaces.
– A regular map $s \colon X \to E$ is called a *regular section* of ξ if $\pi \circ s = \mathrm{id}_X$. If such a section s is defined only in a Zariski open dense subset $U \subset X$, then it is called a *rational section* of ξ. Locally in U_i a regular (resp., rational) section is given by an r-vector of regular (resp., rational) functions on U_i. There is a distinguished *zero section* passing through the origin in each fiber of π.
– A *line bundle* over X is a vector bundle of rank 1.
– A *projective bundle* over X or rank r is the projectivization of a vector bundle over X of rank $r + 1$. Its fiber is isomorphic to $\mathbb{P}^r$ (see Exercise 14.2).

Exercises 13.2. Let X be a smooth variety. Define

- the *tangent bundle* $T(X)$, its dual *cotangent bundle* $T^*(X)$, and the *tensor bundles* $T^{\otimes k}(X) \otimes T^{*\otimes l}(X)$;
- the direct (*Whitney*) sum $\xi \oplus \eta$ of two vector bundles ξ and η over X;
- the (tensor) inverse ξ^{-1} of a line bundle ξ on X (written usually as $-\xi$);
- the *determinant* of a rank r vector bundle ξ on X, that is, the top exterior power $\det \xi = \Lambda^r \xi$. This is a line bundle;
- for a morphism $\varphi \colon Y \to X$ and a vector bundle ξ on X, the *induced vector bundle* $\varphi^* \xi$ on Y.

Show that

- the line bundles on X form an Abelian group with respect to the tensor product (however, it is common to apply the additive notation in this group);
- the regular (resp., rational) sections of a vector bundle ξ on X form a vector space denoted by $H^0(X, \mathcal{O}_X(\xi))$; this is an $\mathcal{O}(X)$-module.

Definition 13.3. The line bundle $K_X := \det T^*(X)$ on X is called the *canonical bundle*, its inverse $-K_X$ the *anticanonical bundle*, and the tensor powers $nK_X := K_X^{\otimes n}$ (resp., $-nK_X := (-K_X)^{\otimes n}$), $n > 0$, the *pluricanonical bundles* (resp., *plurianticanonical bundles*). It $\varphi \colon X \to X'$ is an isomorphism, then $\varphi^* K_{X'} = K_X$. In particular, the pluri(anti)canonical bundles are invariant under the automorphisms of X. This justifies the adjective 'canonical'.

14. Line bundles and linear systems

Exercises 14.1. Let X be a projective variety and ξ be a line bundle over X. Show that

- given a nonzero rational section s of ξ, $D(s) = \operatorname{div}_0 s - \operatorname{div}_\infty s \in \operatorname{CDiv} X$ is a well defined Cartier divisor (called the *divisor of* s);
- $D(s)$ is effective if and only if s is a regular section of ξ;
- for any two nonzero rational sections s_1, s_2 of ξ, the ratio s_1/s_2 is a well defined rational function on X, and the divisors $D(s_1)$ and $D(s_2)$ are linearly equivalent;
- for any Cartier divisor $D' \sim D(s)$ on X there exists a rational section s' of ξ such that $D' = D(s')$;
- for any nonzero effective divisor $D' \in |D(s)|$ there exists a nonzero regular section s' of ξ such that $D' = D(s')$;
- for any Cartier divisor $D \in \operatorname{CDiv}(X)$ there exists a line bundle ξ and a rational section s of ξ such that $D = D(s)$;
- D is a principal divisor if and only if ξ is trivial;
- the divisor $-D$ corresponds to a rational section of the inverse line bundle ξ^{-1};
- if $D_1, D_2 \in \operatorname{CDiv} X$ are Cartier divisors and ξ_1, ξ_2 the corresponding line bundles on X, then $D_1 \sim D_2$ if and only if the line bundles ξ_1 and ξ_2 are equivalent;
- the divisor $D_1 + D_2$ is the divisor $D(s)$ of a rational section s of the tensor product $\xi_1 \otimes \xi_2$.

Deduce that

- the group of equivalence classes of line bundles on X with the tensor product is isomorphic to the Picard group $\operatorname{Pic} X$;
- given a nonzero rational section s of ξ, there is an isomorphism of vector spaces $\mathcal{L}(D(s)) \cong H^0(X, \mathcal{O}_X(\xi))$ (both of them are finite-dimensional due to the Cartan-Serre Theorem).

Exercises 14.2. Let X be a projective variety.

- Given a vector bundle $\xi = (\pi\colon V \to X)$ of rank $r+1$, define its *projectivization* $\mathbb{P}V \to X$ as an algebraic fiber bundle with a general fiber $\mathbb{P}^r$.
- Show that $\mathbb{P}\mathcal{E} \cong \mathbb{P}(\mathcal{E} \otimes \eta)$ for any vector bundle $\mathcal{E}$ on X and any line bundle η on X.
- Establish a one-to-one correspondence between the members of the complete linear system $|D(s)|$ and the points of the projectivization $\mathbb{P}H^0(X, \mathcal{O}_X(\xi))$.

- Show that any line bundle over $\mathbb{P}^n$ is a tensor power of the *tautological line bundle* $\mathcal{O}(-1)$. The latter bundle is the dual of the *Serre twisting line bundle* $\mathcal{O}(1)$.
- Show that: each point of $\mathbb{P}^n$ corresponds to a copy of the punctured line A^1_*, and these copies of A^1_* can be assembled into a A^1_*-bundle over $\mathbb{P}^n$ so that, by adjoining a point to each fiber, we get the tautological line bundle on $\mathbb{P}^n$.

Theorem 14.3 (Grothendieck). *Any vector bundle ξ over $\mathbb{P}^1$ is a direct sum of line bundles:*

$$\xi = \mathcal{O}_{\mathbb{P}^1}(n_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(n_r)\,,$$

where $n_i \in \mathbb{Z}$ and $r = \operatorname{rk}\xi$.

Exercises 14.4. Show that

- any projective bundle with 1-dimensional fibers over $\mathbb{P}^1$ is isomorphic to the projection of a *Hirzebruch surface* $\mathbb{F}_n$ for some $n \geq 0$, that is, the projectivization
$$\pi_n \colon \mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)) \to \mathbb{P}^1\,;$$
- the summands $\mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{O}_{\mathbb{P}^1}(n)$ give rise to the sections s_0 and s_∞ of $\pi_n \colon \mathbb{F}_n \to \mathbb{P}^1$, respectively, where $s_0^2 = n$, $s_\infty^2 = -n$, and $s_0 \cdot s_\infty = 0$ on $\mathbb{F}_n$.

15. Rational maps

Definitions 15.1. Let X be an affine variety.

- A *rational map* $F \colon X \dashrightarrow \mathbb{A}^m$ is an m-vector of rational functions $F = (f_1, \ldots, f_m)$, where $f_i \in k(X)$.
- The *domain* $\operatorname{dom} F$ of F is the maximal Zariski open subset in X, where all the f_i are regular.
- A *rational map* $F \colon X \dashrightarrow Y$, where $Y \subset \mathbb{A}^m$ is an affine variety, is a rational map $\mathbb{F} \colon X \dashrightarrow \mathbb{A}^m$ such that $F(\operatorname{dom} F) \subset Y$.
- A *rational map* $\mathbb{F} \colon X \dashrightarrow \mathbb{P}^m$ is given by $x \mapsto (f_0(x) : \cdots : f_m(x))$, where $f_i \in k(X)$ are not all zero; the vector-function $(f_0, \ldots, f_n)$ is defined up to a nonzero common factor $f \in k(X)^*$.
- Such a map F is called *regular* at a point $x \in X$ if for a suitable $f \in k(X)^*$ the functions $f \cdot f_i$ are regular at x for all $i = 0, \ldots, m$ and not vanishing simultaneously in a neighborhood of x.
- The *domain* $\operatorname{dom} F$ is the Zariski open set of all points $x \in X$, where F is regular.
- A *rational map* $F \colon X \dashrightarrow Y$, where $Y \subset \mathbb{P}^m$ is a projective variety, is a rational map $\mathbb{F} \colon X \dashrightarrow \mathbb{P}^m$ such that $F(\operatorname{dom} F) \subset Y$.
- One call Y the *image of* X if Y coincides with the Zariski closure of $F(\operatorname{dom} F)$.

Definitions 15.2. Let $X \subset \mathbb{P}^n$ be a projective variety, and let $(U_j)_j$ be an affine cover of X.

– A *rational map* $F\colon X \dashrightarrow Y$, where $Y \subset \mathbb{P}^m$ is a projective variety, is a collection of rational maps $F_j\colon U_j \dashrightarrow Y$ such that $F_i|_{U_i \cap U_j} = F_j|_{U_i \cap U_j}$ $\forall i, j$.
– The *domain* $\operatorname{dom} F \subset X$ is the union $\bigcup_j \operatorname{dom} F_j$, and the *image* of F is the image of $F|_{U_i}$ for any i.
– A rational map $F\colon X \dashrightarrow Y$ which is generically one-to-one is called *birational.*
– A projective or affine variety birational to some $\mathbb{P}^n$ is called *rational.*

Exercises 15.3. Let $F\colon X \dashrightarrow Y$ be a rational map between normal projective varieties $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$. Show that

- the *indeterminacy set* $I(F) := X \setminus \operatorname{dom} F$ is a Zariski closed subset of codimension at least 2 in X;
- any rational map from a smooth curve is regular;
- any birational map between smooth projective curves is an isomorphism;
- any smooth rational projective curve is isomorphic to $\mathbb{P}^1$;
- the map $F\colon X \dashrightarrow Y$ extends to a rational map $\tilde{F}\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^m$;
- any rational map $\Phi\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ can be given by $x \mapsto (\varphi_0(x) : \cdots : \varphi_m(x))$, where φ_i are homogeneous forms in $n+1$ variables of the same degree and without a common factor of positive degree;
- the indeterminacy set $I(\Phi)$ is the set of common zeros of the forms ϕ_i, $i = 0, \ldots, m$;
- $F\colon X \dashrightarrow Y$ is birational if and only if it induces an isomorphism of rational function fields $F^*\colon k(Y) \xrightarrow{\cong} k(X)$;
- birationally equivalent varieties are of the same dimension;
- the birational transformations $X \dashrightarrow X$ form a group (denoted $\operatorname{Bir} X$) isomorphic to the group of field automorphisms $\operatorname{Aut}_k k(X)$;
- any smooth quadric hypersurface in $\mathbb{P}^{n+1}$ is a rational variety.

The indeterminacies of rational maps can be resolved in the following manner.

Theorem 15.4 (Hironaka). *Let X and Y be smooth projective varieties, and $F\colon X \dashrightarrow Y$ be a birational map. Then F fits in a commutative diagram*

$$\begin{tikzcd} & W \arrow[dl, "g"'] \arrow[dr, "f"] & \\ X \arrow[rr, dashed, "F"] & & Y \end{tikzcd} \tag{4}$$

where W is a smooth projective variety and f and g are birational morphisms.

16. Rational maps vs line bundles vs linear systems

Definition 16.1 (Rational map associated with a linear system). Let X be a projective variety, and let L be a linear system of effective divisors on X without fixed components. One may interpret L as a subspace of the vector space $H^0(X, \mathcal{O}_X(\xi))$ of regular sections of a suitable line bundle $\xi = (\pi\colon E \to X)$ on X. Fix an affine coordinate in the fiber $\pi^{-1}(x) \cong \mathbb{A}^1$ over a point $x \in X$. The evaluation $\mathrm{eval}_x\colon s \mapsto s(x)$ defines a linear form $L \to \mathbb{A}^1$, that is, an element of the dual vector space L^*. The affine coordinate in the fiber $\pi^{-1}(x)$ is defined uniquely up to a nonzero factor. Hence the image of eval_x in the projectivization $\mathbb{P}L^*$ does not depend on the choice of an affine coordinate in $\pi^{-1}(x)$. This gives a well defined map $\Phi_L\colon X \setminus \mathrm{Bs}\, L \to \mathbb{P}L^*$, $x \mapsto [\mathrm{eval}_x]$. Indeed, the base point locus $\mathrm{Bs}\, L$ coincides with the set of common zeros of the sections in L.

Exercises 16.2.

- Verify that Φ_L extends to a rational map $X \dashrightarrow \mathbb{P}L^*$ such that $\mathrm{dom}\, \Phi_L = X \setminus \mathrm{Bs}\, L$.
- Choosing a basis $s_0, \ldots, s_m$ of L one may identify the dual projective space $\mathbb{P}L^*$ with $\mathbb{P}^m$. Using this identification, obtain a presentation
$$\Phi_L\colon x \mapsto (s_0(x) : \cdots : s_m(x))\,.$$
- Given a rational map $\Phi\colon X \dashrightarrow \mathbb{P}^m$, show that $\Phi = \Phi_L$, where $L = \Phi^*\mathcal{O}_{\Phi(X)}(1)$ is induced by the linear system of hyperplane sections of the image of X.
- Show that $\Phi_L\colon X \dashrightarrow \mathbb{P}^m$ is a morphism if and only if the linear system L is base point free: $\mathrm{Bs}\, L = \emptyset$.
- Let $Y = \overline{\Phi(X \setminus \mathrm{Bs}\, L)} \subset \mathbb{P}^m$ (the Zariski closure) be the image of X. Consider the diagram (4). Explain the meaning of the expression "*$g\colon W \to X$ is a resolution of the base point set* $\mathrm{Bs}\, L$".
- Establish the bijections

$$\begin{aligned}&\{\text{linear systems on } X \text{ of projective dimension } m \text{ without fixed components}\}\\ &\qquad\longleftrightarrow \quad \{\text{Riemann-Roch spaces on } X \text{ of dimension } m+1\}\\ &\longleftrightarrow \quad \{\text{rational maps } X \dashrightarrow \mathbb{P}^m \text{ modulo projective linear transformations}\}\,.\end{aligned}$$

17. Linear normality

Definition 17.1. Given a projective variety $X \subset \mathbb{P}^n$, one says that X is *linearly normal* if the linear system of hyperplane sections $|H|_X|$ on X is complete, where $H \subset \mathbb{P}^n$ is a hyperplane. In other words, X is linearly normal if the natural embedding $X \hookrightarrow \mathbb{P}^n$ coincides with an embedding $\Phi_{|D|}$ produced by a base point free complete linear system $|D|$ on X.

Example 17.2. A *rational normal curve* Γ_n in $\mathbb{P}^n$ is the image of the embedding
$$\Phi_{|nP|}\colon \mathbb{P}^1 \hookrightarrow \mathbb{P}^n, \quad (u:v) \mapsto (u^n : u^{n-1}v : \cdots : v^n)\,,$$
where $P \in \mathbb{P}^1$ is a point. Clearly, Γ_n is linearly normal.

Exercises 17.3. Show that

- $\deg \Gamma_n = n$;
- any linearly non-degenerate curve of degree n in $\mathbb{P}^n$ can be obtained from Γ_n by a projective linear transformation;
- any linearly non-degenerate rational curve C of degree n in $\mathbb{P}^{n-k}$, where $k > 0$, is the image of Γ_n under the projection $\pi\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-k}$ with center in a linear subspace $E \subset \mathbb{P}^n$ of dimension $k-1$ disjoint with Γ_n, so that $\pi|_{\Gamma_n}\colon \Gamma_n \to C$ is a birational morphism;
- any linearly non-degenerate projective variety $X \subset \mathbb{P}^n$, which is not linearly normal, can be obtained as the image of a linearly normal and linearly non-degenerate one $Y \subset \mathbb{P}^{n+k}$, where $k > 0$, under the projection $\pi\colon \mathbb{P}^{n+k} \dashrightarrow \mathbb{P}^n$ with center in a linear subspace $E \subset \mathbb{P}^{n+k}$ of dimension $k-1$ disjoint with Y, so that $\pi|_Y\colon Y \to X$ is a birational morphism.
- The *Veronese surface* is the image of the Veronese embedding $v_2\colon \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ given by the complete linear system $|2h|$, where $h \subset \mathbb{P}^2$ is a line. Show that there exists a projection $\pi\colon \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$ from a point $P \notin V_2$ such that the image $\pi(V_2) \subset \mathbb{P}^4$ (called a *Steiner surface*) is isomorphic to $\mathbb{P}^2$. *Hint*: An embedding of $\mathbb{P}^2$ onto a Steiner surface in $\mathbb{P}^4$ can be given via
$$(x : y : z) \mapsto (x^2 : y^2 : z^2 : y(x-z) : z(x-y)) .$$
(Note that, up to projective linear transformations, the Steiner surface is the only *Severi variety* of dimension 2, that is, the only linearly nondegenerate smooth projective surface embedded in a projective space by a non-complete linear system.)

Definition 17.4. A projective variety $X \subset \mathbb{P}^n$ is called *projectively normal* if the affine cone $\widehat{X} \subset \mathbb{A}^{n+1}$ over X is a normal affine variety.

Proposition 17.5. *A projective variety $X \subset \mathbb{P}^n$ is projectively normal if and only if for any $d \geq 1$ the restriction $|\mathcal{O}_X(d)|$ to X of the complete linear system $|dH|$ on $\mathbb{P}^n$, where $H \subset \mathbb{P}^n$ is a hyperplane, is a complete linear system on X.*

Definition 17.6. Let X be a smooth projective variety, and $K = K_X \in \operatorname{Pic} X$ be the canonical divisor class of X. The rational map $\Phi_{|nK|}\colon X \dashrightarrow \mathbb{P}^{N(n)}$, where $N(n) := \dim |nK| > 0$, is called

– *canonical* if $n = 1$;
– *pluricanonical* if $n > 1$;
– *anticanonical* if $n = -1$;
– *plurianticanonical* if $n < -1$.

Exercise 17.7. Provide examples in all four cases.

18. Contractions and blowups

Definitions 18.1. Let X and Y be projective varieties, and let $E \subsetneq X$, $Z \subsetneq Y$ be Zariski closed subsets.

– Given a morphism $\sigma\colon X \to Y$, we say that σ is a *contraction of E to Z* if
  1) $\sigma|_{X\setminus E}\colon X \setminus E \to Y \setminus Z$ is an isomorphism;
  2) $\sigma(E) = Z$;
  3) for each irreducible component E_i of E there is an irreducible component $Z_{j(i)}$ of Z such that $\sigma(E_i) = Z_{j(i)}$ and $\dim Z_{j(i)} < \dim E_i$.
– E is called the *exceptional set* of σ.
– For any subvariety $V \subset Y$ not contained in Z, the *proper* (or *strict*) *transform* V' of V in X is the Zariski closure in X of the preimage $\sigma^{-1}(V \setminus Z)$.

Exercises 18.2.

- Let $\sigma\colon X \to Y$ be a contraction with exceptional set E. Establish the equality $E = \sigma^{-1}(Z)$. Show that any irreducible component Z_j of $Z = \sigma(E)$ has codimension at least 2 in Y.
- Show that any birational morphism of normal projective varieties over $\mathbb{C}$ is a contraction. *Hint*: apply the Zariski Main Theorem.
- Give an example of a birational morphism of projective varieties (resp., of smooth affine varieties) which is not a contraction.

Theorem 18.3 (**Castelnuovo**). *Let X be a smooth (abstract) algebraic surface, and let E be a smooth projective curve in X. There exists a contraction $\sigma\colon X \to Y$ with exceptional curve E, where Y is a smooth algebraic surface, if and only if E is a* (-1)-curve, *that is, $E \cong \mathbb{P}^1$ and $E^2 = -1$.*

Definition 18.4. The inverse of the contraction $\sigma\colon X \to Y$ of a (-1)-curve E on X is called the *blowup of the point $P = \sigma(E) \in Y$*, and σ is called the *blowdown of E*.

Exercises 18.5. Let $k = \mathbb{C}$. Let Y be a smooth algebraic surface, $\sigma\colon X \to Y$ be the blowup of a point $P \in Y$ with exceptional curve E, and U be a classical neighborhood of P in Y with analytic coordinates (x, y) centered at P.

- Consider the surface $S \subset U \times \mathbb{P}^1$ given by equation $vx - uy = 0$, where $(u : v)$ are homogeneous coordinates in $\mathbb{P}^1$. Verify that S is the closure in $U \times \mathbb{P}^1$ of the graph of the meromorphic function $f(x, y) = x/y$ on U.
- Consider the restriction $\pi\colon S \to U$ to S of the natural projection $U \times \mathbb{P}^1 \to U$. Show that $C = \pi^{-1}(0, 0) \subset S$ is a (-1)-curve on S. *Hint*: replace U by $\mathbb{A}^2$ and S by the blowup of $\mathbb{A}^2$ at a point. Apply the Castelnuovo Theorem.
- Verify that there is an isomorphism $\Phi\colon S \xrightarrow{\cong} \sigma^{-1}(U)$ such that $\Phi(C) = E$.
- Let C_1 and C_2 be two curves on Y, and let C_1' and C_2' be their proper transforms in X. Show that $C_1' \cdot C_2' = C_1 \cdot C_2 - m_1 m_2$, where $m_i = \operatorname{mult}_P C_i$.
- Deduce that for any curve C on Y such that P is a point of multiplicity m of C one has $(C')^2 = C^2 - m^2$.

Assuming that Y is a smooth projective surface, show that

- X is a smooth projective surface too;

- if H is a hyperplane section of $X \subset \mathbb{P}^n$ and $D := H + mE$, where $m := H \cdot E$, then the complete linear system $|D|$ is base point free, and the morphism $\varphi_{|D|} \colon X \to \mathbb{P}^N$ contracts the curve E to a point. (In fact, after replacing H by lH with $l \gg 1$, the image $\varphi_{|D|}(X)$ is isomorphic to Y, so that $\varphi_{|D|} = \sigma$ under this isomorphism);
- $\sigma^* K_Y = K_X - E$;
- $\sigma^* C = C' + mE$ for any curve C on Y, where $m = \operatorname{mult}_P C$;
  the following *projection formulas* hold:
- $\sigma^* D_1 \cdot \sigma^* D_2 = D_1 \cdot D_2$ for any $D_1, D_2 \in \operatorname{Pic} Y$;
- $\sigma^* D \cdot E = 0$ for any $D \in \operatorname{Pic} Y$;
- $\sigma^* D_1 \cdot D_2 = D_1 \cdot \sigma_* D_2$ for any $D_1 \in \operatorname{Pic} Y$, $D_2 \in \operatorname{Pic} X$, where the *pushforward* $\sigma_* D$ is defined as folllows.

Definition 18.6. Let $\varphi \colon X \to Y$ be a dominant morphism of projective varieties of the same dimension. For a prime divisor D on X, we let

- $\varphi_*(D) = 0$ if $\dim \varphi(D) < \dim D$, and
- $\varphi_*(D) = r\varphi(D)$ otherwise, where r is the degree of the finite map $\varphi|_D \colon D \to \varphi(D)$.

For a divisor $D = \sum_i a_i D_i$ on X its *pushforward* is $\varphi_* D = \sum_i a_i \varphi_*(D_i)$.

Theorem 18.7 (**Decomposition of birational maps of surfaces**). *Let X and Y be smooth projective surfaces.*

(a) *Any birational morphism $f \colon X \to Y$ can be decomposed into a sequence of blowdowns of (-1)-curves.*

(b) *Any birational map $F \colon X \dashrightarrow Y$ can be decomposed into a sequence of blowups of points followed by a sequence of blowdowns of (-1)-curves. The first sequence yields the birational morphism $g \colon W \to X$ in diagram (4), and the second the birational morphism $f \colon W \to Y$ in (4).*

Exercises 18.8.

- Let $\sigma \colon X \to Y$ be the contraction of a (-1)-curve E on a smooth projective surface X. Show that $\operatorname{Pic} X \cong \operatorname{Pic} Y \oplus \langle E \rangle$. Deduce that if $\operatorname{Pic} Y$ is a lattice of finite rank, then $\operatorname{Pic} X$ is as well, and $\operatorname{rk} \operatorname{Pic} X = \operatorname{rk} \operatorname{Pic} Y + 1$.
- Show that $C^2 \geq 0$ for any curve C on $\mathbb{P}^1 \times \mathbb{P}^1$.
- Let $\mathbb{F}_1$ be the surface obtained as a result of a blowup of a point in $\mathbb{P}^2$. Show that $\mathbb{F}_1 \not\cong \mathbb{P}^1 \times \mathbb{P}^1$, while the ranks of the corresponding Picard groups equal 2 for both surfaces.
- Let Y be a smooth projective surface. Given a reduced normal crossing divisor $D = \sum_{i=1}^n D_i$ on Y, consider the symmetric matrix
  $$M = M(D) = (D_i \cdot D_j)_{i,j=1,\dots,n}$$
  of the bilinear intersection form on the lattice $\bigoplus_{i=1}^n \mathbb{Z} D_i$. Let $\sigma \colon X \to Y$ be the blowup of a point $P \in Y$ with the exceptional (-1)-curve E, and

let $D' = \sigma^{-1}(D) = E + \sum_{i=1}^n D'_i$. Show that $\det(-M) = \det(-M')$, where $M' = M(D')$. *Hint*: apply a suitable change of a basis.

- Verify that the linear string $D = \sum_{i=1}^k D_i$ of smooth rational curves on a smooth projective surface with $[D_1^2, \ldots, D_k^2] = [-2, \ldots, -2, -1]$ can be contracted to a smooth point. Write the matrix $M(D)$ and compute the determinant $\det(-M)$.
- Assume that $D = \sum_i D_i$ can be contracted to a smooth point. Show that $\det(-M(D)) = 1$, and D is a *tree of rational curves*, or *rational tree*. The latter means that D is an SNC (simple normal crossings) divisor, all components of D are rational curves, and the *dual graph* Γ_D is a *tree*, that is, a connected graph without cycles. The dual graph Γ_D of D is defined as follows: its vertices are the irreducible components D_i of D, and for $i \neq j$ the segment $[D_i, D_j]$ is an edge of Γ_D if and only if $D_i \cdot D_j > 0$.
- One defines the *wighted dual graph* Γ_D by attributing to any vertex D_i of Γ_D the integer D_i^2. Show that if D can be contracted to a smooth point, then there is no vertex of Γ_D of weight -1 and of degree ≥ 3, where the *degree* of a vertex is the number of incident edges.
- Deduce that, if D can be contracted to a smooth point, then this contraction can start by contracting an arbitrary (-1)-component D_i of D.

Remark 18.9. Let X be a complex manifold of dimension 2, and let $E = \sum_i E_i$ be a compact connected analytic set in X such that any irreducible component E_i of E is a curve. Due to a theorem of Grauert, there exists a contraction of E to a (singular, in general) point of a complex analytic surface Y if and only if the intersection form on $H_2(X, \mathbb{Z})$ is negative definite on the lattice $\operatorname{span}(E_i)$, that is, the bilinear form with matrix $M(E)$ is negative definite. However, for a smooth projective surface X and a curve E on X which satisfies the Grauert criterion, the analytic surface resulted from the contraction of E is not biholomorphically equivalent to any algebraic surface, in general.

Definition 18.10. Let X be an affine variety, and let $I \subset \mathcal{O}(X)$ be an ideal generated by nonzero regular functions $f_0, \ldots, f_r$ on X. The *blowup* $\mathrm{Bl}_I X$ *of* X *with center* I is the Zariski closure in $X \times \mathbb{P}^r$ of the graph $\Gamma(\varphi)$ of the rational map

$$\varphi\colon X \dashrightarrow \mathbb{P}^r, \quad x \mapsto (f_0(x) : \cdots : f_r(x))$$

together with the projection to the first factor $\sigma\colon \mathrm{Bl}_I X \to X$.

Exercises 18.11. Let X be an affine variety. Show that

- the blowup of a smooth point $P \in X$ (see 18.4) is the blowup with center the maximal ideal $\mathfrak{m}_P \subset \mathcal{O}(X)$;
- $\sigma\colon \mathrm{Bl}_I X \to X$ is a birational surjective morphism with connected projective fibers, that is, any fiber of σ is a connected Zariski closed set in a projective space.

- Is it true that any birational surjective morphism $\tilde{X} \to X$ with connected projective fibers is a blowup with center an ideal $I \subset \mathcal{O}(X)$?

Remarks 18.12.

1. Given a finitely generated graded k-domain $R = \bigoplus_{i\geq 0} R_i$, consider the set $\operatorname{Proj} R$ of all homogeneous (or graded) prime ideals of R different from the *irrelevant* (or *augmentation*) ideal $I_0 := \bigoplus_{i>0} R_i \subset R$. This set possesses a natural structure of an abstract algebraic variety. The subdomain $R_0 \subset R$ is finitely generated, hence $X = \operatorname{Specm} R_0$ is an affine variety with $\mathcal{O}(X) = R_0$. The algebra R is an R_0-module; the embedding $R_0 \subset R$ defines a morphism $\pi\colon \operatorname{Proj} R \to X$ with projective fibers.
2. Given an ideal $I \subset \mathcal{O}(X)$, the graded domain $R := R_0 \oplus \bigoplus_{k>0} I^k$, where $R_0 := \mathcal{O}(X)$, is called the *Rees algebra* of I. We have $\mathrm{Bl}_I X = \operatorname{Proj} R$. The morphism $\pi\colon \operatorname{Proj} R \to X$ induced by the inclusion $R_0 \hookrightarrow R$ coincides with the morphism of blowup $\sigma\colon \mathrm{Bl}_I X \to X$.
3. An analog of the construction "blowup of an ideal" exists also in the context of projective varieties. However, this needs introducing the language of schemes. Such a blowup of a projective variety X restricts to each element of an affine cover of X yielding the blowup of an ideal.

19. Resolution of singularities

Theorem 19.1 (Hironaka). (a) *Given a projective variety Y, there exists a smooth projective variety Y' and a birational morphism $\varphi\colon Y' \to Y$ (called a* desingularization *of Y) such that φ is an isomorphism over the smooth locus* $\operatorname{reg} Y$, *that is,*

$$\varphi|_{\varphi^{-1}(\operatorname{reg} Y)}\colon \varphi^{-1}(\operatorname{reg} Y) \xrightarrow{\cong} \operatorname{reg} Y\,. \tag{5}$$

(b) *Given a smooth projective variety X and a projective subvariety $Y \subset X$, there exists a smooth projective variety W and a birational morphism $\varphi\colon W \to X$ such that $\varphi^{-1}(Y) \subset W$ is a divisor with simple normal crossings.*

(c) *If $Y \subset X$ as in (b) is a hypersurface, then there is a unique irreducible component Y' of the exceptional set $\varphi^{-1}(Y)$ which dominates Y. The restriction $\varphi|_{Y'}\colon Y' \to Y$ is a desingularization satisfying* (5) *(in this case φ is called an* embedded desingularization, *or* embedded resolution *of Y).*

(d) *The morphism φ in* (a) *and* (b) *can be chosen to be a composition of blowups with smooth reduced centers.*

In the case of surfaces, we have the following facts.

Theorem 19.2 (Zariski). *Any projective surface can be desingularized via a sequence of transforms, which repeatedly alternates normalizations and blowups of maximal ideals. The resulting smooth surface is again projective.*

Theorem 19.3 (**Minimal resolution of surface singularities**). *Any normal projective surface Y admits a unique* minimal resolution of singularities, *that is, a birational morphism $\varphi\colon X \to Y$ such that*

1) *X is a smooth projective surface;*
2) *φ is an isomorphism over the regular locus* $\operatorname{reg} Y$*;*
3) *any other desingularization $\varphi'\colon X' \to Y$ satisfying* 1) *and* 2) *admits a factorization $\varphi' = \sigma \circ \varphi$, where $\sigma\colon X' \to X$ is a composition of blowups of points over the exceptional locus of φ.*

Example 19.4 (**Embedded resolution of singularities of a curve**). Let X be a smooth projective surface, and let C be a curve in X. The Hironaka resolution of singularities of C consists in a sequence of blowups of X in the singular points of C and their *infinitely near points*, that is, in points of the proper transforms of C on the successive projective surfaces blown up starting with X. By the Hironaka Theorem, after a finite number of steps this process terminates with an SNC divisor as the total preimage of C; see [16, Ch. V, Theorem 3.9] for an elementary proof.

Exercise 19.5. Show that the singularity of the Neil parabola $\{y^2 - x^3 = 0\}$ in $\mathbb{A}^2$ resolves after a single blowup at the origin, whereas an embedded resolution needs at least 3 blowups over the origin.

20. Riemann-Roch Theorems and Serre Duality

Notation 20.1. For a divisor D on a variety X we let $h^0(D) := \dim H^0(X, \mathcal{O}_X(D))$.

Definitions 20.2.

– Let C be a smooth projective curve, and let K_C be the canonical divisor class of C. The *genus* $g = g(C)$ is defined by the formula
$$g := \dim \mathcal{L}(K_C) = h^0(K_C)\,.$$
– The *geometric genus* $g = g(C)$ of a projective curve C is the genus of the normalization C_{norm}. The *genus* of an affine curve C is the geometric genus of the projective closure $\bar{C}$ of C.
– A curve C of geometric genus 1 is called an *elliptic curve*.

Exercises 20.3.

- Show that the canonical divisor of the projective space $\mathbb{P}^n$ satisfies $K_{\mathbb{P}^n} \sim -(n+1)H$, where $H \subset \mathbb{P}^n$ is a hyperplane.
- Deduce that $g(\mathbb{P}^1) = h^0(K_{\mathbb{P}^1}) = 0$.
- Show that birationally equivalent curves share the same geometric genus, and
- the projective curves of geometric genus zero are exactly the rational projective curves.
- Show that any smooth cubic in $\mathbb{P}^2$ is an elliptic curve, and any smooth projective elliptic curve is isomorphic to a smooth plane Weierstrass cubic (2).

- **(Riemann-Hurwitz Formula)** Given a surjective morphism $f\colon X \to Y$ of smooth projective curves X and Y, prove that
$$K_X = f^*(K_Y) + R_f\,,$$
where an effective divisor R_f on X (called the *ramification divisor* of f) is defined as follows. For a point $x \in X$ and the image $y = f(x) \in Y$ one choses local analytic coordinates z on X centered at x, resp., w on Y centered at y such that $f\colon z \mapsto w = z^{\nu(x)}$ near x. The integer $\nu(x) \geq 1$ is called the *ramification index* of f at x. We let
$$R_f := \sum_{x\in X}(\nu(x) - 1)x\,.$$
The sum is finite since $\nu(x) > 1$ just in a finite set of *branch points* of f, which are the critical points of f.
- Deduce the *Riemann-Hurwitz inequality* $g(X) \geq g(Y)$. Show that if $g(Y) \geq 2$, then the equality $g(X) = g(Y)$ forces f to be an isomorphism. Show that the latter is not any longer true if $g(Y) \in \{0, 1\}$. *Hint*: The Riemann-Hurwitz inequality in case $k = \mathbb{C}$ is equivalent to the inequality $e(X) \leq e(Y)$ for the topological Euler characteristics.[3] Indeed, we have $e(X) = 2 - 2g(X)$, and similarly for Y. The inequality $e(X) \leq e(Y)$ can be checked using the facts that the Euler characteristic is additive for disjoint partitions and multiplicative for unramified topological coverings.

Remark 20.4.

Theorem 20.5 (The Riemann-Roch Formula for curves). *Let C be a smooth projective curve of genus $g = g(C)$. For a divisor D on C of degree d one has*
$$h^0(D) - h^0(K_C - D) = d - g + 1\,.$$

Exercise 20.6. Show that
- $\deg K_C = 2g - 2$ and $\dim |K_C| = g - 1$;
- if $d \geq g$ then the linear system $|D|$ is nonempty;
- for a divisor D on C of degree $d \geq 2g - 1$ one has
$$h^0(D) = d - g + 1\,;$$
- if $g(C) = 1$, i.e., if C is a smooth elliptic curve, then $K_C \sim 0$ and there is a bijection between C and $\mathrm{Pic}^0(C)$, where $\mathrm{Pic}^0(C) := \ker(\deg\colon \mathrm{Pic}\, C \to \mathbb{Z})$.

Remark 20.7. For a smooth projective curve C of genus $g = g(C) \geq 1$, the Abelian group $\mathrm{Pic}^0(C)$ is isomorphic to the *Jacobian* $\mathrm{Jac}\, C$ of C, which is an *Abelian variety* of dimension g, that is, a smooth projective variety of dimension g equipped with a group structure such that the group operations are morphisms. For $k = \mathbb{C}$, topologically, $\mathrm{Jac}\, C$ is a real torus of dimension $2g(C)$. For an elliptic curve, $\mathrm{Jac}\, C$ is a real 2-torus.

[3]Recall that the topological Euler characteristic is the alternating sum of Betti numbers.

Theorem 20.8 (Adjunction formula). *Let X be a smooth projective variety, D be a smooth prime divisor in X, and K_X and K_D be the canonical divisors of X and D, respectively. Then*

$$K_D = (K_X + D)|_D\,.$$

Consequently, for a smooth curve C of genus $g(C)$ in a smooth projective surface X one has

$$(K_X + C) \cdot C = \deg K_C = 2g(C) - 2\,. \tag{6}$$

Exercises 20.9. Show that

- for a smooth curve $C \subset \mathbb{P}^2$ of degree d,
$$g(C) = \frac{(d-1)(d-2)}{2}\,;$$
- any smooth quartic $C \subset \mathbb{P}^2$ is a *canonical curve*, that is, the image of a smooth projective curve under the canonical embedding. *Hint*: verify that the canonical linear system on C is the system of hyperplane sections;
- vice versa, any canonical curve in $\mathbb{P}^2$ is a smooth quartic;
- if $C \subset \mathbb{P}^n$ is a smooth canonical curve of degree d and of genus g, then $g \geq 3$, $d = 2g - 2$, and $n = g - 1$;
- letting C be a smooth curve in $\mathbb{P}^1 \times \mathbb{P}^1$ of type (bidegree) (a, b), where $a = C \cdot (\{pt\} \times \mathbb{P}^1)$ and $b = C \cdot (\mathbb{P}^1 \times \{pt\})$ for a point $pt \in \mathbb{P}^1$, one has
$$g(C) = (a-1)(b-1)\,.$$
Hint: show first that
$$K_{\mathbb{P}^1 \times \mathbb{P}^1} \sim -2(\mathbb{P}^1 \times \{pt\}) - 2(\{pt\} \times \mathbb{P}^1)\,.$$
- Let $X \subset \mathbb{P}^3$ be a smooth surface of degree d.
  - Show that $K_X \sim (d-4)h$, where h is a hyperplane section of X. *Hint*: use the Adjunction Formula;
  - suppose X contains a line l. Show that $l^2 = 2 - d$ on X. *Hint*: consider a hyperplane section h of X passing through l;
  - deduce that for $d \geq 3$ there is at most finite number of lines in X. *Hint*: show that a curve with negative selfintersection in X is not movable, that is, is a unique curve in its class in $N^1(X)$;
  - in particular, $l^2 = -1$ for a line l in a smooth cubic surface in $\mathbb{P}^3$. How many lines lie on such a cubic? Justify your answer.

Remark 20.10. A generic hypersurface of degree d in $\mathbb{P}^n$

i) does not contain lines if $d \geq 2n - 2$;
ii) contains a finite number $N(n)$ of lines if $d = 2n - 3$;
iii) contains a family of lines if $d < 2n - 3$.

Definition 20.11 (Arithmetic genus of a curve). Let X be a smooth projective surface. For a divisor D on X the *arithmetic genus* $p_a(D)$ can be defined by the formula

$$p_a(D) := \frac{1}{2}D(D+K_X)+1\,.$$

Exercises 20.12. Let X be a smooth projective surface. Show that

- The arithmetic genus $p_a(D)$ depends only on the linear equivalence class of D;
- deduce that for a projective curve C of degree d in $\mathbb{P}^2$,
$$p_a(C) = \frac{1}{2}(d-1)(d-2)\,;$$
- for a smooth curve C on X one has $p_a(C) = g(C)$. *Hint*: apply the Adjunction Formula; see (6);
- birationally equivalent curves can have different arithmetic genera;
- for any divisors D, D_1, D_2 on X,
$$p_a(-D) = D^2 - p_a(D) + 2, \quad \text{and} \quad p_a(D_1+D_2) = p_a(D_1)+p_a(D_2)+D_1\cdot D_2 - 1\,;$$
- for a smooth projective curve C of genus $g \geq 0$, letting $\Delta \subset C\times C$ be the diagonal, one has $\Delta^2 = 2-2g$;
- let C be a curve on X, and let $P \in C$ be a point of multiplicity m on C. Let $\sigma\colon X' \to C$ be the blowup of P in X with exceptional (-1)-curve E, and let C' be the proper transform of C in X'. Then
$$K_{X'} = \sigma^* K_X - E, \quad \sigma^*(C) = C' + mE, \;\text{ and }\; p_a(C') = p_a(C) - \frac{1}{2}m(m-1)\,;$$
- furthermore,
$$g(C) = p_a(C) - \frac{1}{2}\sum_i m_i(m_i-1)\,,$$
the summation over all infinitely near singular points of C, that is, the singular points of C and of all its proper transforms on the intermediate surfaces in the resolution of singularities of C;
- deduce that $p_a(C) \geq 0$, and $p_a(C) = 0$ if and only if $C \cong \mathbb{P}^1$;
- **(Genus Formula)** for a curve $C \subset \mathbb{P}^2$, deduce the classical formula:
$$g(C) = \frac{1}{2}(d-1)(d-2) - \frac{1}{2}\sum_i m_i(m_i-1)\,.$$

Theorem 20.13 (The Riemann-Roch Formula for surfaces). *Let X be a smooth projective surface over $\mathbb{C}$, and let $K = K_X$ be the canonical divisor class of X. For a divisor D on X one has*

(7) $$h^0(D) + h^0(K-D) = \frac{1}{2}D(D-K) + \frac{1}{12}(K^2 + e(X)) + s(D)\,,$$

where $e(X)$ stands for the topological Euler characteristic of X, and

$$s(D) := \dim H^1(X, \mathcal{O}_X(D)) \geq 0\,.$$

Exercise 20.14. Show that if $D^2 > 0$ then either $h^0(nD) > 0$ or $h^0(-nD) > 0$ for $n \gg 0$, and exactly one of the linear systems $|nD|$ and $|-nD|$ is nonempty.

Remark 20.15 (Holomorphic Euler characteristic). The Riemann-Roch Formula (7) goes back to the 19th century. In the 20th century it was rewritten in a more elegant way, using the language of sheaves and sheaf cohomology:

$$\chi(\mathcal{O}_X(D)) - \chi(\mathcal{O}_X) = \frac{1}{2}D(D - K_X)\,, \tag{8}$$

where $\mathcal{O}_X$, resp., $\mathcal{O}_X(D)$ stands for the sheaves of germs of regular functions on X, resp., of germs of regular sections of the line bundle associated with D. For a sheaf $\mathcal{F}$ on X, the integer $\chi(\mathcal{F}) := \sum_{i\geq 0}(-1)^i h^i(\mathcal{F})$ is called the *holomorphic Euler characteristic* of $\mathcal{F}$, where $h^i(\mathcal{F}) := \dim H^i(X, \mathcal{F})$. There are analogs of (8) in all dimensions; these are the celebrated *Hirzebruch-Riemann-Roch* and *Grothendieck-Riemann-Roch Formulas.*

The *arithmetic genus* is defined for an arbitrary projective variety X as

$$p_a(X) = \chi(\mathcal{O}_X) - 1\,.$$

For a smooth projective surface X over $\mathbb{C}$, the arithmetic genus can be expressed via the *Noether Formula*

$$p_a(X) = \frac{1}{12}(K_X^2 + e(X)) - 1\,,$$

where the right hand side represents the *Todd genus*. The latter participates in (7). Another important ingredient that relates (7) and (8) is the *Serre Duality*:

$$h^i(D) = h^{n-i}(K_X - D), \quad i = 0, \dots, n,$$

for any smooth projective variety X of dimension n and any divisor D on X. For a smooth projective curve C (surface X, respectively) this yields the equality $h^1(D) = h^0(K_C - D)$ ($h^2(D) = h^0(K_X - D)$, respectively).

21. Ample divisors and ample cone

Definitions 21.1 (Ample divisors). Let X be a projective variety.

- A divisor D on X is said to be *very ample* if the rational map $\Phi_{|D|}\colon X \dashrightarrow \mathbb{P}^n$ defined by the complete linear system $|D|$ on X is a regular embedding.
- A divisor D is said to be *ample* if mD is very ample for some $m \in \mathbb{N}$.
- A *polarized variety* is a projective variety equipped with a (very) ample divisor called a *polarization*.

Exercises 21.2. Show that

- a divisor D on X is very ample if and only if $D = \varphi^* H$ for a projective embedding $\varphi\colon X \hookrightarrow \mathbb{P}^n$ and a hyperplane $H \subset \mathbb{P}^n$;
- if D is a very ample divisor on X, then D is Cartier and the complete linear system $|D|$ on X is base point free;
- a divisor D on a smooth projective curve X is very ample if and only if for any points $P, Q \in X$,
$$\dim |D - P - Q| = \dim |D| - 2\,,$$
and ample if and only if $\deg D > 0$;
- any projective curve $C \subset \mathbb{P}^2$ represents a very ample (prime) divisor on $\mathbb{P}^2$;
- any Cartier divisor on X is a difference of two very ample divisors;
- $D^2 > 0$ for any ample divisor D on a smooth projective surface X.

Due to the following theorem, ampleness is an invariant of the numerical equivalence.

Theorem 21.3 (**Nakai-Moishezon Criterion**). *A divisor D on a smooth projective surface X is ample if and only if $D^2 > 0$ and $D \cdot C > 0$ for any projective curve C on X.*

Exercise 21.4. Let X be a projective variety, and let D be an ample effective divisor on X. Show that the Zariski open set $U = X \setminus \operatorname{supp} D$ is affine.

For smooth surfaces, the converse is also true; see [13] or [15, Ch. 2, §2].

Theorem 21.5 (**Goodman**). *Let X be a smooth projective surface, and let $U \subset X$ be a Zariski open affine subset in X. Then there exists an ample effective divisor D on X such that $U = X \setminus \operatorname{supp} D$.*

Remark 21.6. Zariski's example (see Remark 12.7) shows that this theorem does not hold any longer in dimension 3 and higher.

Useful applications of ampleness to the Riemann-Roch Formula go through vanishing theorems, that ensure vanishing of the term $s(D) = h^1(D)$ in (7). The first vanishing theorems for surfaces are as follows.

Theorem 21.7. *Let X be a smooth projective surface, and D be a divisor on X.*

a) **(Kodaira Vanishing Theorem)** *If D is ample then*
$$h^1(K_X + D) = h^1(-D) = 0\,.$$
b) **(Ramanujam Vanishing Theorem)** *The same conclusions hold if D is nef and $D^2 > 0$ (see 21.14 for the notion of a nef divisor).*

Definitions 21.8 (**$\mathbb{Q}$-divisors**). Let X be a normal projective variety.

– A $\mathbb{Q}$-*divisor* $D = \sum_i r_i D_i$ on X is a formal linear combination of prime divisors D_i with rational coefficients $r_i \in \mathbb{Q}$.

A $\mathbb{Q}$-divisor $D = \sum_i r_i D_i$ is called

- *effective* if $r_i \geq 0\ \forall i$;
- $\mathbb{Q}$-*Cartier* if mD is Cartier for some $m \in \mathbb{N}$.
- X is called $\mathbb{Q}$-*factorial* if every $\mathbb{Q}$-divisor on X is $\mathbb{Q}$-Cartier.

Definitions 21.9 (Neron-Severi vector space). Let again X be a normal projective variety.

- Let $Z_1(X)$ be the group of 1-cycles on X, i.e., the free Abelian group generated by the curves in X. Two 1-cycles C_1, C_2 are said to be *numerically equivalent* (written $C_1 \equiv C_2$) if $L \cdot C_1 = L \cdot C_2$ for any $L \in \operatorname{Pic} X$.
- Define

$$N^1(X) = (\operatorname{Pic} X/ \equiv) \otimes \mathbb{R} \quad \text{and} \quad N_1(X) = (Z_1(X)/ \equiv) \otimes \mathbb{R}\,.$$

Then $N^1(X)$ (called the *Neron-Severi vector space*) and $N_1(X)$ are dual finite dimensional $\mathbb{R}$-vector spaces with respect to the non-degenerate pairing induced by

$$\operatorname{Pic} X \times Z_1(X) \to \mathbb{Z}, \quad (L, C) \mapsto L \cdot C := \deg_C(L|_C)\,.$$

The integer $\rho(X) := \dim_{\mathbb{R}} N^1(X)$ is called the *Picard number of* X.

Theorem 21.10 (Hodge Index Theorem). *Let X be a smooth projective surface, H be an ample divisor on X, and D be a divisor on X. Then $H \cdot D = 0$ implies $D^2 \leq 0$. Moreover, if $D^2 = 0$ then $D \equiv 0$.*

Exercise 21.11.

- Deduce that the intersection pairing on $N^1(X)$ is a nondegenerate symmetric bilinear form of signature

$$(+1, -(\rho(X) - 1))\,,$$

where the positive part corresponds, e.g., to the one-dimensional subspace $\langle H \rangle \subset N^1(X)$ with H being a hyperplane section of X.
- Show that for generic elliptic curves E_1 and E_2 we have $\rho(E_1 \times E_2) = 2$. Give an example of two elliptic curves E_1 and E_2 such that $\rho(E_1 \times E_2) > 2$.

Definition 21.12. One says that a surface $X \subset \mathbb{P}^3$ of degree d is *very generic* if its vector of coefficients is chosen outside a certain countable union of proper Zariski closed subsets in the affine space of the corresponding dimension $N(d)$.

Theorem 21.13 (Noether-Lefschetz). *Let X be a very generic surface in $\mathbb{P}^3$ of degree $d \geq 4$. Then $\rho(X) = 1$, and $N^1(X)$ is generated by the class of a hyperplane section of X.*

Definitions 21.14 (Mori cones). Let X be a smooth projective surface.

– We let $\mathrm{NE}(X) \subset N_1(X)$ be the cone of effective 1-cycles, that is,

$$\mathrm{NE}(X) = \{C = \sum_i r_i C_i \in N_1(X) \,|\, r_i \in R_{\geq 0}\}\,,$$

where C_i are curves.

– The *Kleiman-Mori cone* of X is the closure $\overline{\mathrm{NE}(X)}$ of $\mathrm{NE}(X)$ in the classical topology on $N_1(X)$.

– A $\mathbb{Q}$-divisor D on X is called *pseudoeffective* if the class of D (considered as a 1-cycle on X) in $N_1(X)$ belongs to the cone $\overline{\mathrm{NE}(X)}$.

– We use the following notation: $\overline{\mathrm{NE}(X)}_{H\geq 0} := \overline{\mathrm{NE}(X)} \cap H_{\geq 0}$, and similarly for $> 0, \leq 0, < 0$.

– An element $H \in N^1(X)$ is called *numerically effective* or *nef* if $H \cdot C \geq 0$ for any curve C on X, in other words, if $\overline{\mathrm{NE}(X)}_{H\geq 0} = \overline{\mathrm{NE}(X)}$.

– The set of all classes of nef divisors in $N^1(X)$ form a closed convex cone called the *nef cone*. This is the dual cone of the Kleiman-Mori cone $\overline{\mathrm{NE}(X)}$.

– The *ample cone* is the convex cone in $N^1(X)$ generated (over $\mathbb{R}$) by the classes of ample divisors on X.

The following theorem is due to Zariski for effective divisors, and to Fujita [10] in the general case.

Theorem 21.15 (**Zariski Decomposition for surfaces**). *Let X be a smooth projective surface. Any pseudoeffective $\mathbb{Q}$-divisor D on X admits a unique* Zariski decomposition *$D = H + N$, where*

i) *H and N are $\mathbb{Q}$-divisors on X;*
ii) *H is nef and $N = \sum_i a_i E_i$ is effective, where $a_i \in \mathbb{Q}$;*
iii) *either $N = 0$ or the symmetric matrix $M(N) = (E_i \cdot E_j)_{i,j}$ is negative definite;*
iv) *$H \cdot E_i = 0\ \forall i$.*

Theorem 21.16 (**Kleiman Amplness Criterion**). *Let X be a smooth projective variety. A divisor $D \in \operatorname{Pic} X$ is ample if and only if $D\cdot C > 0$ for all $C \in \overline{\mathrm{NE}(X)} \setminus \{0\}$.*

Definition 21.17 (**Extremal rays**). Let $V \subset \mathbb{R}^n$ be a closed convex cone. A *ray* $r \subset R$ is a one-dimensional closed subcone. The ray r is called *extremal* if

$$(u, v \in V \,\&\, u + v \in r) \Rightarrow (u, v \in r)\,.$$

Exercises 21.18. Show that

- any closed convex cone in $\mathbb{R}^n$ is the convex hall of its extremal rays;
- the Kleiman-Mori cone $\overline{\mathrm{NE}(X)}$ does not contain any line;
- if $\rho(X) = 2$, that is, $N_1(X) \cong \mathbb{R}^2$, then the cone $\overline{\mathrm{NE}(X)}$ has exactly two extremal rays with angle $< \pi$ inside the cone;
- a Cartier divisor D on X is ample (nef) if and only if it takes positive (resp., non-negative) values on any extremal ray of the cone $\overline{\mathrm{NE}(X)}$;

- deduce the following corollary.

Corollary 21.19. *The cone of ample divisors in $\mathbb{N}^1(X)$ is the interior of the nef cone.*

Exercises 21.20. Let X be a smooth projective surface. Show that

- if $z \in N_1(X)$ is such that $z^2 > 0$ and $z \cdot H > 0$ for an ample divisor H on X then z is an interior point of the cone $\overline{\mathrm{NE}(X)}$;
- let C be a curve in X. If $C^2 \le 0$ then the class $[C] \in N_1(X)$ belongs to the boundary of $\overline{\mathrm{NE}(X)}$, and if $C^2 < 0$ then $[C]$ lies on an extremal ray;
- for an extremal ray R of the cone $\overline{\mathrm{NE}(X)}$ the following conditions are equivalent:
  i) $R^2 < 0$;
  ii) $R \cdot C < 0$ for a curve C in X;
  iii) $R^2 < 0$ and R contains the class of a curve in X;
- if there is an extremal ray $R \subset \overline{\mathrm{NE}(X)}$ such that $R^2 > 0$, then $\rho(X) = 1$;
- if R is K_X*-negative*, that is, $K_X \cdot R < 0$, then one of the following holds:
  i) $R^2 < 0$ and $R = \mathbb{R}_{\ge 0}[C]$ for a (-1)-curve C in X;
  ii) $R^2 = 0$ and X admits a ruling with the class f of a general fiber sitting in R;
  iii) $R^2 > 0$, $\rho(X) = 1$, and $-K_X$ is ample, hence $X \cong \mathbb{P}^2$.

22. Minimal Model Program for surfaces

Theorem 22.1 (Mori Cone Theorem for surfaces). *Let X be a smooth projective surface. Then*

$$\overline{\mathrm{NE}(X)} = \overline{\mathrm{NE}(X)}_{K_X \ge 0} + \sum_i R_i \,,$$

where R_i are the extremal rays of $\overline{\mathrm{NE}(X)}$ contained in $\overline{\mathrm{NE}(X)}_{K_X < 0}$. Moreover, for any ample divisor H and any $\varepsilon > 0$ there are only finitely many extremal rays R_i such that $(K_X + \varepsilon H) \cdot R_i \le 0$.

Remark 22.2. The theorem can be interpreted as follows. Consider the linear form on $N_1(X)$ defined by the canonical divisor K_X. Then the part of the Kleiman-Mori cone $\overline{\mathrm{NE}(X)}$ which sits in the negative halfspace defined by K_X (if not empty) is locally polyhedral and is spanned by a countable set of extremal rays R_i. Moreover, moving an ε away from the hyperplane $\{C \in N_1(X) \,|\, K_X \cdot C = 0\} \subset N_1(X)$ in the K_X-negative direction, the number of extremal rays becomes finite.

Proposition 22.3 (Supporting nef divisor). *For any extremal ray $R \subset \overline{\mathrm{NE}(X)}$ with $K_X \cdot R < 0$ there exists a nef divisor H on X such that $Hz = 0$ if and only if $z \in R$.*

Definition 22.4. Let R be an extremal ray of $\overline{\mathrm{NE}(X)}$. The *extremal contraction associated to* R is a projective morphism $\varphi\colon X \to W$ onto a normal projective variety Z such that

i) for any irreducible curve $C \subset X$, $\varphi(C)$ is a point if and only if $C \in R$;
ii) φ has connected fibers;
iii) for some ample Cartier divisor A on Z, $H = \varphi^*(A) \in \operatorname{Div} X$ is a supporting nef divisor for R.

Theorem 22.5 (Contraction theorem). *Let X be a smooth projective surface. For any extremal ray R of $\overline{\mathrm{NE}(X)}$ in the halfspace $N_1(X)_{K_X<0}$ there exists the associated extremal contraction $\varphi_R\colon X \to Z$. Moreover, φ_R is one of the following types:*

(1) *Z is a smooth surface and $\varphi_R\colon X \to Z$ is the blowdown of a (-1)-curve in X, so that $\rho(Z) = \rho(X) - 1$ for the Picard numbers;*
(2) *Z is a smooth curve and $\varphi_R\colon X \to Z$ is a* minimal ruled surface over Z *(with Picard number $\rho(X) = 2$), that is, the projection $\mathbb{P}V \to Z$ of the projectivization of a rank 2 vector bundle $V \to Z$;*
(3) *Z is a point, $\rho(X) = 1$, and $-K_X$ is ample; in this case $X \cong \mathbb{P}^2$.*

Definitions 22.6. A smooth projective variety X is called

– a *Fano variety* if the anticanonical divisor $-K_X$ is ample;
– a *del Pezzo surface* if X is Fano of dimension 2;
– *ruled* if there is a surjective morphism $X \to Z$ with a general fiber $\mathbb{P}^1$;
– *uniruled* if X is covered by rational curves;
– a *minimal model* if the canonical divisor K_X is nef;
– a *minimal surface* if $\dim X = 2$ and X does not contain any (-1)-curve.

Exercise 22.7. Show that a smooth cubic surface in $\mathbb{P}^3$ and a smooth intersection of two smooth quadric hypersurfaces in $\mathbb{P}^4$ are del Pezzo surfaces.

22.8 (Minimal Model Conjecture). It suggests the following dichotomy:

Any projective variety is either uniruled or birationally equivalent to a minimal model.

The uniruledness can be detected numerically as follows.

Theorem 22.9 (Mori-Miyaoka). *Let X be a smooth projective variety. If X contains a Zariski open set U such that through any point $x \in U$ passes a curve C in X with $K_X \cdot C < 0$, then X is uniruled.*

Exercises 22.10.

- Confirm the Minimal Model Conjecture for curves. *Hint*: replace any projective curve by its normalization, and then show that the canonical divisor is nef unless the curve is rational (it is zero for an elliptic curve, and ample for a curve of genus ≥ 2).
- Let X be a smooth projective surface. Show that

– X is a minimal model if and only if $\overline{\mathrm{NE}(X)} = \overline{\mathrm{NE}(X)}_{K_X \geq 0}$, if and only if the Kleiman-Mori cone $\overline{\mathrm{NE}(X)}$ has no K_X-negative extremal ray;
– if X is a minimal model, than X is minimal, that is, it contains no (-1)-curve. Is the converse also true? *Hint*: verify, and then use, the following characterization: *a curve C on X is a (-1)-curve if and only if $C^2 < 0$ and $K_X \cdot C < 0$.*

It is common to use the abbreviation MMP for the Minimal Model Program.

Theorem 22.11 (MMP for surfaces). *Let X be a smooth projective surface. After finitely many blowdowns of (-1)-curves,*

$$X = X_n \to X_{n-1} \to \cdots \to X_0 \,,$$

one reaches a smooth surface X_0 satisfying one of the following:
1) *K_{X_0} is nef, that is, X_0 is a minimal model;*
2) *X_0 is a minimal ruled surface;*
3) *$X_0 \cong \mathbb{P}^2$.*

Remark 22.12. The MMP for surfaces consists in performing a sequence of *extremal contractions*, that is, contractions of K_X-negative extremal rays, in order to make the K_X-negative part of the Kleiman-Mori cone as small as possible. The extremal contractions of (-1)-curves are contractions of *divisorial type*, or *divisorial contractions*. Such a contraction $\sigma\colon X \to Y$ is a birational morphism onto a variety of the same dimension, which contracts a prime divisor E in X to a subvariety $\sigma(E)$ of Y of codimension at least 2, and so, decreases the Picard number by 1. An extremal contraction $X \to Y$, where $\dim Y < \dim X$, is called a *Mori fiber space*. In particular, case 1) of Contraction Theorem 22.5 represents a divisorial contraction, while cases 2) and 3) represent Mori fiber spaces with an extremal contraction to a curve, resp., to a point. Theorem 22.11 can be translated in this language as follows.

Corollary 22.13. *Running the MMP for a smooth projective surface, after a finite number of extremal divisorial contractions one arrives either at a minimal model, or at a Mori fiber space.*

Exercises 22.14. Let X be a smooth projective surface. Show that
- if X is uniruled, then it is not birationally equivalent to a minimal model, and either $X \cong \mathbb{P}^2$, or X is ruled;
- the Minimal Model Conjecture holds for projective surfaces;
- if X is rational, then X is uniruled and birationally dominates either $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_n$ with $n \neq 1$ (see 14.4).

The following two theorems provide useful supplements to Theorem 22.11.

Theorem 22.15. a) *Two smooth projective surfaces which are minimal models are birationally equivalent if and only if they are isomorphic.*

b) *A ruled surface does not admit any minimal model, and is birationally equivalent to a countable number of pairwise non-isomorphic minimal smooth ruled surfaces.*

Theorem 22.16 (Abundance Theorem for surfaces). *Let X be a smooth projective surface. If X is a minimal model, then the linear system $|mK_X|$ is base point free for a sufficiently large $m \in \mathbb{N}$.*

One can deduce from Theorem 22.11 the following classical results (see [1]).

Theorem 22.17. *Let X be a smooth projective surface.*

(Lüroth Theorem for surfaces) *If X admits a dominant morphism $\varphi\colon \mathbb{P}^2 \to X$, then X is a rational surface.*
(Castelnuovo rationality criterion) *X is rational if and only if $h^1(\mathcal{O}_X) = h^0(2K_X) = 0$.*
(Noether theorem on the Cremona group) *Any birational automorphism of $\mathbb{P}^2$ is composed of linear transformations and the quadratic transformation $(x_0 : x_1 : x_2) \mapsto (x_1x_2 : x_0x_2 : x_0x_1)$.*

Remark 22.18 (MMP in higher dimensions). In higher dimensions the uniqueness of a minimal model in its birational equivalence class (see Theorem 22.15) fails. However, two birationally equivalent minimal models differ in codimension ≥ 2 only. Moreover, the very existence of a *smooth* minimal model for a non-uniruled smooth projective variety fails as well in dimension 3.

However, the Minimal Model Conjecture was modified in dimensions ≥ 3 by allowing minimal models with certain moderate singularities. With this modification the Conjecture was established in dimension 3 by Mori (1988), and in higher dimensions for varieties of general type by Birkar, Cascini, Hacon, and McKernan (2010). They also established that any uniruled variety X is birationally equivalent to a *Fano fiber space* X', that is, a projective variety X' with a morphism $\pi\colon X' \to Y$ such that the fibers are Fano varieties.

23. Ruled surfaces

Definitions 23.1.

- A smooth projective surface X is called a *ruled surface* if it admits a *ruling*, that is, a surjective morphism $\pi\colon X \to Y$ to a smooth projective curve Y with general fiber isomorphic to $\mathbb{P}^1$.
- A fiber of a ruling $\pi\colon X \to Y$ over a point $y \in Y$ is called *non-degenerate* (or *general*) if it is reduced and irreducible, that is, $\pi^*(y)$ is a prime divisor, and *degenerate* (or *special*) otherwise.
- The ruling $\pi\colon X \to Y$ is called *relatively minimal* if all its fibers are non-degenerate.

Example 23.2. Any Hirzebruch surface $\pi_n \colon \mathbb{F}_n \to \mathbb{P}^1$ is a relatively minimal ruled surface; it is minimal if $n \neq 1$ (see 14.4).

Exercises 23.3. Let f be the class in $N^1(\mathbb{F}_n)$ of a fiber F and s_∞ the class of the exceptional section S_∞ with $S_\infty^2 = -n$. Show that

- $\rho(\mathbb{F}_n) = 2$ and $\operatorname{Pic}\mathbb{F}_n = \langle [S_\infty], [F] \rangle$;
- $K_{\mathbb{F}_n} \sim -(2S_\infty + (n+2)F)$;
- if C is a curve on X such that $C \equiv S_\infty + nF$, then C is a section of π_n with $C \cdot S_\infty = 0$;
- the linear system $|S_\infty + aF|$ is nonempty if and only if either $a = 0$ or $a \geq n$;
- in the latter case this linear system is base point free;
- this system is ample if and only if $a > n$.

Proposition 23.4. *Any relatively minimal ruling $\pi \colon X \to Y$ on a smooth projective surface X is the projectivization of a vector bundle of rank 2 on Y.*

The proof exploits the following fact.

Theorem 23.5 (Tsen). *Any ruled surface $\pi \colon X \to Y$ admits a regular section $s \colon Y \to X$.*

Exercises 23.6. Show that

- a ruled surface $\pi \colon X \to Y$ cannot have a multiple fiber (a fiber $\pi^*(y) = \sum_{i=1}^r m_i F_i$ is called *multiple* if $\gcd(m_1, \ldots, m_r) > 1$);
- if $\pi \colon X \to Y$ is a relatively minimal ruled surface, then
  - $\rho(X) = 2$;
  - the Neron-Severi space $N^1(X)$ is spanned by the fiber class f and the class s of a section;
  - $\operatorname{Pic} X = \pi^* \operatorname{Pic} Y \oplus \langle f \rangle$;
  - there is at most one curve C on X with $C^2 < 0$, and such a curve is a section of π (called the *exceptional section*);
- if X is a smooth projective surface and C is a smooth rational curve in X such that $C^2 \geq 0$, then $h^0(nK_X) = 0$ for all $n > 0$, and X is either $\mathbb{P}^2$ or a ruled surface. *Hint*: deduce first that $K_X \cdot C < 0$.

The next two lemmas deal with general fibrations on surfaces. In particular, they can be applied to rulings.

Lemma 23.7. *Let X be a smooth projective surface over $\mathbb{C}$, let Y be a smooth projective curve, and let $\pi \colon X \to Y$ be a surjective morphism with a connected (topological) general fiber F. For a point $y \in Y$ we let $X_y := \pi^{-1}(y)$. Then there is the following relation for the topological Euler characteristics:*

$$e(X) = e(Y)e(F) + \sum_{y \in Y} (e(X_y) - e(F)) .$$

Furthermore, $e(X_y) \geq e(F)$ $\forall y \in Y$, with equality if and only if either the fiber X_y is nondegenerate, or this is an irreducible multiple fiber of an elliptic fibration, that is, $g(F) = 1$ and $\pi^*(y) = mE$, where $E = X_y$ is a smooth elliptic curve in X.

Lemma 23.8 (Zariski). *Let X be a smooth projective surface, Y be a smooth projective curve, and $\pi \colon X \to Y$ be a surjective morphism with connected fibers. Consider the fiber $X_y = \pi^{-1}(y) = \sum_i F_i$ over a point $y \in Y$. Let $V \subset N^1(X)$ be the subspace spanned by the components F_i of X_y. Then the intersection form on V is negative semidefinite, and its isotropic cone is the line $\langle \pi^*(y) \rangle \subset V$.*

Exercises 23.9. Let X be a smooth projective surface. Show that

- if X is ruled with a ruling $\pi \colon X \to Y$, then it birationally dominates a relatively minimal ruled surface equipped with an induced ruling. *Hint*: proceed by induction on the number r of components of a degenerate fiber X_y, where $\pi^*(y) = \sum_{i=1}^r m_i F_i$. Show first that $F_i^2 < 0$ $\forall i$, and X_y contains a (-1)-component F_j. The latter follows from the Adjunction Formula and the equality
$$-K_X \cdot f = \sum_{i=1}^r m_i (F_i^2 + 2) = 2\,.$$
- Let $X_y = \sum_i F_i$ be a fiber of a ruling $\pi \colon X \to Y$. Show that
  - X_y is a divisor with simple normal crossings;
  - the wighted dual graph Γ_{X_y} is a rational tree (see 18.8) without branching (-1)-vertices, that is, (-1)-vertices of degree at least 3;
  - any contraction of a (-1)-component F_i yields again such a divisor.

Lemma 23.10 (Gizatullin). *Let $\pi \colon X \to Y$ be a ruling, where X and Y are smooth projective surface resp., curve. Then for any (-1)-component F_j of multiplicity $m_j = 1$ in the divisor $\pi^*(y) = \sum_i m_i F_i$, where $m_i > 0$ $\forall i$, the rest of the fiber X_y can be contracted to a point of a smooth ruled surface X', so that the fiber X'_y is reduced and irreducible.*

Exercises 23.11.

- Prove the Gizatullin lemma. *Hint*: show that any (-1)-fiber component of multiplicity 1 is a vertex of degree 1 in the dual graph of the fiber.
- Let X_y be a fiber of a ruling $\pi \colon X \to Y$ with weighted dual graph being the string $[-2, -1, -2]$. Show that the only (-1)-component F_i of X_y has multiplicity 2 in $\pi^*(y)$.

Definition 23.12. Let X be a smooth projective surface, Y be a smooth projective curve, and $\pi \colon X \to Y$ be a ruling. An *elementary (birational) transformation* in a non-degenerate fiber X_y of π consists in the blowing up a point $P \in X_y$ followed by the blowdown of the proper transform of the curve X_y.

Theorem 23.13 (Nagata-Maruyama). *Given a relatively minimal ruling* $\pi\colon X \to Y$ *on a smooth projective surface* X*, by a finite sequence of elementary transformations one can transform it into the trivial ruling* $\mathrm{pr}_1\colon Y \times \mathbb{P}_1 \to Y$*.*

Exercise 23.14. Deduce that any ruled surface X over Y is birationally equivalent to the product $Y \times \mathbb{P}_1$, and $p_a(X) = -g(Y)$.

24. Kodaira dimension

Definitions 24.1. Let X be a normal projective variety, and let D be a Cartier divisor on X.

– The *Kodaira dimension* of D denoted $\kappa(D)$ is defined as follows:

$$\kappa(D) := \begin{cases} \kappa(D) = -\infty & \text{if } |mD| = \emptyset \ \forall m > 0 \\ \max_{m\geq 1}\{\dim \varphi_{|mD|}(X)\} \in \{0, \dots, \dim X\} & \text{otherwise}. \end{cases}$$

Thus, in the second case the rational map $\varphi_{|mD|}\colon X \dashrightarrow \mathbb{P}^{N(m)}$ is well defined for certain values of $m > 0$, and $\kappa(D)$ is the maximal dimension of the nimage of X for these values.

– D is called *big* if $\kappa(D) = \dim X$. If D is big, then for some $m \in \mathbb{N}$ the map $\varphi_{|mD|}\colon X \dashrightarrow \mathbb{P}^{N(m)}$ is birational onto its image.

– If X is smooth, then $\kappa(X) := \kappa(K_X)$ is called the *Kodaira dimension of* X.

– For a singular X, the Kodaira dimension $\kappa(X)$ is by definition the Kodaira dimension of any smooth model of X (resulting from a resolution of singularities of X; this definition does not depend of the choice of a resolution).

– X is called a *variety of general type* if $\kappa(X) = \dim X$, that is, the canonical divisor K_X is big. Otherwise, X is called *special.*

The following criteria can be useful in order to detect the bigness of a divisor.

Proposition 24.2. *Let* X *be a normal projective variety.*

a) *A* $\mathbb{Q}$*-Cartier divisor* D *on* X *is big if and only if* $D = A + E$*, where* A *is an ample and* E *an effective* $\mathbb{Q}$*-Cartier divisors.*

b) *If* X *is a surface and* H *is an ample divisor on* X*, then* D *is big if* $D^2 > 0$ *and* $D \cdot H > 0$*. If* $D^2 = 0$*,* $D \cdot H > 0$*, and* $D \cdot K_X < 0$*, then* $\kappa(D) = 1$*.*

The Kodaira dimension possesses the following properties.

Theorem 24.3.

a) *The Kodaira dimension is a birational invariant.*

b) $\kappa(X_1 \times X_2) = \kappa(X_1) + \kappa(X_2)$.

c) **(Easy Addition Theorem)** *Let* $f\colon X \to Y$ *be a surjective morphism of projective varieties with connected fibers. Then*

$$\kappa(X) \leq \kappa(F) + \dim Y,$$

where F *is a generic fiber of* f*.*

d) **(Kawamata-Vieweg)** *Let $f\colon X \to Y$ and F be as before. If X is a surface and Y is a curve, then*

$$\kappa(X) \geq \kappa(F) + \kappa(Y)\,.$$

e) *If $f\colon X \to Y$ is an unramified covering, then $\kappa(X) = \kappa(Y)$.*
f) *If X is a uniruled projective variety, then $\kappa(X) = -\infty$.*

Exercise 24.4. Let X be a smooth projective curve of genus g. Show that $\kappa(X) = -\infty$ if $g = 0$, $\kappa(X) = 0$ if $g = 1$, and $\kappa(X) = 1$ if $g \geq 2$.

The Kodaira dimension determines the outcome of the Minimal Model Program for surfaces in the following sense.

Theorem 24.5. *Running the MMP for a smooth projective surface X has as outcome*

– *a minimal model if $\kappa(X) \geq 0$,*
– *a Mori fiber space if $\kappa(X) = -\infty$.*

In the latter case X is a ruled surface or $X \cong \mathbb{P}^2$.

25. Castelnuovo-Enriques classification of projective surfaces

Definitions 25.1. Let X be a minimal smooth projective surface.

– The *geometric genus* of X is the integer $p_g(X) := h^0(K_X)$;
– the *irregularity* of X is the integer $q(X) := h^1(\mathcal{O}_X) = p_g(X) - p_a(X)$.
– An *elliptic fibration* on X is a surjective morphism $X \to Y$ to a smooth projective curve Y with elliptic curves as general fibers.

Let $(p_g, q) = (p_g(X), q(X))$. Then X is called

– *Enriques surface* if $K_X \neq 0$, $2K_X = 0$, and $(p_g, q) = (0, 0)$;
– *$K3$ surface* if $K_X = 0$ and $(p_g, q) = (1, 0)$;
– *hyperelliptic surface* (or also *bielliptic surface*) if $\kappa(X) = 0$ and $(p_g, q) = (0, 1)$;
– *Abelian surface* if $\kappa(X) = 0$ and $(p_g, q) = (1, 2)$;
– *elliptic surface* if X admits an elliptic fibration.

The projective surfaces are classified according to the Kodaira dimension as follows.

Theorem 25.2 (Castelnuovo-Enriques). *For a minimal smooth projective surface X the following hold.*

– *$\kappa(X) = -\infty$ if and only if $|12K_X| = \emptyset$, if and only if X is either $\mathbb{P}^2$, or a minimal ruled surface;*
– *$\kappa(X) = 0$ if and only if $|12K_X| = \{0\}$, if and only if the surface X is either $K3$, or Enriques, or Abelian, or hyperelliptic;*
– *if $\kappa(X) = 1$, then X is an elliptic surface;*
– *otherwise $\kappa(X) = 2$, and so, X is a surface of general type.*

Exercises 25.3. Let $k = \mathbb{C}$.

- Verify that the integers $p_a(X)$, $p_g(X)$, $q(X)$, $P_m(X) := h^0(mK_X)$, and K_X^2 are birational invariants. Deduce that the Kodaira dimension $\kappa(X)$ is as well.
- Show that $q(X_1 \times X_2) = q(X_1) + q(X_2)$ for any projective varieties X_1 and X_2.
- Apply Abundance Theorem 22.16 to the Abelian, Enriques, and K3 surfaces.
- Are there Abelian surfaces different from the Jacobians of genus 2 curves? non-elliptic Abelian surfaces? Give an example of an elliptic Abelian surface.
- Show that any Abelian surface is a two-dimensional Abelian variety.
- Deduce that the tangent bundle of an Abelian surface is trivial, and the canonical line bundle is as well. *Hint*: use the group structure. (In fact, the same hold for any algebraic group.)
- Show that there are no rational curves in the Abelian and hyperelliptic surfaces (by contrast, all the K3 and Enriques surfaces do contain rational curves). *Hint*: apply the topological Monodromy Theorem, the Riemann-Hurwitz Formula, the facts that a hyperelliptic surface is a factor of a product of two elliptic curves by a finite group of automorphisms, and that the universal covering of an Abelian surface is the affine plane $\mathbb{A}^2$.

Let X be a smooth projective surface. Show that

- a smooth surface in $\mathbb{P}^3$ of degree d is rational if $d \leq 3$, a K3 surface if $d = 4$, and a surface of general type if $d \geq 5$;
- the smooth (complete) intersection of a quadric and a cubic hypersurfaces in $\mathbb{P}^4$, and of three quadrics in $\mathbb{P}^5$, are K3 surfaces;
- any Enriques surface and some K3 surfaces are elliptic surfaces over $\mathbb{P}^1$;
- the quotient of a K3 surface by a fixed point free involution is Enriques, and any Enriques surface appears in this way;
- there are rational elliptic surfaces (with Kodaira dimension $-\infty$). *Hint*: In appropriate affine coordinates, such an elliptic surface can be given by the Weierstrass equation $y^2 + x^3 + a(t)x + b(t) = 0$;
- any elliptic surface over a curve of genus at least 2 has Kodaira dimension 1;
- there are elliptic surfaces X with $\kappa(X) = 1$ over rational and elliptic curves;
- for an elliptic fibration over a curve, the analog of Nagata-Maruyama Theorem 23.13 fails, in general. Explain the reason of the failure;
- X is hyperelliptic if and only if $\kappa(X) = 0$ and $q(X) = 1$, if and only if X admits two elliptic fibrations, one over $\mathbb{P}^1$ and another one over an elliptic curve;
- the product of two smooth projective curves of genera at least 2 is a surface of general type.

Let us mention the following celebrated inequalities involving the topological Euler characteristic $e(X)$ of a smooth projective surface X; see, e.g., [12] and [28] for part a), [23] for part b), and [19] for part c).

Theorem 25.4. *Let X be a smooth projective surface over $\mathbb{C}$.*

a) **(Castelnuovo-de Franchis-Enriques)**

a1) *If X is minimal and $\chi(\mathcal{O}_X) < 0$, then X is a ruled surface.*

a2)-a3) *X is an irrational ruled surface provided that either $e(X) < 0$, or X is minimal and $K_X^2 < 0$.*

b) **(Bogomolov-Miyaoka-Yau inequality)** *If $\kappa(X) \geq 0$, then $K_X^2 \leq 3e(X)$.*

c) **(Kobayashi-Nakamura-Sakai)** *If X is of general type, then $H^2 \leq 3e(X)$, where H is the positive part of the Zariski decomposition $K_X = H + N$ (see Theorem 21.15).*

Remark 25.5. The projective surfaces of non-general type are classified completely. A lot is known on the geometry of surfaces of general type. However, their complete classification is out of reach at present. Even the *surface geography*, that is, the distribution of the principal numerical invariants (say, K_X^2 and e(X)) of minimal surfaces of general type, still has white holes.

26. Quick guide to the literature

Section 1: [8, 14, 16, 22, 25, 32, 33]
Section 2: [8, 14, 16, 22, 25, 31, 32, 33, 34]
Section 3: [2, 22, 25, 31, 32, 33, 34]
Section 4: [4, 8, 14, 16, 22, 25, 31, 32, 33, 34, 36]
Section 5: [8, 14, 25]
Section 6: [2, 8, 9, 14, 22, 25, 31, 32, 33, 34]
Section 7: [2, 7, 8, 14, 25, 31, 32, 34]
Section 8: [2, 14, 25, 27, 29, 31, 32, 33, 34]
Section 9: [7, 12, 25, 32, 33]
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Section 11: [11, 12, 14, 16, 22, 25, 29, 31, 32, 34]
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Section 19: [1, 3, 14, 25, 28, 30, 32, 33]
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Section 24: [12, 17, 28]
Section 25: [3, 6, 12, 16, 19, 23, 28]

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