Continuum Kac–Moody Algebras
&
their quantizations

Francesco Sala (Università di Pisa)

May 15, 2020

based on

- `arXiv:1903.02813` (with Olivier Schiffmann)
- `arXiv:1903.01413` (with Andrea Appel)
- `arXiv:1812.08528` (with Andrea Appel and Olivier Schiffmann)
- `arXiv:1711.07391` (with Olivier Schiffmann; with an appendix by Tatsuki Kuwagaki)

Quivers and their Borcherds-Cartan datum

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$$\overset{1}{\circ} \longrightarrow \overset{2}{\circ} \longrightarrow \cdots \longrightarrow \overset{N-2}{\circ} \longrightarrow \overset{N-1}{\circ} \qquad\qquad \begin{tikzcd} \circ \arrow[loop above] \end{tikzcd}$$

A_{N-1} quiver

one-loop quiver

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Borcherds–Cartan datum for $\mathcal{Q}$

- $\mathfrak{h} = \mathbb{C}^{\mathcal{I}} = \mathsf{span}_{\mathbb{C}}\{\xi_\imath \mid \imath \in \mathcal{I}\}$, $\{\xi_\imath \mid \imath \in \mathcal{I}\}$ = tautological basis of $\mathbb{C}^{\mathcal{I}}$.
- symmetric bilinear form

$$(\cdot,\cdot)\colon \mathbb{C}^{\mathcal{I}} \otimes \mathbb{C}^{\mathcal{I}} \to \mathbb{C}\,, \quad (v,u) := {}^{T}vAu \quad \forall\, v,u \in \mathbb{Z}^{\mathcal{I}}$$

  where

  - $A := 2\mathsf{Id} - C - {}^{T}C$ = Cartan matrix;
  - $C = (c_{\imath,\jmath})$ = adjacency matrix; and
  - $c_{\imath,\jmath} := \#\{e \in \mathcal{E} \mid \mathsf{source}(e) = \imath\,, \mathsf{target}(e) = \jmath\}, \forall\, \imath,\jmath \in \mathcal{I}$.

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We say that $\xi_\imath$ is real / imaginary if $c_{\imath,\imath} = 0$ / $c_{\imath,\imath} \neq 0$.

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We say that $\xi_\imath$ is real / imaginary if $c_{\imath,\imath} = 0$ / $c_{\imath,\imath} \neq 0$. For example:

$$A_{\text{one-loop}} = (0)$$

Definition (Borcherds–Kac–Moody algebra)

$\mathfrak{g}_{\mathcal{Q}}^{\mathsf{BKM}}$ = the Lie algebra generated by elements $x_\imath^\pm$, $\imath \in \mathcal{I}$, and $\mathfrak{h}$ with relations

- Diagonal action: $[\phi, \psi] = 0$, $[\phi, x_\jmath^\pm] = \pm(\phi, \xi_\jmath)\, x_\jmath^\pm$.
- Double relations: $[x_\imath^+, x_\jmath^-] = \delta_{\imath,\jmath}\, \xi_\imath$.
- Serre relations:

$$[x_\imath^+, x_\jmath^+] = 0 = [x_\imath^-, x_\jmath^-] \qquad \text{if } (\xi_\imath, \xi_\jmath) = 0\,,$$

$$(\mathsf{ad}\, x_\jmath^+)^{1-(\xi_\imath,\xi_\jmath)} x_\imath^+ = 0 = (\mathsf{ad}\, x_\jmath^-)^{1-(\xi_\imath,\xi_\jmath)} x_\imath^- \qquad \text{if } \xi_\jmath \text{ is real}\,,$$

for $\phi, \psi \in \mathfrak{h}$ and $\imath, \jmath \in \mathcal{I}$.

Borcherds–Kac–Moody algebra of the quiver $\mathcal{Q}$

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For example:

$$\mathfrak{g}^{\mathsf{BKM}}_{A_{N-1}} = \mathfrak{sl}(N) \quad \text{and} \quad \mathfrak{g}^{\mathsf{BKM}}_{\text{one-loop}} = \text{Lie algebra is generated by } \{x^{\pm}, \xi\} \text{ satisfying:}$$

$$[\xi, x^{\pm}] = 0 \quad \text{and} \quad [x^{+}, x^{-}] = \xi\,.$$

"Continuum" version of $\mathfrak{sl}(N)$: the Lie algebra of the line $\mathbb{R}$

Continuum Borcherds–Cartan datum for $\mathbb{R}$

- $\mathfrak{h} := \mathsf{span}_{\mathbb{C}}\{$characteristic functions $\mathbf{1}_\alpha \ \forall\, \alpha \in \mathsf{Int}(\mathbb{R})\}$

  where $\mathsf{Int}(\mathbb{R}) = \{$open–closed intervals $\alpha = (a, b] \subset \mathbb{R}$ with $a \neq b\}$.

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- $(\cdot, \cdot) \colon \mathsf{Fun}(\mathbb{R}) \times \mathsf{Fun}(\mathbb{R}) \to \mathbb{C}$:

$$\langle f, g \rangle := \sum_x f_-(x) \cdot (g_-(x) - g_+(x)) \,, \quad (f, g) := \langle f, g \rangle + \langle g, f \rangle \quad \forall f, g \in \mathsf{Fun}_{\mathbb{Z}}(\mathbb{R}) \,.$$

  ($h_\pm(x) := \lim_{t>0,\, t\to 0} h(x \pm t)$.)

Explicit examples

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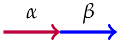
Explicit examples

Attention: $\mathsf{Int}(\mathbb{R})$ has an extra algebraic structure that we have to consider:

concatenation of intervals: 

Continuum Borcherds–Cartan datum for $\mathbb{R}$ (continuation)

- Partial operations on $\mathsf{Int}(\mathbb{R})$:

  $\oplus$ = sum of intervals:

$$\alpha \oplus \beta = \begin{cases} \alpha \cup \beta & \text{if } \alpha \cup \beta \in \mathsf{Int}(\mathbb{R}) \text{ and } \alpha \cap \beta = \emptyset \\ n.d. & \text{otherwise} \end{cases}$$

$$\alpha \oplus \beta = \xrightarrow{\alpha}\xrightarrow{\beta} \quad \text{or} \quad \xrightarrow{\beta}\xrightarrow{\alpha}$$

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  $\ominus$ = difference of intervals:

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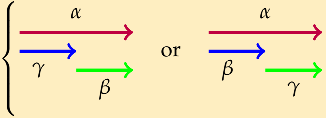
$\alpha \ominus \beta = \gamma$:



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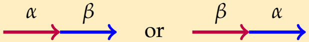

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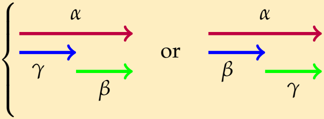

Continuum Borcherds–Cartan datum for $\mathbb{R}$: $((\mathsf{Int}(\mathbb{R}), \oplus, \ominus), \mathsf{Fun}(\mathbb{R}), (\cdot,\cdot))$

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for $\phi, \psi \in \mathsf{Fun}(\mathbb{R})$ and $\alpha, \beta \in \mathsf{Int}(\mathbb{R})$. Here, if $\nexists\, \alpha \oplus \beta$, then $x_{\alpha \oplus \beta}^\pm := 0$.

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- Serre relations are only quadratic (!).
- $\mathfrak{sl}(\mathbb{K})$ has a structure of a (topological) Lie bialgebra $\Longrightarrow$ this will be explained later (!)

'Colimit' realization

Consider the Lie algebra homomorphism $\varphi_k \colon \mathfrak{sl}(2) \mapsto \mathfrak{sl}(k)$ with $k \geq 2$:

$$\left(x^{\pm}, \xi\right) \mapsto \left(x^{\pm}_{\theta}, \xi_{\theta}\right) \quad \text{where} \quad \theta = \text{highest root in } \mathfrak{sl}(k)$$

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“Pack together”:

$$\varphi_{\underline{k}} \colon \mathfrak{sl}(N) \longrightarrow \mathfrak{sl}(k_1 + \cdots + k_{N-1} - 1) \qquad \text{where } \underline{k} = (k_1, \ldots, k_{N-1})$$

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Consider the Lie algebra homomorphism $\varphi_k \colon \mathfrak{sl}(2) \mapsto \mathfrak{sl}(k)$ with $k \geq 2$:

$$(x^{\pm}, \xi) \mapsto (x_\theta^{\pm}, \xi_\theta) \quad \text{where} \quad \theta = \text{highest root in } \mathfrak{sl}(k)$$

"Pack together":

$$\varphi_{\underline{k}} \colon \mathfrak{sl}(N) \longrightarrow \mathfrak{sl}(k_1 + \cdots + k_{N-1} - 1) \qquad \text{where } \underline{k} = (k_1, \ldots, k_{N-1})$$

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The Lie algebra homomorphisms $\varphi_{\underline{k}}$ form a directed system $(\mathfrak{sl}(N), \varphi_{\underline{k}})$. Moreover,

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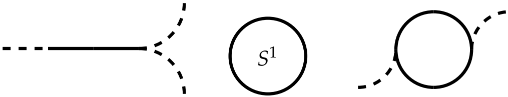

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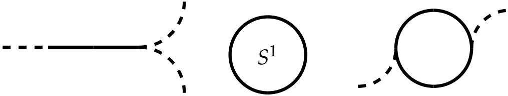
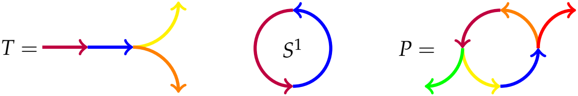


- $\mathsf{Int}(X) = \{$ intervals on X $\}$

  interval on X = any concatenation ($=: \oplus$) of open–closed intervals in $\mathbb{R}$ possibly across different spaghetti–lines



- $\ominus$ as before.

- $\mathsf{Fun}(X)$ is spanned by characteristic functions $\mathbf{1}_\alpha$, $\alpha \in \mathsf{Int}(X)$.

- $(\cdot,\cdot)$ is the natural extension of $(\cdot,\cdot)$ from $\mathbb{R}$ to X such that

$$(\mathbf{1}_{\alpha\oplus\beta}, \mathbf{1}_\gamma) = (\mathbf{1}_\alpha, \mathbf{1}_\gamma) + (\mathbf{1}_\beta, \mathbf{1}_\gamma)\ .$$

The continuum Kac–Moody Lie algebra $\mathfrak{g}(X)$ (Appel–S.–Schiffmann)

Let $\widetilde{\mathfrak{g}}(X)$ be the Lie algebra generated by $x_\alpha^\pm$, $\alpha \in \mathsf{Int}(X)$, and $\mathsf{Fun}(X)$ subject to the relations

- Diagonal action: $[\phi, \psi] = 0$, $[\phi, x_\alpha^\pm] = \pm(\phi, \mathbf{1}_\alpha)\, x_\alpha^\pm$,
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Set

$$\mathfrak{g}(X) := \widetilde{\mathfrak{g}}(X)/\mathfrak{r}_X\,,$$

where $\mathfrak{r}_X$ is the maximal (graded) two–sided ideal in $\widetilde{\mathfrak{g}}(X)$ which intersects trivially the Cartan subalgebra $\mathsf{Fun}(X)$.

Theorem (Appel–S.–Schiffmann)

The ideal $\mathfrak{r}_X$ is generated by the Serre relations

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‘Colimit’ realization of $\mathfrak{g}(X)$ and Borcherds–Kac–Moody algebras

Consider a set of intervals $\mathcal{J} \subset \mathsf{Int}(X)$ such that:

- for any $\alpha \in \mathcal{J} \rightsquigarrow \alpha =$ a ‘tree’ of (contractible) intervals or $\alpha = S^1$;
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We can define 2 Lie algebras:

- $\mathfrak{g}(X;\mathcal{J}) := \langle \mathbf{1}_\alpha, x_\alpha^\pm \mid \alpha \in \mathcal{J} \rangle \subset \mathfrak{g}(X)$
- quiver $\mathcal{Q} = (\mathcal{J}, \mathcal{E})$, Borcherds–Cartan matrix $A_{\mathcal{J}} = \big((\mathbf{1}_\alpha, \mathbf{1}_\beta)\big)_{\alpha,\beta\in\mathcal{J}}$

  $\rightsquigarrow$ corresponding Borcherds–Kac–Moody algebra $\mathfrak{g}^{\mathsf{BKM}}(\mathcal{J})$

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Remark

Our quadratic Serre relations $\Longrightarrow$ Serre relations of the Borcherds–Kac–Moody algebra.

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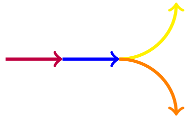	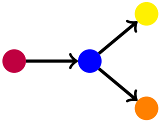
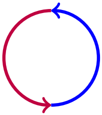	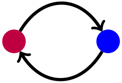
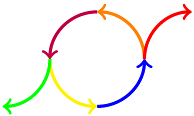	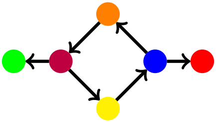

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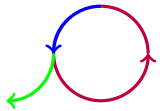	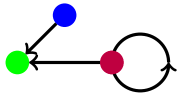
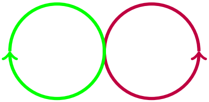	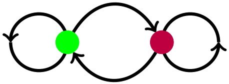

Remark

We obtain always (!) simply-laced diagrams with at most 1 (!) loop at each vertex.

Theorem (Appel-S.-Schiffmann)

Let $\mathcal{J}, \mathcal{J}'$ be two (finite) sets of intervals in X (as before).

- *If $\mathcal{J}' \subseteq \mathcal{J}$, there is a canonical (obvious) embedding*

$$\phi'_{\mathcal{J},\mathcal{J}'} \colon \mathfrak{g}(X;\mathcal{J}') \hookrightarrow \mathfrak{g}(X;\mathcal{J}) \,.$$

- *If*

$$\gamma \in \mathcal{J}' \,, \quad \gamma = \alpha \oplus \beta \,, \quad \mathcal{J} = (\mathcal{J}' \smallsetminus \{\gamma\}) \cup \{\alpha, \beta\} \,,$$

there is a canonical embedding

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which sends

$$x_\alpha^\pm \mapsto x_\alpha^\pm \,, \mathbf{1}_\alpha \mapsto \mathbf{1}_\alpha \quad \text{for } \alpha \in \mathcal{J}' \smallsetminus \{\gamma\} \,,$$

$$\mathbf{1}_\gamma \mapsto \mathbf{1}_\alpha + \mathbf{1}_\beta \,,$$

$$x_\gamma^+ \mapsto (-1)^{\langle \mathbf{1}_\beta, \mathbf{1}_\alpha \rangle} [x_\alpha^+, x_\beta^+] \,, \quad x_\gamma^- \mapsto (-1)^{\langle \mathbf{1}_\alpha, \mathbf{1}_\beta \rangle} [x_\alpha^-, x_\beta^-] \,.$$

- *The collection of all possible embeddings $\phi'_{\mathcal{J},\mathcal{J}'}, \phi''_{\mathcal{J},\mathcal{J}'}$ form a direct system. Moreover,*

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Continuum quantum groups $\mathbf{U}_v\mathfrak{g}(X)$ (S.-Schiffmann, Appel-S.)

Let $\mathcal{Q}_X = (\mathsf{Int}(X), \oplus, \ominus, (\cdot,\cdot))$ be a continuum quiver.

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The continuum Kac–Moody algebra $\mathfrak{g}(X)$ has the structure of a topological (!) quasi–triangular Lie bialgebra such that the topological cobracket

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Theorem (Appel-S.)

There exists a topological quasi–triangular Hopf algebra $\mathbf{U}_v\mathfrak{g}(X)$ which quantizes $\mathfrak{g}(X)$.

Theorem (S.-Schiffmann)

Let $\mathbb{K} = \mathbb{Q}$, $\mathbb{R}$, $\mathbb{Q}/\mathbb{Z}$, $S^1 = \mathbb{R}/\mathbb{Z}$.

There exists an Hall algebra realization of $\mathbf{U}_v\mathfrak{sl}(\mathbb{K})$ *via the abelian category* $\mathsf{Rep}_{\mathbb{F}_q}(\mathbb{K})$ *of finite-dimensional (coherent) representations of* $\mathbb{K}$ *(called also coherent persistent modules of* $\mathbb{K}$*).*

What do we know and what is it open?

Representation theory

- Fock space of $\mathbf{U}_v\mathfrak{sl}(\mathbb{R})$ and corresponding action of $\mathbf{U}_v\mathfrak{sl}(S^1)$ via continuum version of partitions.

  (due to S.-Schiffmann)

- symmetric tensor representations $V_{g,r}$ of $\mathbf{U}_v\mathfrak{sl}(S^1)$.

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- Cartan subalgebra generated by locally–constant functions $f \colon \mathbb{R} \to \mathbb{C}$ with bounded support

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Generalization of $\mathfrak{g}(X)$ via Hall algebras? We need abelian categories of

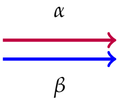
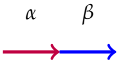
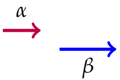
- topology: finite-dimensional representations of X (i.e., persistence modules of X).

- algebraic geometry: generalized parabolic sheaves with weights in X.

  (torsion parabolic sheaves with weights in S^1 = finite-dimensional coherent representations of S^1.)

Thank you for the attention!

Bilinear forms

Intervals	$\langle \mathbf{1}_\alpha, \mathbf{1}_\beta \rangle$	$\langle \mathbf{1}_\beta, \mathbf{1}_\alpha \rangle$	$(\mathbf{1}_\alpha, \mathbf{1}_\beta) := \langle \mathbf{1}_\alpha, \mathbf{1}_\beta \rangle + \langle \mathbf{1}_\beta, \mathbf{1}_\alpha \rangle$
	1	1	2
	-1	0	-1
	0	0	0

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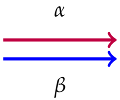
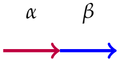
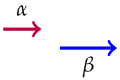
Spelling out the relations

Intervals	Relations
	$[\mathbf{1}_\alpha, x_\alpha^+] = 2\,x_\alpha^+\ , [\mathbf{1}_\alpha, x_\alpha^-] = -2\,x_\alpha^-\ , [x_\alpha^+, x_\alpha^-] = \mathbf{1}_\alpha \ \Rightarrow\ \mathfrak{sl}(2)$-triple
	$\begin{cases} [\mathbf{1}_\alpha, \mathbf{1}_\beta] = 0\,, [\mathbf{1}_\alpha, x_\beta^+] = -x_\beta^+\ , [\mathbf{1}_\alpha, x_\beta^-] = x_\beta^-\ , [x_\alpha^+, x_\beta^-] = 0 \\ \text{Our Serre rel: } [x_\alpha^+, x_\beta^+] = x_{\alpha\cup\beta}^+\,, [x_\alpha^-, x_\beta^-] = -x_{\alpha\cup\beta}^- \\ \text{Our Serre rel} \Rightarrow \text{Usual Serre rel: } [x_\alpha^\pm, [x_\alpha^\pm, x_\beta^\pm]] = \pm[x_\alpha^\pm, x_{\alpha\cup\beta}^\pm] = 0 \end{cases}$ (Attention: $\alpha \cup \beta = \alpha \oplus \beta$ and $\nexists\, \alpha \oplus \alpha \oplus \beta$)
	$\begin{cases} [\mathbf{1}_\alpha, \mathbf{1}_\beta] = 0\,, [\mathbf{1}_\alpha, x_\beta^+] = 0\,, [\mathbf{1}_\alpha, x_\beta^-] = 0\,, [x_\alpha^+, x_\beta^-] = 0 \\ \text{(Our Serre rels = Usual Serre rels)} \ \ [x_\alpha^\pm, x_\beta^\pm] = 0 \end{cases}$

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An example of the morphism $\varphi_{\underline{k}}$

Case: $N = 3$, $k_1 = 2$, and $k_2 = 3$

$$\varphi_{(1,2)} : \mathfrak{sl}(3) \longrightarrow \mathfrak{sl}(2+3-1) = \mathfrak{sl}(4)$$

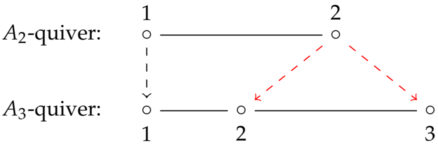
such that

$$(x_1^{\pm}, \xi_1) \mapsto (x_1^{\pm}, \xi_1)$$

$$x_2^{\pm} \mapsto \pm[x_2^{\pm}, x_3^{\pm}]$$

$$\xi_2 \mapsto \xi_2 + \xi_3$$

Pictorially:



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Definition

Serre(X) is the set of pairs (α, β) of intervals such that either

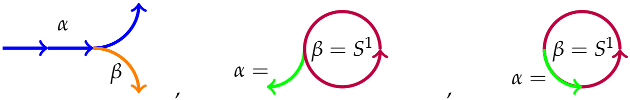
- α is a "tree" of (contractible) intervals, and

$$\alpha \subset \beta = S^1 \text{ , or } \quad \nexists\, \alpha \oplus \beta \text{ , or } \quad \alpha \oplus \beta \text{ does not contain } S^1 \not\subseteq \alpha, \beta \text{ ,}$$

or

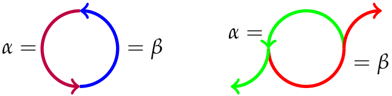
- $\nexists\, \alpha \oplus \beta$ and $\alpha \cap \beta = \emptyset$.

In Serre(X), we do have:



$$[x_\alpha^\pm, x_\beta^\pm] = \mp x_{\alpha\oplus\beta}^\pm \quad , \qquad [x_\alpha^\pm, x_\beta^\pm] = \pm x_{\alpha\oplus\beta}^\pm \quad , \qquad [x_\alpha^\pm, x_\beta^\pm] = 0$$

In Serre(X), we do not have:



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