Weak generalized lifting property, Bruhat intervals and Coxeter matroids

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Virtually Università di Tor Vergata but #IoRestoaCasa, March 13, 2020

Coxeter matroids

- Generalization of Whitney's (ordinary) matroids
- Introduced by I. Gelfand and V. Serganova in 1987
- Studied by many people such as V. Borovik, I. Gelfand, M. Goresky, R. MacPherson, V. Serganova, A. Vince, N. White, A. Zelevinsky...
- Lies at the intersection of Combinatorics, Algebra, Geometry, Optimization Theory

We want to tell you about:
THEOREM (CASELLI-D'ADDERIO-M). Bruhat intervals of finite Coxeter groups are Coxeter matroids

Main new tool in the proof of the theorem:
WEAK GENERALIZED LIFTING PROPERTY
(true for all finite and infinite Coxeter groups)

- (W, S) **Coxeter system**
  - W Coxeter group
  - $S = \{s_1, \ldots, s_n\}$ Coxeter generators
  - relations: $\begin{cases} s_i^2 = e & \text{(involutions)} \\ (s_i s_j)^{m_{ij}} = e & m_{ij} \in \mathbb{N}_{\geq 2} \cup \infty \end{cases}$

- Finite Coxeter groups are Reflection Groups
  W acting on a real vector space V

- $\ell(w) := \min\{k : w \text{ is a product of } k \text{ generators}\}$ **length**

- $\Phi = \Phi^+ \sqcup \Phi^-$ **(positive and negative) roots**

- T **reflections**

$$\begin{array}{ccc} T & \overset{\sim}{\longleftrightarrow} & \Phi^+ \\ t & \mapsto & \alpha_t \end{array} \quad \text{bijection}$$

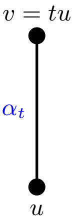
- $\leq$ **Bruhat order** on W

  it is the transitive closure of $u \lhd v \iff \exists t \in T : \begin{cases} v = tu \\ \ell(v) = \ell(u) + 1 \end{cases}$

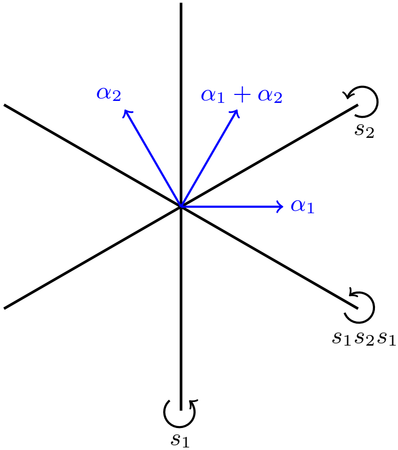
PROPERTIES:

- the identity e is the bottom element
- the poset is ranked by length function ℓ
- there exists a top element iff W is finite

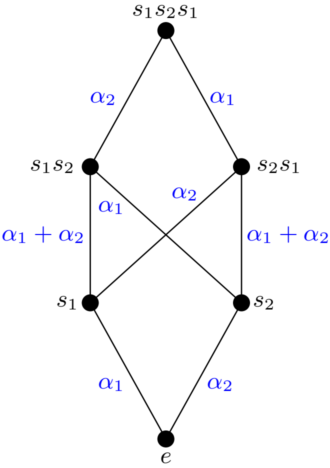
- **Hasse diagram**: upword edge from u to v iff $u \lhd v$
  We also lable the edge with the positive root α_t corresponding to t



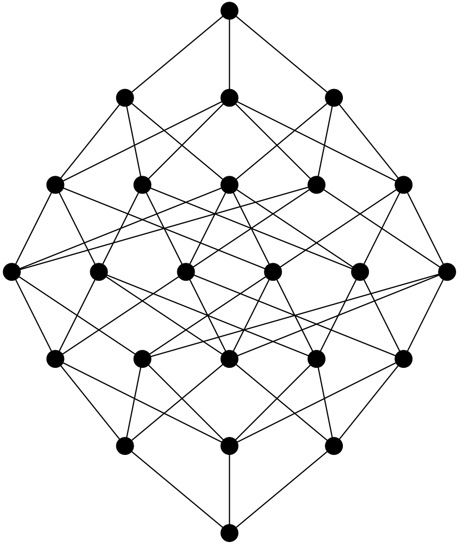
$(W, S = \{s_1, s_2\})$ relations: $s_1^2 = s_2^2 = (s_1 s_2)^3 = e$
$W = \{e, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1 = s_2s_1s_2\} \simeq$ symmetric group S_3



Hasse diagram



W of type A_3: the symmetric group S_4



Let (W, S) be finite, $J \subseteq S$.

- $W_J := \langle J \rangle$ parabolic subgroup generated by J
- $W^J := \{w \in W : ws > w \quad \forall s \in J\}$ minimal left coset representatives

There is a unique decomposition $\begin{array}{ccc} W & \stackrel{\sim}{\longleftrightarrow} & W^J \times W_J \\ w & \mapsto & w^J \cdot w_J \end{array}$

$$W^J \stackrel{\sim}{\longleftrightarrow} W/W_J \quad \text{bijection}$$

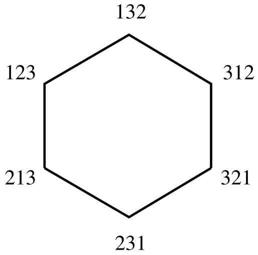
Fix $p \in V$ s. t. $(p, \alpha_s) \begin{cases} = 0 & \text{if } s \in J \\ < 0 & \text{if } s \notin J \end{cases}$

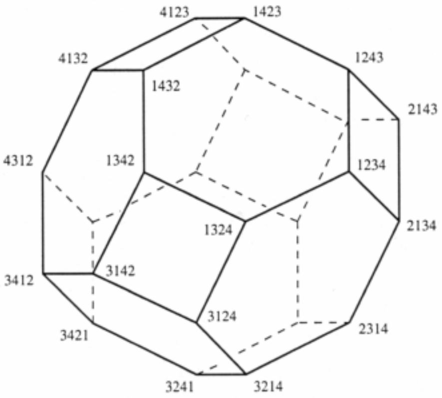
$\begin{array}{cccc} \delta_p : & W/W_J & \to & V \\ & vW_J & \mapsto & v(p) \end{array}$ well-defined since W_J fixes p

Given $\emptyset \neq \mathcal{M} \subseteq W/W_J$, define a polytope associated with $\mathcal{M}$:

$\boxed{\Delta_{\mathcal{M}}(p) = \text{ convex hull of } \delta_p(\mathcal{M})}$ shorthand: $\Delta_{\mathcal{M}}(p) = \Delta_{\mathcal{M}}$

If $W = S_n$, $J = \emptyset$, $\mathcal{M} = W$, then $\Delta_{\mathcal{M}}$ is the classical permutohedron.





w-Bruhat order on W:

$$u \leq^w v \iff w^{-1}u \leq w^{-1}v$$

Note: $\leq^e = \leq$

w-Bruhat order on W/W_J:
THEOREM/DEFINITION: Every $A \in W/W_J$ has a $\min^w$ and a $\max^w$ w.r.t. $\leq^w$.
Let $A, B \in W/W_J$, $w \in W$. TFAE:

- $A \leq^w B$
- $\min^w A \leq^w \min^w B$
- $\max^w A \leq^w \max^w B$
- $a \leq^w b$ for some $a \in A$ and $b \in B$

$\emptyset \neq \mathcal{M} \subseteq W/W_J$ is a **Coxeter matroid for W and J** if it satisfies the **Maximality Property**

- for all $w \in W$, there exists $A \in \mathcal{M}$ s. t. $B \leq^w A$ for all $B \in \mathcal{M}$

- (W, S) of type A_{n-1}
- $W \simeq S_n$ the symmetric group on $[n] := \{1, \ldots, n\}$
- $S = \{s_1, s_2, \ldots, s_{n-1}\}$ with $s_i = (i, i+1)$.

If $J = S \setminus \{s_k\}$ then

- $W_J \simeq S_k \times S_{n-k}$
- every $b \in W/W_J$ corresponds to a subset B of $[n]$ of cardinality k

$$W/W_J \overset{\sim}{\longleftrightarrow} \binom{[n]}{k} \quad \text{bijection}$$

With these choises:

$$\boxed{\{\text{Coxeter matroids for } W \text{ and } J\} = \{\text{ordinary matroids on } [n] \text{ of rank } k\}}$$

The B's are the bases of the matroid

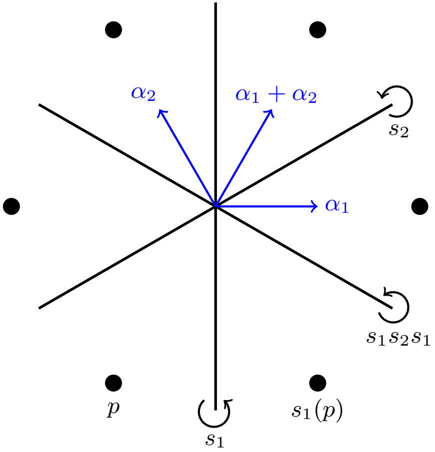
The theorem translates the definition of a Coxeter matroid into geometric terms.

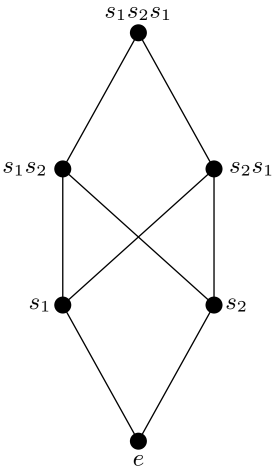
THEOREM (GELFAND–SERGANOVA). Let $\emptyset \neq \mathcal{M} \subseteq W/W_J$. TFAE

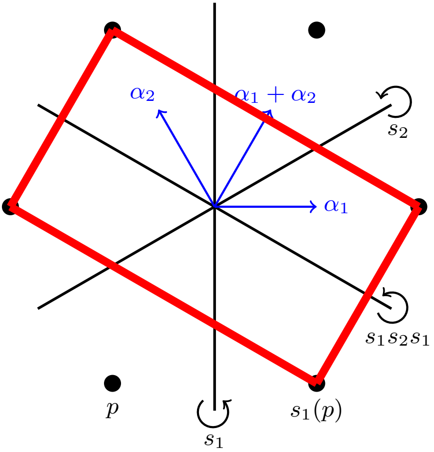
- $\mathcal{M}$ is a Coxeter matroid
- for every edge of $\Delta_{\mathcal{M}}$, there exists a reflection of W that flips that edge
- every edge of $\Delta_{\mathcal{M}}$ is parallel to a root in Φ

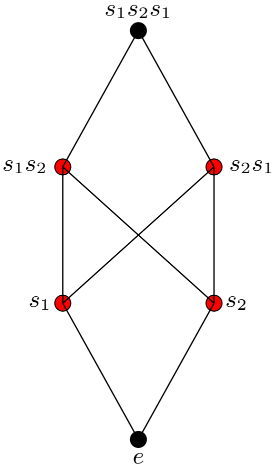
- One of the most important tool of the theory
- Geometric interpretation of Coxeter matroids as polytopes with certain symmetry property
- Surprisingly simple (although cryptomorphic) definition of a Coxeter matroid
- This is why roots play a fundamental role.

$W = S_3 \qquad J = \emptyset$

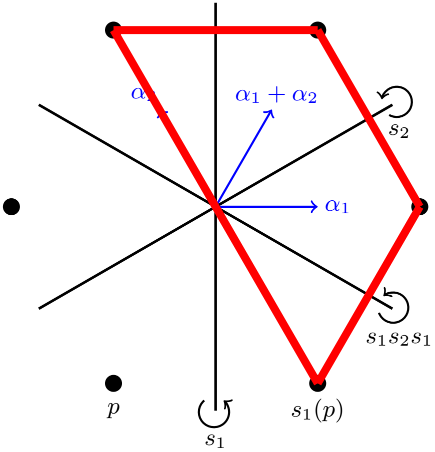


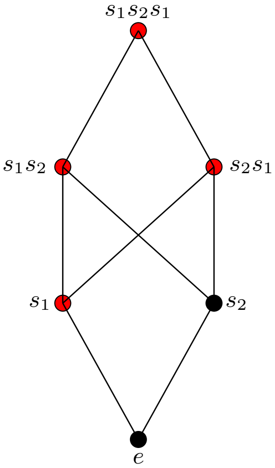






$\mathcal{M} = \{s_1, s_2, s_1 s_2, s_2 s_1\}$ is not a Coxeter matroid.





$\mathcal{M} = \{s_1, s_1 s_2, s_2 s_1, s_1 s_2 s_1\}$ is a Coxeter matroid.

Theorem (Caselli-D'Adderio-M).
Let (W, S) be a finite Coxeter system. For all $J \subseteq S$ and all $x, y \in W^J$ with $x \leq y$, the parabolic Bruhat interval

$$\{z \in W^J : x \leq z \leq y\}$$

is a Coxeter matroid.

- In 2015, Kodama and Williams prove the theorem for W of type A and $J = \emptyset$.

IDEA OF THE PROOF OF THE THEOREM FOR $J = \emptyset$

Let $\mathcal{M}$ be a Bruhat interval:

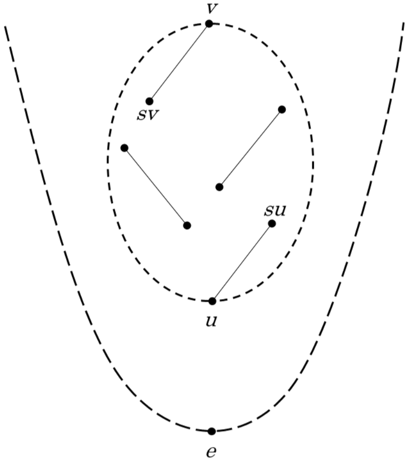
$$\mathcal{M} = [x, y] = \{z \in W : x \leq z \leq y\}$$

$\Delta_{\mathcal{M}}$ is the **Bruhat interval polytope** corresponding to $\mathcal{M}$

To prove the theorem, we

- translate the problem into geometric terms using Gelfand–Serganova Theorem
- need to prove that the edges of $\Delta_{\mathcal{M}}$ are parallel to roots
- study actually all faces of $\Delta_{\mathcal{M}}$
- use several algebro-combinatorial tools in the theory of Coxeter groups
- use a new tool: the Weak Generalized Lifing Property

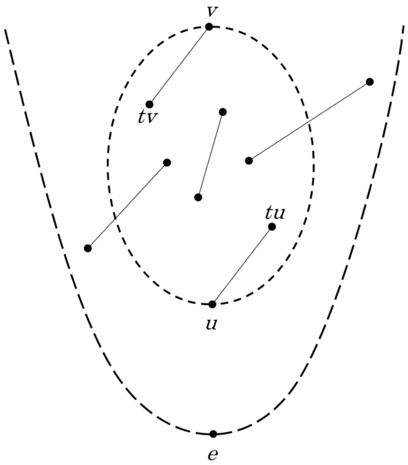
CLASSICAL LIFTING PROPERTY (VERMA). Let $u, v \in W$ with $u < v$ and $s \in S$. If $u \lhd su$ and $sv \lhd v$, then $su \leq v$ and $u \leq sv$



PROS. Characterizes Bruhat order. Has many consequences: e.g. the interval is closed under multiplication by s

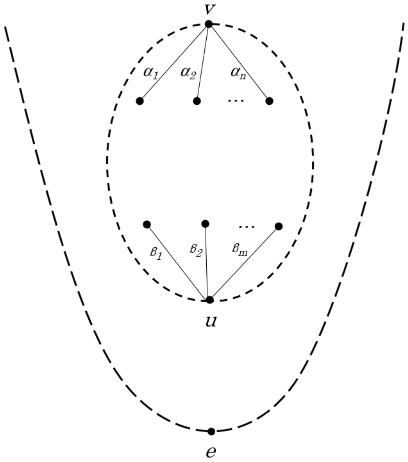
CONS. For some $u, v \in W$, there are no such $s \in S$.

Generalized Lifting Property For all $u, v \in W$ with $u < v$, there exists $t \in T$ s.t. $u \lhd tu \leq v$ and $u \leq tv \lhd v$



Pros. Existence of such t. It holds for $W = S_n$ (Tsukerman–Williams '15) and, more generally, for W simply laced (Caselli–Sentinelli '17)
Cons. It doesn't hold for W not simply laced (Caselli–Sentinelli '17)

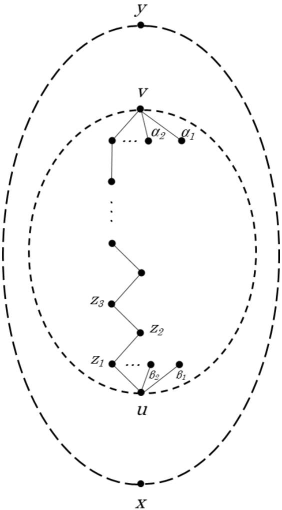
Weak Generalized Lifting Property (C-D-M) Given $u, v \in W$ with $u < v$, let $R_v = \{\alpha_t \in \Phi^+ : u \leq tv \lhd v\}$ and $R_u = \{\beta_r \in \Phi^+ : u \lhd rv \leq v\}$. Then $\mathrm{Cone}(R_v) \cap \mathrm{Cone}(R_u) \neq \{0\}$.



Cons. It is "weak"

Pros. It holds for all (finite and infinite) Coxeter systems

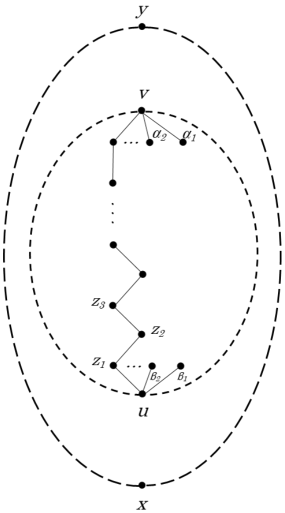
LEMMA. Let F be a face of $\Delta_{[x,y]}(p)$. If F contains $u(p)$ and $v(p)$ for some subinterval $[u, v]$, then there exists a complete chain C from u to v such that $z(p) \in F$ for all $z \in C$.



- By induction on $\ell(v) - \ell(u)$.
- Let $f \in V^*$ be such that $f = c$ is the hyperplane containing F, and $f < c$ is the halfspace containing $\Delta_{[x,y]}(p) \setminus F$.
- By the Weak Generalized Lifting Property

$$\sum_{i \in I} b_i \beta_{r_i} = \sum_{j \in J} a_j \alpha_{t_j} \neq 0$$

with $u \lhd r_i u \leq v$, $u \leq t_j v \lhd v$, and $a_j, b_i > 0$



- Recall
  $r_i(u(p)) = u(p) + c_i\beta_{r_i}$
  $t_j(v(p)) = v(p) - d_j\alpha_{t_j}$
  with $c_i, d_j > 0$
- Since all points $r_i(u(p))$ and $t_j(v(p))$ belong to $\Delta_{[x,y]}(p)$

  $$f(\beta_{r_i}) \leq 0 \qquad f(\alpha_{t_j}) \geq 0$$

  Thus $f(\sum b_i\beta_{r_i}) = f(\sum a_j\alpha_{t_j}) = 0$, and

  $$f(\beta_{r_i}) = f(\alpha_{t_j}) = 0$$

  therefore, all points $r_iu(p)$ and $t_jv(p)$ lie in F.
- By the induction hypothesis, there is a complete chain C' from r_1u to v such that $z(p) \in F$ for all $z \in C'$. Take the chain $C = C' \cup \{u\}$.

GRAZIE!