# The Hirzebruch-Riemann-Roch theorem in the fancy language of Spectra

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# What is the HRR thm about?

Suppose that $E$ is an holomorphic vector bundle over a compact complex manifold $X$ and suppose that we are trying to solve the problem of finding a global invariant for $E$.

Naive candidate: Let $\mathcal{E}$ be the sheaf on $X$ of holomorphic sections of $E$. Our first candidate for a global invariant of $E$ is the complex vector space of global sections of $\mathcal{E}$,

$$H^0(X;\mathcal{E}) := \{holomorphic\ sections\ s : X \to E\}$$

Notice that $H^0(X,\mathcal{E})$ is finite dimensional complex vector space.

# What is the HRR thm about?

$H^0(X;\mathcal{E})$ is obviously a global invariant of $E$: if $E \cong F$ then

$$H^0(X;\mathcal{E}) \cong H^0(X;\mathcal{F}),$$

but unfortunately is not quite well behaved (i.e. hard to compute): in general taking the global section of a sheaf is an operation that does not respect exactness of sequence.

# What is the HRR thm about?

## Example

Denote by: $\mathbb{R}$ the sheaf of constant functions on the circle $S^1$, by $C^\infty$ the sheaf of smooth functions on $S^1$ and by $\Omega^1_{S^1}$ the sheaf of 1-forms on $S^1$. The sequence

$$0 \to \mathbb{R} \hookrightarrow C^\infty(S^1) \xrightarrow{d} \Omega^1_{S^1} \to 0$$

is short exact, but if we take global sections

$$0 \to \mathbb{R} \hookrightarrow C^\infty(S^1;\mathbb{R}) \xrightarrow{d} \Omega^1_{S^1}(S^1)$$

this is not exact on the right anymore.

# What is the HRR thm about?

How to fix the problem of "non-exactness" of $H^0(X; -)$? It is a very well know fact that for any exact sequence of sheaves

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

there exists "higher cohomological groups" $H^i(X; -), i > 0$ such that there exist a long exact sequence

$$0 \to H^0(X; \mathcal{F}) \to H^0(X; \mathcal{G}) \to H^0(X; \mathcal{H}) \to H^1(X; \mathcal{F}) \to H^1(X; \mathcal{G}) \to \cdots$$

where $H^0(X; -)$ is the global section functor.

# What is the HRR thm about?

So to define a good behaved (at least with respect short exact sequence of bundles) global invariant of $E$ we need to take account of this higher degree cohomological corrections.

## Definition (Less naive candidate)

Let $E$ be an holomorphic vector bundle over a compact complex manifold $X$. The *holomorphic Euler characteristic* of $X$ with coefficients in $E$ $\chi_{hol}(X; E)$ is defined to be

$$\chi_{hol}(X; E) := \sum_i (-1)^i \dim H^i(X; \mathcal{E})$$

# What is the HRR thm about?

Obviously if $E$ and $F$ are isomorphic vector bundles then $H^i(X;\mathcal{E}) \cong H^i(X;\mathcal{F})$, hence $\chi_{hol}(X;E) = \chi_{hol}(X;F)$ so $\chi_{hol}$ is an invariant. Moreover,

**Proposition (Additivity of $\chi_{hol}$)**

For three holomorphic vector bundles $F, E, G$ over $X$, such that $F = E \oplus G$ as complex vector bundles, we have that

$$\chi_{hol}(X;F) = \chi_{hol}(X;E) + \chi_{hol}(X;G).$$

# What is the HRR thm about?

The Hirzebruch-Riemann-Roch solves the problem of computing the holomorphic Euler characteristic of an holomorphic vector bundle. It is an integral formula for $\chi_{hol}(X; E)$.

**Theorem (Hirzebruch-Riemann-Roch)**

*Let $E$ be an holomorphic vector bundle over a compact complex manifold $X$. Then the following identity holds:*

$$\chi_{hol}(X; E) = \int_X \mathrm{ch}_X(E)\mathrm{td}(X)$$

*where $\mathrm{ch}_X$ is the Chern character of $E$ and $\mathrm{td}(X) := \mathrm{td}(T^{hol}X)$ is the Todd class of the holomorphic tangent bundle over $X$.*

# What is the (topological) HRR thm about?

Fix a compact smooth manifold $X$. The key players of the HRR theorem are:

- The complex topological $K$-theory of $X$, $K(X)$.

# Digression: $K$-theory in a nutshell

Let $(Vect_{\mathbb{C}}(X), \otimes, \oplus)$ be the semiring of isomorphism classes of complex vector bundles over $X$. $K(X)$ is made up from $Vect_{\mathbb{C}}(X)$ by formally inverting the operation of direct sum, i.e. $K(X)$ is the ring of pairs

$$([E],[F]) =: E - F$$

subjected to the identification

$$([E],[F]) \sim ([E'],[F'])$$

if there exists $[G] \in Vect_{\mathbb{C}}(X)$ such that

$$[E \oplus F' \oplus G] = [E' \oplus F \oplus G]$$

# What is the (topological) HRR thm about?

Fix a compact smooth manifold $X$. The key players of the HRR theorem are:

- The complex topological $K$-theory of $X$, $K(X)$.
- The even 2-periodic rational singular cohomology of $X$, $HP_{\mathrm{ev}}\mathbb{Q}(X) := \prod_{i\in\mathbb{Z}} H^{2i}(X;\mathbb{Q})$.
- The Chern character $\mathrm{ch}$, a ring homomorphism $K(X) \stackrel{\mathrm{ch}_X}{\to} HP_{\mathrm{ev}}\mathbb{Q}(X)$.

$$\mathrm{ch}_X(E) = \sum_i \exp(\gamma_i)$$

  where $\gamma_i$ are the Chern roots of $E$.
- Integration maps $K(X) \stackrel{\int_X^K}{\to} K(pt)$ and $HP_{\mathrm{ev}}\mathbb{Q}(X) \stackrel{\int_X^{HP_{\mathrm{ev}}\mathbb{Q}}}{\longrightarrow} HP_{\mathrm{ev}}\mathbb{Q}(pt)$ (defined only for certain $X$).

# What is the (topological) HRR thm about?

With all of this data we can build a square diagram

$$\begin{tikzcd} K(X) \arrow[r, "\mathrm{ch}_X"] \arrow[d, "\int_X^K"'] & HP_{\mathrm{ev}}\mathbb{Q}(X) \arrow[d, "\int_X^{HP_{\mathrm{ev}}\mathbb{Q}}"] \\ K(pt) \cong \mathbb{Z} \arrow[r, hook, "\mathrm{ch}_{\mathrm{pt}}"] & \mathbb{Q} \cong HP_{\mathrm{ev}}\mathbb{Q}(pt) \end{tikzcd} \tag{1}$$

and it natural to ask if this diagram is commutative for all (nice) $X$ such that the integration maps are defined.
<u>Hint:</u> The integration maps in $K$-theory agrees with the holomorphic Euler characteristic for compact complex $X$ and holomorphic $E \downarrow X$.

# What is the (topological) HRR thm about?

**Theorem ((topological) Hirzebruch-Riemann-Roch)**

*The diagram (1) is not commutative. There exists a class* $\mathrm{td}(X) \in HP_{\mathrm{ev}}\mathbb{Q}(X)$ *such that the following diagram is commutative*

$$\begin{tikzcd} K(X) \arrow[r, "\mathrm{ch}_X \cdot \mathrm{td}(X)"] \arrow[d, "\int_X^K"'] & HP_{\mathrm{ev}}\mathbb{Q}(X) \arrow[d, "\int_X^{HP_{\mathrm{ev}}\mathbb{Q}}"] \\ K(pt) \cong \mathbb{Z} \arrow[r, hook] & \mathbb{Q} \cong HP_{\mathrm{ev}}\mathbb{Q}(pt) \end{tikzcd}$$

*i.e.* $\forall a \in K(X)$ *we have*

$$\int_X^K a = \int_X^{HP_{\mathrm{ev}}\mathbb{Q}} \mathrm{ch}_X(a) \cdot \mathrm{td}(X)$$

# Why spectra?

There are at least three reasons for which the HRR theorem can be derived from the language of spectra:

- Both the functors $K$ and $HP_{\mathrm{ev}}\mathbb{Q}$ are the restrictions to the $0^{\mathrm{th}}$ degree group of two generalized cohomology theories $K^*$ and $HP_{\mathrm{ev}}\mathbb{Q}^*$ .

**Definition (Generalized cohomology theory)**

A *Generalized cohomology theory* $E^*$ is a contravariant functor

$$\mathsf{Top} \xrightarrow{E^*} \mathsf{Ab}^{\mathbb{Z}}$$

from the category of topological spaces to the cateogry of graded abelian groups, subject to a set of axioms called Eilenberg-Steenrod axioms.

- Spectra represent generalized cohomology theories.
- Manifolds can be seen as a special type of spectra.

# What a spectrum is: a brave analogy

The slogan to understand the main properties of the category of spectra is:

*Topological spaces stands to spectra as rational numbers stands to real numbers.*

$\mathbb{Q} : \mathbb{R}$

- There is an inclusion $\mathbb{Q} \hookrightarrow \mathbb{R}$.
- There is a multiplication on $\mathbb{R}$ that agrees to the one on $\mathbb{Q}$ when restricted.

Topological spaces : Spectra

- There is a functor $\mathrm{Top} \stackrel{\Sigma^\infty_+}{\to} \mathrm{Sp}$.
- Sp is a symmetric monoidal category and $\Sigma^\infty_+$ is a symmetric monoidal functor.

# What a spectrum is: a brave analogy

$$\mathbb{Q} : \mathbb{R}$$

- There exists a unit element for the multiplication which is the rational number 1.
- Every real number $a \in \mathbb{R}$ can be shifted by an integer $k \in \mathbb{Z}$ by sending $a \mapsto a + k$.

Topological spaces : Spectra

- There exists a unit spectrum $\mathbb{S}$ called the sphere spectrum which is given by the one point space pt considered as a spectrum.
- For every $k \in \mathbb{Z}$ and every spectrum $E$ there exists the shifting of $E$ by $k$ denoted by $E[k]$.

# What a spectrum is

There are two other properties the are fundamental for our purposes:

- For every spectrum $E$ there exists a notion of the "dual spectrum" of $E$, denoted by $DE$. This construction reassemble in a contravariant functor $\mathrm{Sp} \stackrel{D}{\to} \mathrm{Sp}$. Moreover $D\mathbb{S} \cong \mathbb{S}$.
- For a fixed spectrum $E$ and a topological space $X$, the association

$$X \mapsto [X[k], E] := \hom_{\mathrm{Sp}}(X[k], E)$$

that goes from topological spaces to graded abelian groups is a generalized cohomology theory called *$E$-cohomology*.

# What a $E_\infty$-spectrum is

We will restrict our attention to a particular class special objects in Sp, namely, $E_\infty$-spectra.

**Definition ($E_\infty$-spectra)**

A spectrum $E$ is called an $E_\infty$*-spectrum* if it is equipped with two maps;

$$\mu_E : E \otimes E \to E$$

called *multiplication map*, and

$$1_E : \mathbb{S} \to E$$

called *unit map of E*, subjected the standard associative, commutative and unital conditions.

# What an $E_\infty$-spectrum is

If the spectrum $E$ is an $E_\infty$-spectrum the associated generalized cohomology theory is multiplicative, i.e. comes with a product:

$$[X[m], E] \otimes [X[n], E] \longrightarrow [X[n+m], E].$$

For $f \in [X[m], E], g \in [X[n], E]$ then the composition

$$X[m+n] \xrightarrow{\Delta} X[m] \otimes X[n] \xrightarrow{f \otimes g} E \otimes E \xrightarrow{\mu} E$$

is defined to be the product $f \cdot g \in [X[n+m], E]$.

# What an $E_\infty$-spectrum is

The algebraic/categorical fact hidden behind the previous computation is the following:

- Every topological space $X$ is a cocommutative, coassociative comonoid object in Sp, via

$$\underbrace{X \xrightarrow{\Delta} X \otimes X}_{\text{comultiplication}}, \qquad \underbrace{X \xrightarrow{t_X} \mathbb{S}}_{\text{counit}}$$

- $E_\infty$-spectra are exactly the commutative, associative monoid objects in Sp.
- The hom set $[X, E]$ from the comonoid $X$ to the monoid $E$ is a monoid.

# Orientations and integration: an overview

$$\begin{gathered} \textit{Orientation of a spectrum} \\ \Downarrow \\ \textit{Orientation of a manifold with respect to } E \\ \Downarrow \\ \textit{Integration maps in } E-\textit{cohomology} \end{gathered}$$

# Orientations of spectra:Thom Spaces

Let $V$ be a (real) rank $d$ vector bundle over a (nice) compact topological space $X$.

**Definition (Thom Space)**

The *Thom space* of $V$, denoted by $X^V$ is the pointed space given by the one point compactification of the total space of the bundle.

The *Thom spectrum* of $V$ is simply the pointed topological space $X^V$ thinked as a spectrum and will be denoted by the same symbol.

# Orientations of spectra:Thom Spaces

## Example

The trivial bundle of rank $d$ over $X$ is $X \times \mathbb{R}^d$. The one point compactification of $X \times \mathbb{R}^d$ is given by the quotient $X \times S^d/(X \times \{\infty\})$ where we think $S^d$ as $\mathbb{R}^d \cup \{\infty\}$ and the Thom space of d is given by

$$X^{X\times\mathbb{R}^d} = \frac{X \times S^d}{X \times \{\infty\}} = \Sigma^d X_+$$

A very useful fact implied by this simple example is that the Thom spectrum of the rank $d$ trivial bundle $X^{X\times\mathbb{R}^d}$ is just $X$ (again thinked as a spectrum) shifted by $d$, i.e.

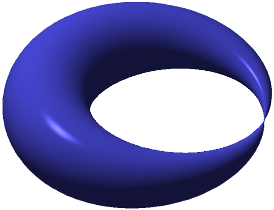
$$X^{X\times\mathbb{R}^d} \cong X[d]$$

# Orientations of spectra:Thom Spaces

## Example: Thom space of rank 1 trivial bundle on $S^1$

The (total space of the) rank 1 trivial bundle on $S^1$ is the infinite cylinder $S^1 \times \mathbb{R}$. So the Thom space $(S^1)^{S^1 \times \mathbb{R}}$ is given by the quotient $S^1 \times S^1 / S^1 \times \{\infty\}$ where we have collapsed the subspace $S^1 \times \{\infty\}$ to a point. The resulting space is a pinched torus.

Figure: The Thom space of the rank 1 trivial bundle on $S^1$, $\Sigma S^1_+$



# Orientations of spectra:Thom Spaces

One of the most important feature of Thom spectra is that given a vector bundle $V$ over $X$, then for every spectrum $E$, the $E$-cohomology of the Thom space $X^V$ is a (graded) module over the (graded) $E$-cohomology ring of the base space $X$, which means that are defined multiplication maps

$$[X[n], E] \otimes [X^V[m], E] \longrightarrow [X^V[n+m], E]$$

which allows to multiply an element of the $E$-cohomology of $X^V$ by an element of the $E$-cohomology of $X$ to get an element of the $E$-cohomology of $X^V$ again.

# Orientations of spectra:Thom Spaces

Again, the hidden algebraic/categorical fact behind the previous construction is the following: the Thom spectrum functor $Th$ can be refined to a functor

$$\mathrm{VectBun}_{\mathbb{R}} \stackrel{Th}{\to} \mathrm{Comod}(\mathrm{Sp})$$

which means that for a real vector bundle $V \downarrow X$ the Thom spectrum $X^V$ is a comodule over the comonoid object $X \in \mathrm{Sp}$.
This gives a (graded) module structure to the $E$-cohomology of $X^V$.

# Integration for $E$-oriented manifolds

Let $X$ be a compact smooth manifold. The *negative tangent bundle* of $X$ is by definition the virtual vector bundle of (real) rank $\dim X$ given by $-TX$. It is possible to define a Thom spectrum $X^{-TX}$ by defining

$$X^{-TX} := X^{\nu_\iota}[-N]$$

where $\nu_\iota : X \hookrightarrow \mathbb{R}^N$ is an embedding of $X$ into a (sufficiently large) euclidean space.

The condition for a manifold $X$ to be oriented in the generalized cohomology theory induced by a spectrum $E$ is expressed in terms of (the module structure of the $E$-cohomology of the) Thom spectrum $X^{-TX}$.

# Integration for $E$-oriented manifolds

Let $E$ be a spectrum.

**Definition ($E$-Oriented manifolds)**

- A compact smooth manifold $X$ is said to be *E-orientable* if the $[X, E]$-module $[X^{-TX}[\dim X], E]$ is isomorphic to the free module of rank 1.
- An *E-oriented manifold* is a pair $(X, \sigma)$, where $X$ is an $E$-orientable manifold and $[X, E] \xrightarrow{\sigma} [X^{-TX}[\dim X], E]$ is an isomorphism of $[X, E]$ modules.

# Integration for $E$-oriented manifolds

Let $X$ be a smooth manifold, as a particular type of spectrum via the infinite suspension $X$ has a Spanier-Whitehead dual $DX$.
$DX$ receives a map from $\mathbb{S}$ given by applying the duality functor to the terminal map $X \xrightarrow{t_X} \mathbb{S}$ (considered as a map between spectra) so for every manifold $X$ we have a natural map

$$\mathbb{S} \xrightarrow{Dt_X} DX$$

In particular, by taking the pullback in $E$-cohomology of the above map, we get maps

$$[DX[k], E] \xrightarrow{(Dt_X)^*} [\mathbb{S}[k], E]$$

# Integration for $E$-oriented manifolds

So for a compact smooth manifold $X$, it is possible to define two spectra: the dual $DX$ and the Thom spectrum of the negative tangent bundle $X^{-TX}$.

**Theorem (Atiyah)**

*Let $X$ be a compact smooth manifold, then $DX$ and $X^{-TX}$ are isomorphic.*

The integration map associated to an $E$-oriented manifold $(X, \sigma)$ is defined to be the composition

$$\begin{tikzcd}
\int^E_{(X,\sigma)} : [X, E] \arrow[r, "\sigma"] & {[X^{-TX}[\dim X], E]} \arrow[r] \arrow[d, "\cong"] & {[\mathbb{S}[\dim X], E]} \\
 & {[DX[\dim X], E]} \arrow[ur, "(Dt_X)^*"'] &
\end{tikzcd}$$

# Integration for $E$-oriented manifolds

- Done: We have shown that for every spectrum $E$ the definition of $E$-oriented manifold gives, by Atiyah's result $DX \cong X^{-TX}$, integration map in $E$-cohomology.
- To do: Show that there exists a general definition of orientation of spectra, such that a certain class of manifold is automatically oriented with respect to an oriented spectrum.

# Complex oriented spectra

A complex orientation for a spectrum $E$ is basically a theory of Thom isomorphisms for the generalized cohomology induced by $E$.

## Definition

A vector bundle $V$ over $X$ is *oriented in $E$-cohomology* if it is equipped with an isomorphism of $[X, E]$-modules

$$\sigma_V : [X, E] \xrightarrow{\sim} [X^V[-\mathrm{rk}\, V], E]$$

The isomorphism $\sigma_V$ is called the orientation of $V$ in $E$-cohomology.

## Definition (Complex oriented spectrum)

A spectrum $E$ is called *complex oriented* if every complex vector bundle is coherently oriented in $E$-cohomology.

# Complex oriented spectra

Obviously, it is possible for the same spectrum $E$ to be complex oriented in two different ways, i.e. to have two different set of isomorphism $\sigma_V, s_V : [X, E] \xrightarrow{\sim} [X^V[-\mathrm{rk}\, V], E]$ for each complex vector bundle $V \downarrow X$. The two isomorphism are related by the multiplication of an element $m_V \in GL_1[X, E]$ such that

$$\begin{tikzcd} {[X,E]} \arrow[rr, "- \cdot m_V"] \arrow[dr, "\sigma_V"'] & & {[X,E]} \arrow[dl, "s_V"] \\ & {[X^V[-\mathrm{rk}\, V], E]} & \end{tikzcd}$$

Two different complex orientations of the same spectrum defines a set of multipliers $\{m_V\} \subset GL_1[X, E]$.

# Complex oriented spectra

In order to define integration maps for compact manifolds the cohomology induced by a complex oriented spectrum $E$ we may ask for the manifold to have a complex negative tangent bundle.

**Definition (Stably complex manifold)**

A *stably complex manifold* $X$ is a manifold $X$ such that the normal bundle of an embedding of $X$ into a (sufficiently large) euclidean space is a complex vector bundle.

Every stable complex manifold is oriented for a complex oriented spectrum by, hence we can build integration maps for every stably complex manifold.

# Complex oriented spectra

## Example (Stably complex manifold)

An example is given by an odd sphere $S^{2n+1}$. The normal bundle $NS^{n+1}$ with respect to the standard inclusion $S^{2n+1} \hookrightarrow \mathbb{R}^{2n+2}$ is the trivial line bundle 1.

$$TS^{2n+1} \oplus NS^{2n+1} = TS^{2n+1} \oplus 1 \cong 2n + 2$$

This means that the normal bundle $\nu_\iota$ with respect to the inclusion

$$\iota : S^{2n+1} \hookrightarrow \mathbb{R}^{2n+2} \xhookrightarrow{x_{2n+3}=0} \mathbb{R}^{2n+3},$$

is the rank two trivial vector bundle 2, which has a standard complex structure. So $S^{2n+1}$ is a stably complex manifold.

# Examples: $KU$ and $HP_{\mathrm{ev}}\mathbb{Q}$

As we said before $K$ and $HP_{\mathrm{ev}}\mathbb{Q}$ are the restriction to the $0^{\mathrm{th}}$-degree group of two generalized cohomology theories $K^*$ and $HP_{\mathrm{ev}}\mathbb{Q}^*$. These two generalized cohomology theories are represented by two spectra, respectively $KU$ and $HP_{\mathrm{ev}}\mathbb{Q}$, so we have isomorphism, for each topological space $X$:

$$K(X) \cong [X, KU]; \quad HP_{\mathrm{ev}}\mathbb{Q}(X) = \prod_{i \in \mathbb{Z}} H^{2i}(X; \mathbb{Q}) \cong [X, HP_{\mathrm{ev}}\mathbb{Q}]$$

Both $KU$ and $HP_{\mathrm{ev}}\mathbb{Q}$ are examples of $E_\infty$-spectra and most important they are both complex oriented. We will refer to these complex orientations on $KU$ and $HP_{\mathrm{ev}}\mathbb{Q}$ as *standard orientations*.

# Examples: $KU$ and $HP_{\mathrm{ev}}\mathbb{Q}$

The reason why both $KU$ and $HP_{\mathrm{ev}}\mathbb{Q}$ are standardly complex oriented lies in a surprisingly fact: the datum of a complex orientation for a spectrum $E$ (coherent choice of orientation isomorphism for every complex vector bundle) can be condensed into a single object.

**Theorem (Universal Euler class)**

*A spectrum $E$ is complex oriented iff it admits an element $x_E \in [\mathbb{P}^\infty[-2], E]$ such that*

$$[\mathbb{P}^\infty[-2], E] \xrightarrow{\iota^*} [\mathbb{P}^1[-2], E] \cong [\mathbb{S}, E]$$

$$x_E \longmapsto 1_E,$$

*where $\mathbb{P}^1 \xrightarrow{\iota} \mathbb{P}^\infty$ is the standard inclusion map.*

# Examples: KU and $HP_{\mathrm{ev}}\mathbb{Q}$

The elements $x_{KU}$ and $x_{HP_{\mathrm{ev}}\mathbb{Q}}$ that gives the standard complex orientation of $KU$ and $HP_{\mathrm{ev}}\mathbb{Q}$ are given by

$$x_{KU} := 1_{\mathbb{C}} - \mathcal{O}(1)^* \in [\mathbb{P}^\infty[-2], KU]$$

$$x_{HP_{\mathrm{ev}}\mathbb{Q}} := c_1(\mathcal{O}(1)) \in [\mathbb{P}^\infty[-2], HP_{\mathrm{ev}}\mathbb{Q}],$$

where $\mathcal{O}(1)$ is the universal line bundle on $\mathbb{P}^\infty$.

# Examples: $KU$ and $HP_{\mathrm{ev}}\mathbb{Q}$

Complex orientations of spectra can be push forward along maps of spectra, by pushing forward the orientation isomorphisms.

## Observation (Pushing forward complex orientations)

Let $E \xrightarrow{\phi} F$ be a map of complex oriented spectra, the for every complex vector bundle $V \downarrow X$ the diagram

$$\begin{tikzcd}
{[X, E]} \arrow[r, "\phi_*"] \arrow[d, "\cong"'] & {[X, F]} \arrow[d, "\cong"] \\
{[X^V[-\mathrm{rk}\, V], E]} \arrow[r, "\phi_*"'] & {[X^V[-\mathrm{rk}\, V], F]}
\end{tikzcd}$$

commute iff the complex orientation on $F$ is the one on $E$ pushed forward along $\phi$.

# Examples: $KU$ and $HP_{\mathrm{ev}}\mathbb{Q}$

In our case the Chern character can be refined to a map of spectra $KU \xrightarrow{\mathrm{ch}} HP_{\mathrm{ev}}\mathbb{Q}$, so that the complex orientation of $KU$ can be pushed forward along the Chern character to get a new complex orientation of $HP_{\mathrm{ev}}\mathbb{Q}$.

For a complex vector bundle $V \downarrow X$ denote by:

- $\sigma_V$ the isomorphisms in $HP_{\mathrm{ev}}\mathbb{Q}$ given by the standard complex orientation,
- $s_V$ the isomorphisms in $HP_{\mathrm{ev}}\mathbb{Q}$ given by pushing forward the complex orientation of $KU$ along $\mathrm{ch}$,
- $k_V$ the isomorphisms in $K$-theory given by the standard complex orientation on $KU$.

# Examples: $KU$ and $HP_{\mathrm{ev}}\mathbb{Q}$

*The Hirzebruch-Riemann-Roch theorem is completely encoded in the fact that the standard complex orientation on $HP_{\mathrm{ev}}\mathbb{Q}$ does not coincide with the one given by the standard complex orientation on $KU$ pushed forward along the Chern character.*

# Beautiful friend, this is the end

Let $X$ be a stably complex manifold, we end up with three integration maps associated to $X$:

- $[X, HP_{\mathrm{ev}}\mathbb{Q}] \overset{\sigma_{-TX}}{\to} [X^{-TX}[\dim X], HP_{\mathrm{ev}}\mathbb{Q}] \overset{(Dt_X)^*}{\to} [\mathbb{S}[\dim X], HP_{\mathrm{ev}}\mathbb{Q}]$
- $[X, HP_{\mathrm{ev}}\mathbb{Q}] \overset{s_{-TX}}{\to} [X^{-TX}[\dim X], HP_{\mathrm{ev}}\mathbb{Q}] \overset{(Dt_X)^*}{\to} [\mathbb{S}[\dim X], HP_{\mathrm{ev}}\mathbb{Q}]$
- $[X, KU] \overset{k_{-TX}}{\to} [X^{-TX}[\dim X], KU] \overset{(Dt_X)^*}{\to} [\mathbb{S}[\dim X], KU]$

And all of this orientation maps assemble into a unique diagram.

# Beautiful friend, this is the end

$$
\begin{tikzcd}
{[X, KU]} \arrow[r, "\mathrm{ch}_*"] \arrow[d, "k_{-TX}"] & {[X, HP_{\mathrm{ev}}\mathbb{Q}]} \arrow[r, "\cdot m_{-TX}"] \arrow[d, "s_{-TX}"] & {[X, HP_{\mathrm{ev}}\mathbb{Q}]} \arrow[dl, "\sigma_{-TX}"] \arrow[ddl, bend left] \\
{[X^{-TX}[\dim X], KU]} \arrow[r, "\mathrm{ch}_*"] \arrow[d, "(Dt_X)^*"] & {[X^{-TX}[\dim X], HP_{\mathrm{ev}}\mathbb{Q}]} \arrow[d, "(Dt_X)^*"] & \\
{[\mathbb{S}[\dim X], KU]} \arrow[r, "\mathrm{ch}_*"] & {[\mathbb{S}[\dim X], HP_{\mathrm{ev}}\mathbb{Q}]} &
\end{tikzcd}
$$

# Beautiful friend, this is the end

$$
\begin{tikzcd}
{[X, KU]} \arrow[r, "\mathrm{ch}_*"] \arrow[d, "k_{-TX}"] & {[X, HP_{\mathrm{ev}}\mathbb{Q}]} \arrow[r, "\cdot m_{-TX}"] \arrow[d, "s_{-TX}"] & {[X, HP_{\mathrm{ev}}\mathbb{Q}]} \arrow[dl, "\sigma_{-TX}"] \arrow[ddl, bend left] \\
{[X^{-TX}[\dim X], KU]} \arrow[r, "\mathrm{ch}_*"] \arrow[d, "(Dt_X)^*"] & {[X^{-TX}[\dim X], HP_{\mathrm{ev}}\mathbb{Q}]} \arrow[d, "(Dt_X)^*"] & \\
{[\mathbb{S}[\dim X], KU]} \arrow[r, "\mathrm{ch}_*"] & {[\mathbb{S}[\dim X], HP_{\mathrm{ev}}\mathbb{Q}]} &
\end{tikzcd}
$$

# Beautiful friend, this is the end

$$
\begin{tikzcd}
{[X, KU]} \arrow[r, "\mathrm{ch}_*"] \arrow[d, "k_{-TX}"] & {[X, HP_{\mathrm{ev}}\mathbb{Q}]} \arrow[r, "\cdot m_{-TX}"] \arrow[d, "s_{-TX}"] & {[X, HP_{\mathrm{ev}}\mathbb{Q}]} \arrow[dl, "\sigma_{-TX}"] \arrow[ddl, bend left] \\
{[X^{-TX}[\dim X], KU]} \arrow[r, "\mathrm{ch}_*"] \arrow[d, "(Dt_X)^*"] & {[X^{-TX}[\dim X], HP_{\mathrm{ev}}\mathbb{Q}]} \arrow[d, "(Dt_X)^*"] & \\
{[\mathbb{S}[\dim X], KU]} \arrow[r, "\mathrm{ch}_*"] & {[\mathbb{S}[\dim X], HP_{\mathrm{ev}}\mathbb{Q}]} &
\end{tikzcd}
$$

So for every $a \in [X, KU]$, we have that

$$\int_{(X, k_{-TX})}^{KU} a$$

# Beautiful friend, this is the end

$$\begin{tikzcd}
{[X, KU]} \arrow[r, "\mathrm{ch}_*"] \arrow[d, "k_{-TX}"] & {[X, HP_{\mathrm{ev}}\mathbb{Q}]} \arrow[r, "\cdot m_{-TX}"] \arrow[d, "s_{-TX}"] & {[X, HP_{\mathrm{ev}}\mathbb{Q}]} \arrow[dl, "\sigma_{-TX}"] \arrow[ddl, bend left] \\
{[X^{-TX}[\dim X], KU]} \arrow[r, "\mathrm{ch}_*"] \arrow[d, "(Dt_X)^*"] & {[X^{-TX}[\dim X], HP_{\mathrm{ev}}\mathbb{Q}]} \arrow[d, "(Dt_X)^*"] & \\
{[\mathbb{S}[\dim X], KU]} \arrow[r, "\mathrm{ch}_*"] & {[\mathbb{S}[\dim X], HP_{\mathrm{ev}}\mathbb{Q}]} &
\end{tikzcd}$$

So for every $a \in [X, KU]$, we have that

$$\int_{(X, k_{-TX})}^{KU} a = \int_{(X, s_{-TX})}^{HP_{\mathrm{ev}}\mathbb{Q}} \mathrm{ch}_*(a)$$

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$$\begin{tikzcd}
{[X, KU]} \arrow[r, "\mathrm{ch}_*"] \arrow[d, "k_{-TX}"] & {[X, HP_{\mathrm{ev}}\mathbb{Q}]} \arrow[r, "\cdot m_{-TX}"] \arrow[d, "s_{-TX}"] & {[X, HP_{\mathrm{ev}}\mathbb{Q}]} \arrow[dl, "\sigma_{-TX}"] \arrow[ddl, bend left] \\
{[X^{-TX}[\dim X], KU]} \arrow[r, "\mathrm{ch}_*"] \arrow[d, "(Dt_X)^*"] & {[X^{-TX}[\dim X], HP_{\mathrm{ev}}\mathbb{Q}]} \arrow[d, "(Dt_X)^*"] & \\
{[\mathbb{S}[\dim X], KU]} \arrow[r, "\mathrm{ch}_*"] & {[\mathbb{S}[\dim X], HP_{\mathrm{ev}}\mathbb{Q}]} &
\end{tikzcd}$$

So for every $a \in [X, KU]$, we have that

$$\int_{(X, k_{-TX})}^{KU} a = \int_{(X, s_{-TX})}^{HP_{\mathrm{ev}}\mathbb{Q}} \mathrm{ch}_*(a) = \int_{(X, \sigma_{-TX})}^{HP_{\mathrm{ev}}\mathbb{Q}} \mathrm{ch}_*(a) m_{-TX}$$

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The commutativity of the previous diagram reads:

**Theorem (Hirzebruch-Riemann-Roch in the language of spectra)**

*Let $KU$ and $HP_{\mathrm{ev}}\mathbb{Q}$ be the spectra representing respectively complex $K$-theory and even 2-periodic rational singular cohomology. Then for every stably complex oriented $X$ and every $a \in [X, KU]$ we have that there exist a class $m_{-TX} \in [X, HP_{\mathrm{ev}}\mathbb{Q}]$ such that*

$$\int_X^{KU} a = \int_X^{HP_{\mathrm{ev}}\mathbb{Q}} \mathrm{ch}_*(a) m_{-TX}.$$

*where the integral are made with respect to the standard complex orientations on both $KU$ and $HP_{\mathrm{ev}}\mathbb{Q}$.*

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But why this is a proof of the topological HHR theorem?
The multiplier $m_{-TX} \in [X, HP_{\mathrm{ev}}\mathbb{Q}]$, as said before, measure the difference from the standard complex orientation on $HP_{\mathrm{ev}}\mathbb{Q}$ with another complex orientation (also in $HP_{\mathrm{ev}}\mathbb{Q}$) but I did not said anything on how to compute it. It can be shown that

$$m_{-TX} = \mathrm{td}(X) \in HP_{\mathrm{ev}}\mathbb{Q}(X)$$

for every stably complex manifold $X$. The proof of the latter identity is basically an application of the splitting principle for complex vector bundles.

# Beautiful friend, this is (really) the end

For good (=stably complex) maps between compact smooth manifolds $X \xrightarrow{f} Y$, it is possible to define pushforward maps

$$K(X) \xrightarrow{f_!} K(Y)$$

$$HP_{\mathrm{ev}}\mathbb{Q}(X) \xrightarrow{f_*} HP_{\mathrm{ev}}\mathbb{Q}(Y)$$

and build the same diagram

$$\begin{tikzcd} K(X) \arrow[r, "\mathrm{ch}_X"] \arrow[d, "f_!"'] & HP_{\mathrm{ev}}\mathbb{Q}(X) \arrow[d, "f_*"] \\ K(Y) \arrow[r, "\mathrm{ch}_Y"'] & HP_{\mathrm{ev}}\mathbb{Q}(Y). \end{tikzcd}$$

which is, of course, *not* commutative.

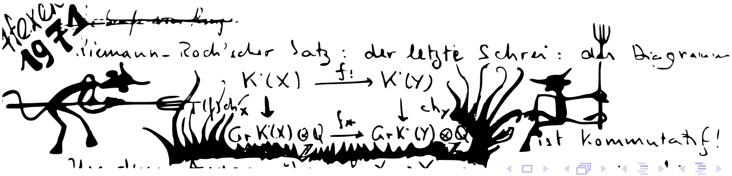
# Beautiful friend, this is (really) the end

**Theorem (Grothendieck-Hirzebruch-Riemann-Roch)**

*For every stable complex map $X \xrightarrow{f} Y$, there exists a cohomology class $\mathrm{td}(f) \in HP_{\mathrm{ev}}\mathbb{Q}(X)$, such that the following diagram is commutative:*

$$\begin{tikzcd} K(X) \arrow[r, "\mathrm{ch}_X \cdot \mathrm{td}(f)"] \arrow[d, "f_!"'] & HP_{\mathrm{ev}}\mathbb{Q}(X) \arrow[d, "f_*"] \\ K(Y) \arrow[r, "\mathrm{ch}_Y"'] & HP_{\mathrm{ev}}\mathbb{Q}(Y). \end{tikzcd}$$

Figure: Grothendieck's drawing on the GHRR theorem



# Beautiful friend, this is (really) the end

Thank you for your attention.

[0] A complete treatment and an exhaustive list of references to these arguments can be found in the ArXiv preprint: Mattia Coloma, Domenico Fiorenza, and Eugenio Landi; An exposition of the topological half of the Grothendieck-Hirzebruch-Riemann-Roch theorem in the fancy language of spectra, 2019. https://arxiv.org/abs/1911.12035