Weak Quasi-Hopf Algebras, Vertex Operator Algebras and Conformal Nets

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Introduction

- Rational QFTs, when seen from a representation theory point of view, naturally give rise to fusion categories. When the QFT is unitary the corresponding representation category should also be unitary.
- In particular rational chiral CFTs are an important source very interesting fusion categories. In fact it has been conjectured that every unitary modular fusion category comes from a rational chiral CFT.

- Weak quasi-Hopf algebras are a generalization Drinfelds' quasi-Hopf algebras. Every fusion category is tensor equivalent to the representation category of a weak quasi-Hopf algebra.
- In this talk I will discuss some recent results showing that weak quasi-Hopf algebras are a useful and natural tool to understand certain aspects of the representation theory of rational VOAs especially for the unitarity and the relations to the theory of conformal nets.

Tensor categories

- I will denote the objects of a category $\mathcal{C}$ by $X, Y, Z, \cdots \in \mathrm{Obj}(\mathcal{C})$ and the corresponding hom-spaces by $\mathrm{Hom}(X, Y) \cdots \subset \mathrm{Hom}(\mathcal{C})$.
- In a linear category the hom-spaces are vector spaces (finite-dimensional and over $\mathbb{C}$ in this talk), the composition is bilinear and we have direct sums.
- In a tensor category we have a tensor product of objects $X, Y \mapsto X \otimes Y$ and a corresponding tensor product of arrows $T \in \mathrm{Hom}(X_1, Y_1), S \in \mathrm{Hom}(X_2, Y_2) \mapsto$ $T \otimes S \in \mathrm{Hom}(X_1, \otimes X_2, Y_1 \otimes Y_2)$.

- We have a unit object $\iota \in \mathrm{Obj}(\mathcal{C})$ that is simple, i.e. it is non-zero and it has no non-trivial subobjects, and that here we assume to be strict i.e. $\iota \otimes X = X \otimes \iota = X$ for all $X \in \mathrm{Obj}(\mathcal{C})$. Moreover, we have associativity isomorphisms $\alpha_{X,Y,Z} \in \mathrm{Hom}\big((X \otimes Y) \otimes Z), X \otimes (Y \otimes Z)\big)$ satisfying the so called pentagon equation. A tensor category is called strict if the tensor product is (strictly) associative and the associativity isomorphisms are the identity isomorphisms.
- To simplify the exposition I will only consider fusion categories. These are semisimple tensor categories with finitely many isomorphism classes of simple objects and which are rigid i.e. every object X has a (two-sided) dual object $X^\vee$.
- The Grothendieck ring $\mathrm{Gr}(\mathcal{C})$ generated by the isomorphism classes of simple objects is the fusion ring of the fusion category $\mathcal{C}$.

- A fusion category is called braided if it admits a natural family of isomorphisms $c_{X,Y} \in \mathrm{Hom}(X \otimes Y, Y \otimes X)$ satisfyinfg the so called hexagon equations. Braided fusion categories give rise to representations of the braid group.
- A braided fusion category with a compatible twist $X \mapsto \theta_X \in \mathrm{Hom}(X, X)$ is called a ribbon fusion category. A ribbon fusion category with a non-degenerate grading is called a modular fusion category. The latter defines a (projective) representation of the modular group $\mathrm{SL}(2, \mathbb{Z})$ trough the modular matrices S, T.
- Some examples of fusion categories are: Vec; Vec_G; $\mathrm{Rep}(G)$ (G finite group); Vec_G^ω (ω 3-cocycle on G); $\mathrm{Rep}(A)$ (A finite dimensional semisimple Hopf algebra).

Unitary fusion categories

- A C*-category (with f.d. Hom spaces) is a linear category with a *-structure on the Hom spaces. This means that there is an anti-linear involutive map $\mathrm{Hom}(X, Y) \ni T \mapsto T^* \in \mathrm{Hom}(Y, X)$ such that $(TS)^* = S^* T^*$. Moreover we have the positivity condition $T^* T = 0 \Rightarrow T = 0$.
- A unitary (or C*) fusion category is a fusion category which is also a C*-category and such that $(T \otimes S)^* = T^* \otimes S^*$. Moreover the associativity isomorphisms are unitary, i.e. $\alpha^*_{X,Y,Z} = \alpha^{-1}_{X,Y,Z}$.
- Some examples of unitary fusion categories are: Hilb; Hilb_G; Hilb^ω_G; $\mathrm{Rep}^u(G)$ ...

Weak quasi-Hopf algebras

- A quasi-Hopf algebra is a quintuple $(A, \Delta, \varepsilon, S, \Phi)$. Here A is a unital associative algebra (over $\mathbb{C}$ in this talk), the coproduct $\Delta : A \to A \otimes A$ is a unital homomorphism, the counit $\varepsilon : A \to \mathbb{C}$ is a nonzero homomorphism, the antipode $S : A \to A$ is an antiautomorphism, $\Phi \in A \otimes A \otimes A$ is the associator + axioms
- The coproduct gives a tensor structure on Rep(A). The tensor product $\underline{\otimes}$ on the objects of Rep(A) is then given by $\pi_1 \underline{\otimes} \pi_2 := \pi_1 \otimes \pi_2 \circ \Delta \in \mathrm{Rep}(A)$ and Φ gives rise to the associativity isomorphisms.
- If A is finite dimensional and semisimple then Rep(A) is a fusion category.

- Quasi-Hopf algebras are not sufficiently general to describe many interesting fusion categories related to QFT.
- This is because, when A is semisimple, the function D on the fusion ring $\mathrm{Gr}(\mathrm{Rep}(A))$ defined by $D([\pi]) := \dim(V_\pi)$, where V_π is the representation space of π, is a (positive) integral valued dimension function and hence it must agree with the Frobenius-Perron dimension of the category which in general is not integer valued. For example it can take the values $D([\pi]) = 2\cos(\frac{\pi}{n})$, n=3, 4, 5, ...
- Mack and Schoumerus suggested the following solution to the above problem: give up to the request that Δ is unital so that a weak quasi-Hopf algebra is again a quintuple $(A, \Delta, \varepsilon, S, \Phi)$ with a possibly non-unital coproduct.

- In this way $\Delta(1_A)$ is an idempotent in $A \otimes A$ commuting with $\Delta(A)$ but typically different from $1_A \otimes 1_A$.
- The tensor product $\pi_1 \underline{\otimes} \pi_2$ in $\mathrm{Rep}(A)$ is now defined by the restriction of $\pi_1 \otimes \pi_2 \circ \Delta$ to $\pi_1 \otimes \pi_2 \circ \Delta(1_A) V_{\pi_1} \otimes V_{\pi_2}$.
- Now, for a given (f.d., semisimple) A, the additive function $D : \mathrm{Gr}(\mathrm{Rep}(A)) \to \mathbb{Z}$ defined by $D([\pi]) := \dim(V_\pi)$ is only a (positive) weak dimension function i.e. it satisfies $D([\pi_1 \underline{\otimes} \pi_2]) \leq D([\pi_1])D([\pi_2])$, $D([\iota]) = 1$ and $D([\overline{\pi}]) = D([\pi]) > 0$ (if π is non-zero) and this gives no important restrictions.

Tannakian results

- The following result are due mainly due to Häring-Oldenburg (1997).
- Let $\mathcal{C}$ be a fusion category and $D : \mathrm{Gr}(\mathcal{C}) \to \mathbb{Z}$ be an integral weak dimension then there exists a finite dimensional semisimple weak quasi-Hopf algebra $(A, \Delta, \varepsilon, S, \Phi)$ and a tensor equivalence $\mathscr{F} : \mathcal{C} \to \mathrm{Rep}(A)$ such that $D([X]) = \dim(V_{\mathscr{F}(X)})$ for all $X \in \mathrm{Obj}(\mathcal{C})$.
- Extra structure on $\mathcal{C}$ gives extra structure on A: brading $\leftrightarrow$ R-matrix ; C*-tensor structure on $\mathcal{C} \leftrightarrow \Omega$ - involutive structure on A (in particular A is a C*-algebra).
- The weak quasi-Hopf algebra associated to a fusion category $\mathcal{C}$ is highly non-unique: it depends on the choice of D and, once D is fixed is only defined up to a "twist" $F \in A \otimes A$.

Fusion categories from chiral CFT

- There are two main approaches to chiral (2D) CFT: vertex operator algebras (VOAs) and conformal nets. Under suitable rationality conditions they both give rise to modular fusion categories.
- A vertex operator algebra (VOA) is a vector space V together with a linear map (the state-field correspondence) $a \mapsto Y(a,z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$ from V into the set of formal Laurent series with coefficients $\mathrm{End}(V)$.
- The map is assumed to satisfy certain natural (and physically motivated) conditions: locality; conformal covariance; energy bounded from below; vacuum. The "fields" $Y(a,z)$ are called vertex operators.

- A conformal net $\mathcal{A}$ is an inclusion preserving map $S^1 \ni I \mapsto \mathcal{A}(I)$, where each $\mathcal{A}(I)$ is a von Neumann algebra acting on a fixed Hilbert space $\mathcal{H}$.
- The map is assumed to satisfy various natural assumptions: locality, conformal covariance, positivity of the energy, vacuum.

- Both VOAs and conformal nets have interesting representation theories.
- If V is strongly rational VOA then $\mathrm{Rep}(V)$ is a modular fusion category (Huang 2008).
- If $\mathcal{A}$ is a completely rational then $\mathrm{Rep}(\mathcal{A})$ is a unitary modular fusion category (Kawahigashi, Longo Mueger (2001)).

From VOAs to conformal nets

- A general connection between VOAs and conformal nets has been recently considered by Carpi, Kawahigashi, Longo and Weiner (2018).
- One first need to consider only unitary VOAs (explicitly defined by Dong, Lin and CKLW).
- For sufficiently nice (simple) unitary VOAs called strongly local one can define a map $V \mapsto \mathcal{A}_V$ into the class of conformal nets.

- Conjecture 1: The map $V \mapsto \mathcal{A}_V$ gives a one-to-one correspondence between the class of simple unitary VOAs and the class of conformal nets.
- Conjecture 2: The map $V \mapsto \mathcal{A}_V$ gives gives a one-to-one correspondence between the class of strongly rational unitary VOAs and the class of completely rational conformal nets. Moreover, if V is completely rational we have a tensor equivalence $\mathrm{Rep}(V) \simeq \mathrm{Rep}(\mathcal{A}_V)$.

- Recently it has been suggested by Carpi, Weiner and Xu (in preparation) to consider a strong integrability condition on unitary VOA-modules of a strongly local V which allows to define a map $M \mapsto \pi_M$ from V-modules to representations of $\mathcal{A}_V$. In certain cases this gives an isomorphism of linear C*-categories $\mathscr{F} : \mathrm{Rep}^u(V) \to \mathrm{Rep}(\mathcal{A}_V)$ where $\mathrm{Rep}^u(V)$ is the linear C*-category of unitary V-modules.
- Conjecture 3: Assume that V is strongly rational and strongly local. Then $\mathrm{Rep}^u(V)$ can be upgraded to a unitary modular tensor category such that the forgetful functor : $\mathrm{Rep}^u(V) \to \mathrm{Rep}(V)$ is a braided tensor equivalence. Moreover, the functor $\mathscr{F} : \mathrm{Rep}^u(V) \to \mathrm{Rep}(\mathcal{A}_V)$ discussed above admits a unitary tensor structure.
- Further remarkable results in this direction have been recently obtained by Bin Gui and by James Tener.

From VOAs to unitary fusion categories

- Let $\mathcal{C}^+$ be a linear C*-category, $\mathcal{C}$ be a fusion category and $\mathcal{F} : \mathcal{C}^+ \to \mathcal{C}$ be a linear equivalence
- Theorem (Carpi, Ciamprone, Pinzari): If $\mathcal{C}$ is tensor equivalent to a unitary fusion category $\mathcal{D}^+$ then $\mathcal{C}^+$ can be upgraded to a unitary fusion category in such a way that $\mathcal{F} : \mathcal{C}^+ \to \mathcal{C}$ becomes a tensor equivalence. This structure is unique up to unitary equivalence and makes $\mathcal{C}^+$ unitary tensor equivalent to $\mathcal{D}^+$.
- As a corollary we find a proof of a conjecture by Cesar Galindo: two tensor equivalent unitary fusion categories must be unitary tensor equivalent. A different proof was found independently by David Reutter.

- Let $\mathfrak{g}$ be a complex simple Lie algebra, let k be a positive integer and let $V_{\mathfrak{g}_k}$ be the corresponding simple level k affine VOA. It is known that $V_{\mathfrak{g}_k}$ is a unitary strongly rational VOA and that every $V_{\mathfrak{g}_k}$-module is unitarizable.
- By a result of Finkelberg (1996) based on the work Kazhdan and Lusztig we know that $\mathrm{Rep}(V_{\mathfrak{g}_k})$ is tensor equivalent to the "semisimplified" category $\widetilde{\mathrm{Rep}(G_q)}$ associated to the representations of the quantum group G_q, with G the simply connected compact Lie group associated to $\mathfrak{g}$ and $q = e^{\frac{i\pi}{d(k+h^\vee)}}$, $h^\vee$ = dual Coxeter number, $d = 1$ if $\mathfrak{g}$ is ADE, $d = 2$ if $\mathfrak{g}$ is BCF and $d = 3$ if $\mathfrak{g}$ is G_2.
- It was shown by Wenzl and Xu (1998) that $\widetilde{\mathrm{Rep}(G_q)}$ is tensor equivalent to a unitary fusion category.

- As a consequence we get that $\mathrm{Rep}^u(V_{\mathfrak{g}_k})$ admits an essentially unique structure of unitary fusion category.
- The existence part of this result has been recently also proved by Gui and Tener in a series of papers appeared in the arXiv between 2017 and 2019, by a completely different method based on Connes fusions for bimodules and a deep analysis of the analytic properties of the smeared intertwiners operators for VOA modules.
- Our method works also for many other VOAs such as e.g. lattice VOAs, holomorphic orbifolds, ...

The Zhu algebra as a weak quasi-Hopf algebra

- Let V be strongly rational. In 1998 Zhu introduced a finite-dimensional semisimple algebra $A(V)$ gave a linear equivalence $\mathscr{F}_V : \mathrm{Rep}(V) \to \mathrm{Rep}(A(V))$.
- If $D_V([M]) := \dim(\mathscr{F}_V(M))$ defines a weak dimension function then, it follows from the previously described Tannakian results that $A(V)$ can be upgraded to a weak-quasi Hopf algebra and $\mathscr{F}_V : \mathrm{Rep}(V) \to \mathrm{Rep}(A(V))$ to an equivalence of fusion categories.
- D_V is not always a weak dimension function. A counterexample is given e.g. by the Ising VOA ($c = \frac{1}{2}$ Virasoro). However, D_V is a weak dimension function in many interesting cases e.g. if V is a unitary affine VOA.

Classification of type A ribbon fusion categories

- As another application of the theory of weak quasi-Hopf algebra we give classification of pseudo-unitary type A fusion categories.
- The starting point is the work of Kazhdan and Wenzl (1993) on the classification of type A tensor categories.
- As a consequence of our results we have in particular the following: Let $\mathcal{C}$ be a modular fusion category with modular matrices S, T coinciding with the Kac-Peterson matrices for the $\mathfrak{sl}(n)$ affine Lie algebra at positive integer level k . Then $\mathcal{C}$ is ribbon equivalent to $\mathrm{Rep}(V_{\mathfrak{sl}(n)_k})$

- Consequence 1. Conjecture 3 is true for unitary affine VOAs of type A. In fact we have a unitary ribbon equivalence $\mathscr{F} : \mathrm{Rep}^u(V_{\mathfrak{sl}(n)_k}) \to \mathrm{Rep}(\mathcal{A}_{V_{\mathfrak{sl}(n)_k}})$. The same result has been independently obtained by Bin Gui by different methods (direct analytic proof instead of classification). His proof also works for many other Lie types.
- Consequence 2. We have a proof of Finkelberg equivalence in the type A case.

GRAZIE!