ALGEBRA & REPRESENTATION THEORY SEMINAR

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REAL FORMS of COMPLEX LIE SUPERALGEBRAS and SUPERGROUPS

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— HIGHLIGHTS —

(1) – Basic constructions: real structures & "real forms" on complex super vector spaces / commutative superalgebras / Lie superalgebras: *standard* version (Kac [1977], Parker [1980], Serganova [1983], Chuah [2013]), *graded* version (Pellegrini [2007]) — functorial formulation (Pellegrini [2007])

(1+) – Constructions on supergroups: real structures & "real forms" on complex supergroups (Pellegrini [2007] $\rightsquigarrow$ Fioresi & G. [2020])

(2) – Unitarity & compactness: Hermitian forms on complex superspaces with real structures $\Longrightarrow$ associated *unitary* Lie superalgebras and supergroups $\longrightarrow\!\!\!\!\rightsquigarrow$
$\longrightarrow\!\!\!\!\rightsquigarrow$ *super-compact* $\big/$ *compact* Lie superalgebras and supergroups (Fioresi & G. [2020])

(3) – $\exists!\big/\nexists$ results in the simple contragredient (=basic) cases:

— $\exists!$ *graded* real form which is <u>*super-compact*</u>,

— $\exists!$ *standard* real form which is <u>*compact*</u> in types A and C (i.e., "type 1"),

— $\nexists$ *standard* real form which is <u>*compact*</u> in types B, D, F and G (i.e., "type 2")

(Fioresi & G. [2020])

§ 1.1 – CONSTRUCTIONS ON SUPERSPACES

1.1.1 – Notations. We consider the following categories (with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$):

$(\mathrm{svec})_{\mathbb{K}} :=$ super vector spaces over $\mathbb{K}$

$(\mathrm{salg})_{\mathbb{K}} :=$ commutative superalgebras over $\mathbb{K}$

$(\mathrm{sLie})_{\mathbb{K}} :=$ Lie superalgebras over $\mathbb{K}$

$(\mathbb{Z}_2\text{–vec})_{\mathbb{K}} :=$ $\mathbb{Z}_2$–graded vector spaces over $\mathbb{K}$

$(\mathbb{Z}_2\text{–alg})_{\mathbb{K}} :=$ $\mathbb{Z}_2$–graded commutative algebras over $\mathbb{K}$

$(\mathbb{Z}_2\text{–Lie})_{\mathbb{K}} :=$ $\mathbb{Z}_2$–graded Lie algebras over $\mathbb{K}$

1.1.2 – Real structure on $V \in (\mathrm{svec})_{\mathbb{C}}$: any $\phi \in End_{(\mathrm{svec})_{\mathbb{R}}}(V)$ such that

ϕ is $\mathbb{C}$–antilinear, $\quad \phi^2\Big|_{V_{\bar{0}}} = id_{V_{\bar{0}}}$, $\quad \phi^2\Big|_{V_{\bar{1}}} = \begin{cases} +id_{V_{\bar{1}}} & (\textit{standard}\text{ case}) \\ -id_{V_{\bar{1}}} & (\textit{graded}\text{ case}) \end{cases}$

Rmk: pairs (V, ϕ), with ϕ of fixed type, form a tensor subcategory of $(\mathrm{svec})_{\mathbb{C}}^{\mathbb{Z}_4}$ — the category of super vector spaces with $\mathbb{Z}_4$–action — denoted $(\mathrm{svec})_{\mathbb{C}}^{\bullet}$ for $\bullet \in \{\mathsf{st}\,, \mathsf{gr}\}$.

N.B.: similar constructions apply to complex commutative superalgebras and Lie superalgebras, yielding categories $(\mathrm{salg})_{\mathbb{C}}^{\bullet}$ and $(\mathrm{sLie})_{\mathbb{C}}^{\bullet}$, for all $\bullet \in \{\mathsf{st}\,, \mathsf{gr}\}$.

For $A \in (\mathrm{salg})_{\mathbb{C}}^{\bullet}$ the real structure is called "conjugation" and denoted $a \mapsto \widetilde{a}$.

1.1.3 – What is a "real form" of $(V, \phi) \in (\mathrm{svec})_{\mathbb{C}}^{\bullet}$?

— if $\bullet = \mathsf{st} \implies \begin{cases} V^{\phi} := \left\{ v \in V \,\middle|\, \phi(v) = v \right\} \leq V \quad \text{in } (\mathrm{svec})_{\mathbb{R}} \\ \quad \text{and} \quad \mathbb{C} \otimes_{\mathbb{R}} V^{\phi} \cong V \quad \text{in } (\mathrm{svec})_{\mathbb{C}}^{\mathsf{st}} \end{cases} \implies$ **<u>OK!</u>**

— if $\bullet = \mathsf{gr} \implies$ the above <u>fails</u>, since $V_{\bar{1}}^{\phi} = \{0\}$

$\implies$ we need a new, generalized notion of "real form".

§ 1.2 – FUNCTORIAL FORMULATION

1.2.1 – Superfunctors. Each $V \in (\mathrm{svec})_{\mathbb{K}}$ is described by the "functor of points"

$$\mathcal{L}_V : (\mathrm{salg})_{\mathbb{K}} \longrightarrow (\mathbb{Z}_2\text{–vec})_{\mathbb{K}} \, , \;\; A \mapsto \big(A \otimes_{\mathbb{K}} V \big)_{\bar{0}} = \Big(\underbrace{A_{\bar{0}} \otimes_{\mathbb{K}} V_{\bar{0}}}_{\textit{degree } \bar{0}} \Big) \oplus \Big(\underbrace{A_{\bar{1}} \otimes_{\mathbb{K}} V_{\bar{1}}}_{\textit{degree } \bar{1}} \Big)$$

<u>*Extras:*</u> *(1)* If $V = R \in (\mathrm{salg})_{\mathbb{K}}$, $\quad \Longrightarrow \quad \mathcal{L}_R : (\mathrm{salg})_{\mathbb{K}} \longrightarrow (\mathbb{Z}_2\text{–alg})_{\mathbb{K}}$;

(2) If $V = \mathfrak{g} \in (\mathrm{sLie})_{\mathbb{K}}$, $\quad \Longrightarrow \quad \mathcal{L}_{\mathfrak{g}} : (\mathrm{salg})_{\mathbb{K}} \longrightarrow (\mathbb{Z}_2\text{–Lie})_{\mathbb{K}}$.

Rmks: *(a)* the functor $\mathcal{L}_V$ is *representable*, being represented by $S\big(V^*\big)$;

(b) set $X_A := \underline{Spec}(A) =$ affine superscheme associated with $A \in (\mathrm{salg})_{\mathbb{K}}$. Then

$$\mathcal{L}_V(A) \, = \, Hom_{(\mathrm{salg})}\big(S(V^*)\,, A\big) \, \cong \, Hom_{(\mathrm{ssch})}\big(X_A, V\big) \, =:$$

$=: \big\{$morphisms $X_A \longrightarrow V$ as superschemes$\big\}$ $\Longrightarrow$ $\mathcal{L}_V(A)$ is a "<u>*map space*</u>".

Similarly, $\mathcal{L}_R(A)$ and $\mathcal{L}_{\mathfrak{g}}(A)$ are "map algebras" when $R \in (\mathrm{salg})_{\mathbb{K}}$ and $\mathfrak{g} \in (\mathrm{sLie})_{\mathbb{K}}$.

1.2.2 – Functorial real structures. We have two possible approaches:

[1] Pick $(V, \phi) \in (\mathrm{svec})_{\mathbb{C}}^{\bullet}$ and the forgetful functor $(\mathrm{salg})_{\mathbb{C}}^{\bullet} \xrightarrow{\mathcal{F}} (\mathrm{salg})_{\mathbb{C}}$;

if $\big(A, (\,\widetilde{\ }\,)\big) \in (\mathrm{salg})_{\mathbb{C}}^{\bullet}$, $\quad \Longrightarrow \quad \Big(\, \mathcal{L}_V(A) \,,\, \phi_A := \big((\,\widetilde{\ }\,) \otimes \phi\big)\Big|_{\mathcal{L}_V(A)} \Big)$ lives in

$(\mathbb{Z}_2\text{–vec})_{\mathbb{C}}^{\mathsf{re}} :=$ category of $\mathbb{Z}_2$–graded cplx vect. sp.'s with real struct. (in class. sense);

$\Longrightarrow \quad \big(A, (\,\widetilde{\ }\,)\big) \mapsto \big(\mathcal{L}_V(A) \,, \phi_A\big)$ defines a functor $\widetilde{\mathcal{L}}_V^{\bullet} : (\mathrm{salg})_{\mathbb{C}}^{\bullet} \longrightarrow (\mathbb{Z}_2\text{–vec})_{\mathbb{C}}^{\mathsf{re}}$.

Therefore

$$(V, \phi) \qquad \longleftrightarrow \qquad \widetilde{\mathcal{L}}_V^{\bullet} = \text{“}\mathcal{L}_V \text{ endowed with (involutive) real structure”}$$

Rmks: *(1)* $\widetilde{\mathcal{L}}_V^{\bullet}$ enjoys some “special properties” (SP): *involutivity* and $\mathcal{A}_{\bar{0}}$*–linearity*. *Conversely*, any $\widetilde{\mathcal{L}}^{\bullet} : (\mathrm{salg})_{\mathbb{C}}^{\bullet} \longrightarrow (\mathbb{Z}_2\text{–vec})_{\mathbb{C}}^{\mathsf{re}}$ enjoying (SP) and s. t. $\mathcal{F}_* \circ \widetilde{\mathcal{L}}^{\bullet} = \mathcal{L}_V$, with $(\mathbb{Z}_2\text{–vec})_{\mathbb{C}}^{\mathsf{re}} \xrightarrow{\mathcal{F}_*} (\mathbb{Z}_2\text{–vec})_{\mathbb{C}}$ the forgetful functor, yields a real structure ϕ on V.

(2) Similarly,

$$V = R \in (\mathrm{salg})_{\mathbb{C}}^{\bullet} \qquad \longleftrightarrow \qquad \widetilde{\mathcal{L}}_R^{\bullet} : (\mathrm{salg})_{\mathbb{C}}^{\bullet} \longrightarrow (\mathbb{Z}_2\text{–alg})_{\mathbb{C}}^{\mathsf{re}} \text{ enjoying (SP)}$$

$$V = \mathfrak{g} \in (\mathrm{sLie})_{\mathbb{C}}^{\bullet} \qquad \longleftrightarrow \qquad \widetilde{\mathcal{L}}_{\mathfrak{g}}^{\bullet} : (\mathrm{salg})_{\mathbb{C}}^{\bullet} \longrightarrow (\mathbb{Z}_2\text{–Lie})_{\mathbb{C}}^{\mathsf{re}} \text{ enjoying (SP)}$$

[2] Consider the "scalar restriction" functor $(\mathbb{Z}_2\text{–vec})_{\mathbb{C}} \xrightarrow{\ \mathcal{R}\ } (\mathbb{Z}_2\text{–vec})_{\mathbb{R}}$, and for any $(V, \phi) \in (\mathrm{svec})_{\mathbb{C}}^{\bullet}$ set $\mathcal{L}_V^{\bullet} := \mathcal{R} \circ \mathcal{L}_V \circ \mathcal{F} : (\mathrm{salg})_{\mathbb{C}}^{\bullet} \longrightarrow (\mathbb{Z}_2\text{–vec})_{\mathbb{R}}$.

Then (notation as above)

$$\varphi := \left\{ \varphi_A := \phi_A \right\}_{A \in (\mathrm{salg})_{\mathbb{C}}^{\bullet}} : \mathcal{L}_V^{\bullet} \longrightarrow \mathcal{L}_V^{\bullet}$$

is a ***natural transformation*** enjoying special properties (SP).

DEF.: *real structure* on $\mathcal{L}_V :=$ any natural transformation $\mathcal{L}_V^{\bullet} \xrightarrow{\ \varphi\ } \mathcal{L}_V^{\bullet}$ enjoying (SP).

Proposition: For any $V \in (\mathrm{svec})_{\mathbb{C}}$, $\exists$ bijections between the following:

(a) $\left\{ \text{real structures } \phi \text{ on } V \right\}$

(b) $\left\{ \text{functors } (\mathrm{salg})_{\mathbb{C}}^{\bullet} \xrightarrow{\widetilde{\mathcal{L}}^{\bullet}} (\mathbb{Z}_2\text{–vec})_{\mathbb{C}}^{\mathrm{re}} \,\middle|\, \mathcal{F}_* \circ \widetilde{\mathcal{L}}^{\bullet} = \mathcal{L}_V ,\ \widetilde{\mathcal{L}}^{\bullet} \text{ enjoys (SP)} \right\}$

(c) $\left\{ \text{real structures } \varphi \text{ on } \mathcal{L}_V \right\}$

...what for *"real forms"*, then?

DEF.: $\forall$ real structure φ on $\mathcal{L}_V$, we call *real form* for it the functor of its "fixed points", that is $\mathcal{L}_V^\varphi : (\mathrm{salg})_{\mathbb{C}}^{\bullet} \longrightarrow (\mathbb{Z}_2\text{–vec})_{\mathbb{R}}$ given by $\big(A, (\,\widetilde{\ }\,)\big) \mapsto \mathcal{L}_V(A)^{\varphi_A}$.

Fact: In the *standard* case $\mathcal{L}_V^\varphi = \big((\,\cdot\,)^{\sim} \otimes_{\mathbb{R}} V^{\phi}\big)_{\bar{0}}$, i.e. $\mathcal{L}_V(A)^{\varphi_A} = \big(A^{\sim} \otimes_{\mathbb{R}} V^{\phi}\big)_{\bar{0}}$ for $A \in (\mathrm{salg})_{\mathbb{C}}^{\bullet}$. In the *graded* case instead this <u>fails</u>... *Nevertheless*, the following holds:

Theorem 1 [Fioresi-G. (2020)]: For all $(V, \phi) \in (\mathrm{svec})_{\mathbb{C}}^{\bullet}$ — so $\exists\, \varphi$ on $\mathcal{L}_V$ —

$\mathcal{L}_V^\varphi$ is <u>representable</u>, namely it is represented by $\big(S\big(V^*\big), S\big(\phi^*\big)\big)$ in $(\mathrm{salg})_{\mathbb{C}}^{\bullet}$.

1.2.3 – The case of Lie superalgebras. If $V = \mathfrak{g} \in (\mathrm{sLie})_{\mathbb{C}}$, then in the previous construction "every step must be *Lie-upgraded*", namely

— the functor of points $\mathcal{L}_{\mathfrak{g}}$ takes values in $(\mathbb{Z}_2\text{–Lie})_{\mathbb{K}}$,

— real structures are bracket-preserving,

— the functor $\widetilde{\mathcal{L}}_{\mathfrak{g}}^{\bullet}$ takes values in $(\mathbb{Z}_2\text{–Lie})_{\mathbb{C}}^{\mathrm{re}}$,

— the "fixed points" functor $\mathcal{L}_{\mathfrak{g}}^{\varphi}$ takes values in $(\mathbb{Z}_2\text{–Lie})_{\mathbb{R}}$,

— etc. etc.

§ 1.3 – REAL STRUCTURES & FORMS for SUPERGROUPS

Memo: *(1) (algebraic) supergroup* on $\mathbb{K}$ is any functor $\mathbf{G} : (\mathrm{salg})_{\mathbb{K}} \longrightarrow (\mathrm{grps})$ with extra properties (EP) — roughly, local representability. Set $(\mathrm{sgrps})_{\mathbb{K}}$ for their category.

(2) there exists a natural forgetful functor $\mathcal{F} : (\mathrm{salg})_{\mathbb{C}}^{\bullet} \longrightarrow (\mathrm{salg})_{\mathbb{C}}$.

DEF.: for $\mathbf{G} \in (\mathrm{sgrps})_{\mathbb{C}}$, set $\mathbf{G}^{\bullet} := \mathbf{G} \circ \mathcal{F} : (\mathrm{salg})_{\mathbb{C}}^{\bullet} \longrightarrow (\mathrm{grps})$. We define

real structure on $\mathbf{G}$ any natural transformation $\Phi : \mathbf{G}^{\bullet} \longrightarrow \mathbf{G}^{\bullet}$ enjoying (SP') (=*involutivity* and *"tangent $\mathcal{A}_{\bar{0}}$–linearity"*); $\rightsquigarrow$ this yields the category $(\mathrm{sgrps})_{\mathbb{C}}^{\bullet}$.

Rmk: $\exists$ several alternative characterizations of real structures on $\mathbf{G}$, e. g.:

(a) Φ is a real structure for $\mathbf{G} \iff \Big(d\Phi$ is a real structure for $\mathcal{L}_{Lie(\mathbf{G})}$ & $\Phi^2 = \mathbf{1} \Big)$;

(b) if $\mathbf{G}$ is described/defined as a (classical) algebraic group $\mathbf{G}_{\bar{0}}$ with a structure sheaf $\mathcal{O}$ of commutative superalgebras, then a real structure on it is given by a (classical) real structure on $\mathbf{G}_{\bar{0}}$ coupled with a "real structure" $\varPhi : \mathcal{O} \longrightarrow \mathcal{O}$ on the structure sheaf;

(c) if $\mathbf{G}$ is *affine*, hence described by its Hopf superalgebra $\mathcal{O}(\mathbf{G})$, then a real structure on $\mathbf{G}$ is equivalent on a real structure on the complex Hopf superalgebra $\mathcal{O}(\mathbf{G})$.

(d) similar constructions can be done when $\mathbf{G}$ is a complex *Lie supergroup*.

DEF.: $\forall$ real structure Φ on $\mathbf{G}$, we call *real form* for it the functor of "fixed points"

$$\mathbf{G}^{\Phi} : (\mathrm{salg})_{\mathbb{C}}^{\bullet} \longrightarrow (\mathrm{grps}) \;\; , \qquad \big(A\,,(\,\widetilde{\ }\,)\big) \mapsto \mathbf{G}(A)^{\Phi_A}$$

It is said *standard*, resp. *graded* — as well as the form itself — if $\bullet = \mathrm{st}$, resp. $\bullet = \mathrm{gr}$.

Problem: can we describe the real form $\mathbf{G}^{\Phi}$ of $\mathbf{G}$?

Fact: $\exists$ equivalence $(\mathrm{sgrps})_{\mathbb{K}} \longleftrightarrow (\mathrm{sHCp})_{\mathbb{K}}$ (:= cat. of *super Harish-Chandra pairs*)

$$(\longrightarrow) \qquad \mathbf{G} \mapsto \big(\mathbf{G}_{\bar{0}}\,, \mathit{Lie}(\mathbf{G})\big) \quad \text{with} \quad \mathbf{G}_{\bar{0}} : (\mathrm{alg})_{\mathbb{K}} \longrightarrow (\mathrm{salg})_{\mathbb{K}} \xrightarrow{\mathbf{G}} (\mathrm{grps})$$

$$(\longleftarrow) \qquad (G_+\,, \mathfrak{g}) \mapsto \mathbf{G} := G_+ \times \mathcal{L}_{\mathfrak{g}_{\bar{1}}} \quad \text{i.e.} \quad \mathbf{G}(A) := G_+\big(A_{\bar{0}}\big) \times \big(A_{\bar{1}} \otimes \mathfrak{g}_{\bar{1}}\big)$$

Theorem 2 [Fioresi-G. (2020)]: If $(\mathbf{G}, \Phi) \in (\mathrm{sgrps})_{\mathbb{C}}^{\bullet}$ with $\mathfrak{g} := \mathit{Lie}(\mathbf{G})$, then

(a) if $\mathbf{G}$ is affine (=representable), then $\mathbf{G}^{\Phi}$ is affine too;

(b) $\mathbf{G}^{\Phi} = \mathbf{G}_{\bar{0}}^{\Phi_{\bar{0}}} \times \mathcal{L}_{\mathfrak{g}_{\bar{1}}}^{d\Phi}$, i. e. $\mathbf{G}^{\Phi}(A) = \mathbf{G}_{\bar{0}}^{\Phi_{\bar{0}}}\big(A_{\bar{0}}\big) \times \big(A_{\bar{1}} \otimes \mathfrak{g}_{\bar{1}}\big)^{(\,\widetilde{\ }\,)\otimes\phi} \quad \forall\, A$

2 – UNITARITY & COMPACTNESS

IDEA: "models" of *compact* real groups are unitary groups $U(n)$ $\rightsquigarrow$ "super version"???

— $*$ —

§ 2.1 – SPECIAL FORMS on SUPERSPACES

DEF: $\forall\ V \in (\mathrm{svec})_{\mathbb{C}}$, we say that:

(a) a map $f : V \times V \longrightarrow \mathbb{C}$ is *consistent* iff $f(x\,,y) = 0\ \ \forall\, |x| \neq |y|$.

(b) a $\mathbb{C}$–bilinear form $V \times V \xrightarrow{\langle\ ,\ \rangle} \mathbb{C}$ is *supersymetric* iff $\langle x, y \rangle = (-1)^{|x||y|} \langle y, x \rangle$.

(c) a map $B : V \times V \longrightarrow \mathbb{C}$ is *super Hermitian form* iff

$$B \text{ is } \mathbb{C}\text{–sesquilinear} \quad \text{and} \quad B(x,y) \;=\; (-1)^{|x||y|}\, \overline{B(y,x)}\ .$$

Rmk: every $B : V \times V \longrightarrow \mathbb{C}$ consistent and sesquilinear can be written as $B = B_{\bar{0}} + i\, B_{\bar{1}}$ for some (!) sesquilinear forms $B_{\bar{z}}$ on $V_{\bar{z}} \times V_{\bar{z}}$; then

$$B \text{ is super Hermitian} \iff B_{\bar{0}} \text{ and } B_{\bar{1}} \text{ are (classical) Hermitian}$$

DEF: $\forall\, V \in (\mathrm{svec})_{\mathbb{C}}$ and $B : V \times V \longrightarrow \mathbb{C}$ consistent super Hermitian, we say that B is *non-degenerate*, or that it is *positive definite*, if — writing $B = B_{\bar{0}} + i\, B_{\bar{1}}$ as above — both $B_{\bar{0}}$ and $B_{\bar{1}}$ are non-degenerate, or are positive definite, respectively.

DEF: $\forall\, (V,\phi) \in (\mathrm{svec})_{\mathbb{C}}^{\bullet}$, any bilinear form $\langle\ ,\ \rangle : V \times V \longrightarrow \mathbb{C}$ is called ϕ–*invariant* if it is a morphism of superspaces with real structures (i.e. of $\mathbb{Z}_4$–modules), that is

$$\overline{\langle\, v\, ,\, w\, \rangle} \,=\, \big\langle \phi(v)\, ,\, \phi(w) \big\rangle \qquad \text{for all} \quad v, w \in V \ .$$

The following sets a *link* among the previous notions:

Lemma: Let $(V,\phi) \in (\mathrm{svec})_{\mathbb{C}}^{\bullet}$ and let $\langle\ ,\ \rangle : V \times V \longrightarrow \mathbb{C}$ be $\mathbb{C}$–bilinear, consistent, supersymmetric and ϕ–invariant. Then the formula

$$B_{\phi}^{\pm}(x,y) := (\pm i\,)^{\nu_{\bullet}\,|x|\,|y|} \big\langle x\, ,\, \phi(y) \big\rangle \qquad \text{with} \qquad \nu_{\bullet} := \begin{cases} \bar{0} & \text{if } \ \bullet = \mathrm{st} \\ \bar{1} & \text{if } \ \bullet = \mathrm{gr} \end{cases}$$

defines two consistent super Hermitian forms on V.

§ 2.2 – EXAMPLES

[1] for $V := \mathbb{C}^{m|n}$, there exist consistent, supersymmetric, super Hermitian forms

$$B^{\pm}\big((\mathbf{z},\zeta),(\mathbf{z}',\zeta')\big) \;:=\; \mathbf{z}\cdot\overline{\mathbf{z}'} \pm i\,\zeta\cdot\overline{\zeta'}$$

[2] — *(a)* for $V := \mathbb{C}^{m|2t}$, there exist real structures

$$\phi_{\mathrm{st}}(\mathbf{z}\,,\zeta) \;:=\; \big(\,\overline{\mathbf{z}}\,,\overline{\zeta}\,\big) \quad \textit{(standard)}\,, \qquad \phi_{\mathrm{gr}}(\mathbf{z}\,,\zeta_+\,,\zeta_-) \;:=\; \big(\,\overline{\mathbf{z}}\,,+\overline{\zeta_-}\,,-\overline{\zeta_+}\,\big) \quad \textit{(graded)}$$

— *(b)* there $\exists$ a $\mathbb{C}$–bilinear, consistent, supersymmetric, $\phi_\bullet$–invariant form $\langle\ ,\ \rangle$ on V

$$\big\langle(\mathbf{z}\,,\zeta_+\,,\zeta_-)\,,\,(\mathbf{z}',\zeta'_+\,,\zeta'_-\,)\big\rangle \;:=\; \mathbf{z}\cdot\mathbf{z}' + \zeta_+\cdot\zeta'_- - \zeta_-\cdot\zeta'_+$$

— *(c)* $[$*(a)* $+$ *(b)* $+$ **Lemma**$]$ $\Longrightarrow$ $\exists$ consistent, super Hermitian forms on V

$$B^{\pm}_{\phi_{\mathrm{st}}}\big((\mathbf{z}\,,\zeta_+\,,\zeta_-)\,,\,(\mathbf{z}',\zeta'_+\,,\zeta'_-)\big) \;:=\; \mathbf{z}\cdot\overline{\mathbf{z}'} + \zeta_+\cdot\overline{\zeta'_-} - \zeta_-\cdot\overline{\zeta'_+}$$

$$\begin{aligned} B^{\pm}_{\phi_{\mathrm{gr}}}\big((\mathbf{z}\,,\zeta_+\,,\zeta_-)\,,\,(\mathbf{z}',\zeta'_+\,,\zeta'_-)\big) \;&:=\; \mathbf{z}\cdot\overline{\mathbf{z}'} \mp i\,\big(\,\zeta_+\cdot\overline{\zeta'_+} + \zeta_-\cdot\overline{\zeta'_-}\,\big) \\ &=\; \mathbf{z}\cdot\overline{\mathbf{z}'} \mp i\,\zeta\cdot\overline{\zeta'} \end{aligned}$$

§ 2.3 – FUNCTORIAL FORMULATION

DEF.: $\forall\ (V,\phi) \in (\mathrm{svec})_{\mathbb{C}}^{\bullet}$, we call *Hermitian form* for $\mathcal{L}_V$ — or also for the functor $\mathcal{L}_V^{\bullet} : (\mathrm{salg})_{\mathbb{C}}^{\bullet} \longrightarrow (\mathbb{Z}_2\text{–vec})_{\mathbb{R}}$ — any natural transformation $\mathcal{B} : \mathcal{L}_V^{\bullet} \times \mathcal{L}_V^{\bullet} \longrightarrow \mathcal{L}_{\mathbb{C}}^{\bullet}$ s. t.

(1) $\mathcal{B}$ is "$\mathcal{A}_{\bar{0}}$–sesquilinear", i.e. $\mathcal{B}(a\,X\,,\,Y) = a\,\mathcal{B}(X,Y)$ and $\mathcal{B}(X\,,aY) = \widetilde{a}\,\mathcal{B}(X\,,\,Y)$

(2) $\quad \mathcal{B}(X\,,\,Y) \;=\; (-1)^{(\bar{1}-\nu_{\bullet})[X][Y]}\,\widetilde{\mathcal{B}(Y\,,X)} \qquad \big(\forall\, X, Y \in \mathcal{L}_V^{\bullet}(A)\,,\ a \in A_{\bar{0}}\,\big)$

Such a $\mathcal{B}$ is called *consistent* iff $\mathcal{B}(X\,,\,Y) = 0$ for $[X] \neq [Y]$.

Question: what is the link (if any) with super Hermitian forms on V ?

Proposition: $\forall\ (V,\phi) \in (\mathrm{svec})_{\mathbb{C}}^{\bullet}$, there exists a bijection

$$\left\{ \begin{array}{c} \text{consistent super} \\ \text{Hermitian forms on } V \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{consistent Hermitian} \\ \text{forms on } \mathcal{L}_V \end{array} \right\}$$

given by $\qquad B_V \mapsto \mathcal{B}_{\mathcal{L}_V}\ :\ \mathcal{B}_{\mathcal{L}_V}(a\,x, b\,y) := i^{\,|x||y|}\, a\,\widetilde{b}\,B_V(x,y)$

$\Longrightarrow$ **DEF.:** $\mathcal{B}_{\mathcal{L}_V}$ is called *non-degenerate* or *positive definite* iff the associated B_V is.

§ 2.4 – UNITARITY

DEF.: Given $(V, \phi) \in (\mathrm{svec})_{\mathbb{C}}^{\bullet}$ and a non-degenerate, consistent, Hermitian form $\mathcal{B}$ on $\mathcal{L}_V$, the *adjoint* of $M \in \big(End(V)\big)(A)$ is the unique $M^{\star} \in \big(End(V)\big)(A)$ such that

$$\mathcal{B}\big(X, M^{\star}(Y)\big) \;=\; (-1)^{(\bar{1}-\nu_{\bullet})\,[X]\,[Y]}\,\mathcal{B}\big(M(X)\,,Y\big)$$

Rmk: the sign vanishes in the *graded* case, which then looks more "natural"...

Lemma: (notations as above)

The "adjoint operator" enjoys the following properties:

$$M^{\star\star} = M \quad , \qquad (a\,M)^{\star} = \widetilde{a}\,M^{\star} \quad , \qquad (-M)^{\star} = -\big(M^{\star}\big)$$

$$(M+N)^{\star} = M^{\star} + N^{\star} \quad , \qquad (M\,N)^{\star} = N^{\star}\,M^{\star} \quad , \qquad \big(J^{-1}\big)^{\star} = \big(J^{\star}\big)^{-1}$$

$$[M, N]^{\star} \;=\; (-1)^{(\bar{1}-\nu_{\bullet})\,[M]\,[N]}\,\big[N^{\star}, M^{\star}\big]$$

N.B.: the properties above directly imply the next, key result...

Theorem 3 [Fioresi-G. (2020)]:

The natural transformation $\circledast : \mathcal{L}^{\bullet}_{\mathfrak{gl}(V)} \longrightarrow \mathcal{L}^{\bullet}_{\mathfrak{gl}(V)}$ given on objects by

$$M \mapsto M^{\circledast} := \begin{cases} -M^{\star} & \text{if} \quad [M] = \bar{0} \\ i\,M^{\star} & \text{if} \quad [M] = \bar{1} \end{cases} , \qquad \text{or} \qquad M \mapsto M^{\circledast} := -M^{\star} \quad \forall\ M$$

(_standard_) (_graded_)

is a real structure — standard or graded, following • — on the functor $\mathcal{L}_{\mathfrak{gl}(V)}$, hence it corresponds to a real structure on $\mathfrak{gl}(V)$ and one on $\mathbf{GL}(V)$, both denoted by $\circledast$ again.

DEF.: Let (V, ϕ) and $\mathcal{B}$ be as above.

(a) We call *unitary Lie superalgebra* of $(V, \phi, \mathcal{B})$ the real form of $\left(\mathcal{L}_{\mathfrak{gl}(V)} , \circledast \right)$.

(b) We call *unitary supergroup* of $(V, \phi, \mathcal{B})$ the real form of $\left(\mathbf{GL}(V) , \circledast \right)$.

The notation will be $\mathfrak{u}_{\mathcal{B}}(V) := \mathcal{L}^{\circledast}_{\mathfrak{gl}(V)}$ for *(a)* and $\mathbf{U}_{\mathcal{B}}(V) := \mathbf{GL}(V)^{\circledast}$ for *(b)*.

Rmk ©: the "even part" of a unitary superobject is direct product of unitary (classical) objects, namely $\mathfrak{u}_{\mathcal{B}}(V)_{\bar{0}} = \mathfrak{u}_{\mathcal{B}_{\bar{0}}}\big(V_{\bar{0}}\big) \oplus \mathfrak{u}_{\mathcal{B}_{\bar{1}}}\big(V_{\bar{1}}\big)$ and $\mathbf{U}_{\mathcal{B}}(V)_{\bar{0}} = U_{\mathcal{B}_{\bar{0}}}\big(V_{\bar{0}}\big) \times U_{\mathcal{B}_{\bar{1}}}\big(V_{\bar{1}}\big)$.

§ 2.5 – (SUPER-)COMPACTNESS

Memo: Any Lie group K is *compact* $\iff$ $K \leq U(W)$ for some $W \in (\mathrm{vec})_{\mathbb{C}}^{\mathrm{re}}$ with a positive definite Hermitian form. Then also $\mathfrak{k} = Lie(K)$ for some compact Lie group $K \iff \mathfrak{k} \leq \mathfrak{u}(W)$ for some $W \in (\mathrm{vec})_{\mathbb{C}}^{\mathrm{re}}$ with a positive definite Hermitian form.

DEF.: *(a)* Any $(\mathbf{G}\,,\Phi) \in (\mathrm{sgrps})_{\mathbb{C}}^{\bullet}$ — or its integral form $\mathbf{G}^{\Phi}$ alike — is called

(a.-) *compact* iff $\mathbf{G}_{\bar{0}}^{\Phi} \leq U(W)$ for some $W \in (\mathrm{vec})_{\mathbb{C}}^{\mathrm{re}}$ with positive definite Hermitian form — or, equivalently, iff the (classical) group $\mathbf{G}_{\bar{0}}^{\Phi}$ is compact;

(a.+) *super-compact* iff $\mathbf{G}^{\Phi} \leq \mathbf{U}_{\mathcal{B}}(V)$ for some $V \in (\mathrm{svec})_{\mathbb{C}}^{\bullet}$ with a positive definite super Hermitian form.

(b) Any $(\mathfrak{g}\,,\phi) \in (\mathrm{sLie})_{\mathbb{C}}^{\bullet}$ — or its integral form $\mathcal{L}_{\mathfrak{g}}^{\varphi}$ alike — is called

(b.-) *compact* iff $\left(\mathcal{L}_{\mathfrak{g}}^{\varphi}\right)_{\bar{0}} \leq \mathfrak{u}(W)$ for some $W \in (\mathrm{vec})_{\mathbb{C}}^{\mathrm{re}}$ with a positive definite Hermitian form;

(b.+) *super-compact* iff $\mathcal{L}_{\mathfrak{g}}^{\varphi} \leq \mathfrak{u}_{\mathcal{B}}(V)$ for some $V \in (\mathrm{svec})_{\mathbb{C}}^{\bullet}$ with a positive definite super Hermitian form.

Rmk: due to Rmk © above, we have "super-compact" $\Longrightarrow$ "compact"

3 – EXISTENCE(!) & NON-EXISTENCE RESULTS

HPs: *(a)* $\mathfrak{g} :=$ cmplx (f.d.) contragr. Lie superalgebra, with gen.s x_i^+, h_i, x_i^- $(\, i \in I \,)$

(b) $\mathbf{G} :=$ connected, simply connected, complex supergroup with $Lie(\mathbf{G}) = \mathfrak{g}$

Proposition: $\exists!$ a *graded* real structure ω on $\mathfrak{g}$ given (for all $i \in I$) by

$$\omega(h_i) = -h_i \;, \qquad \omega\big(x_i^\pm\big) = -x_i^\mp \;\text{ if }\; \big|x_i^\pm\big| = \bar{0} \,, \qquad \omega\big(x_i^\pm\big) = \pm x_i^\mp \;\text{ if }\; \big|x_i^\pm\big| = \bar{1}$$

Moreover, $\exists!$ *graded* real structure Ω on $\mathbf{G}$ corresponding to ω.

HPs+: *(a)* $\mathfrak{g} :=$ complex (f.d.) contragredient *simple*, i.e. complex simple of *basic* *type*

(b) $\mathbf{G} :=$ connected, simply connected, complex supergroup with $Lie(\mathbf{G}) = \mathfrak{g}$

Overview: standard real structures on such $\mathfrak{g}$ were *classified* by Kac (1977), Parker (1980), Serganova (1983) and Chuah (2013); Serganova classified also graded ones.

Later on, Pellegrini (2007) looked for *compact* (real) forms, noting (via case by case inspection) that every basic $\mathfrak{g}$ has a *compact graded* real structure (and form).

Our work [Fioresi-G. (2020)] largely improves Pellegrini's results on the existence (and uniqueness) of compact forms, finding a deeper motivation.

Theorem 4 [Fioresi-G. (2020)]: ($\exists!$ super-compact, graded real str. on $\mathfrak{g}/\mathbf{G}$)

Let $\mathfrak{g}$ be complex simple of *basic* type, and $\mathbf{G}$ connected simply conn. s.t. $\mathit{Lie}(\mathbf{G}) = \mathfrak{g}$.

Then the graded real structure ω on $\mathfrak{g}$ given above is *super-compact*; similarly, the graded real structure Ω on $\mathbf{G}$ is *super-compact*.

In particular, $\exists$ super-compact graded real structures on both $\mathfrak{g}$ and $\mathbf{G}$.

Moreover, both structures are unique up to inner automorphisms.

Proof (sketch): Let κ be the Cartan-Killing form of $\mathfrak{g}$: it is $\mathbb{C}$–bilinear, consistent and super-symmetric; moreover, κ is also ω–invariant. Therefore $\Longrightarrow$

$\Longrightarrow$ κ and ω jointly define — applying our general recipe — a consistent, super Hermitian form B_κ on $\mathfrak{g}$; moreover, this B_κ is also positive definite; $\Longrightarrow$

$\Longrightarrow$ letting $\mathcal{B}_\kappa$ be the Hermitian form on $\mathcal{L}_\mathfrak{g}$ corresponding to B_κ on $\mathfrak{g}$, we have a well-defined unitary Lie superalgebra $\mathfrak{u}_{\mathcal{B}_\kappa}(\mathfrak{g})$ inside $\mathfrak{gl}(\mathfrak{g})$.

The form κ is *ad*–invariant $\Longrightarrow$ the image of $\mathit{ad} : \mathfrak{g} \hookrightarrow \mathfrak{gl}(\mathfrak{g})$ sits in $\mathfrak{u}_{\mathcal{B}_\kappa}(\mathfrak{g}) \Longrightarrow$

$\Longrightarrow$ $\mathfrak{g}$ embeds into $\mathfrak{u}_{\mathcal{B}_\kappa}(\mathfrak{g})$, hence $(\mathfrak{g}, \omega)$ is graded super-compact. $\square$

N.B.: The above accounts for the *graded* case, and proves the importance of such a notion. What for *standard* structures instead? The answer is twofold:

Theorem 5 [Fioresi-G. (2020)]: ($\exists! / \nexists$ (super-)compact, standard r. s. on $\mathfrak{g} / \mathbf{G}$)

Let $\mathfrak{g}$ be complex simple of *basic* type $T \in \{A, B, C, D, F, G\}$, and let $\mathbf{G}$ be connected simply connected such that $Lie(\mathbf{G}) = \mathfrak{g}$. Then:

(a) if $T \in \{A, C\}$ — i.e. $\mathfrak{g}$ is "of type 1" — then $\exists$ a standard compact real structure on $\mathfrak{g}$ and on $\mathbf{G}$, that is unique up to inner automorphisms;

(b) if $T \in \{B, D, F, G\}$ — i.e. $\mathfrak{g}$ is "of type 2" — then $\nexists$ standard compact real structures on $\mathfrak{g}$ nor on $\mathbf{G}$.

Proof (sketch): *(a)* We provide a concrete example for both cases.

(b) It follows — with a rather different approach — from Chuah's work. $\square$

Open quest.: in type 1, $\exists$ a real struct. on $\mathfrak{g} \,\big/\, \mathbf{G}$ that is *standard* and *super-compact*?

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— THANKS for your attention! —

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