Gradient estimates for boundary blow-up solutions and applications

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A huge literature has concerned the study of boundary blow-up solutions (also called large-solutions) of elliptic equations like

\[
\begin{aligned}
-\Delta u + g(u) &= f(x) \quad \text{in } \Omega, \\
u(x) &\to +\infty \quad \text{as} \quad d(x) \to 0 \quad [d(x) := \text{dist}(x, \partial \Omega)]
\end{aligned}
\]

since the works of J. Keller, R. Osserman, who proved that a solution exists if and only if

\[
\int_{-\infty}^{+\infty} \frac{1}{\sqrt{G(s)}} \, ds < \infty, \quad G(s) = \int_{0}^{s} g(t) \, dt
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*Keller-Osserman condition*
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Keller-Osserman condition

Fundamental problems: existence, asymptotic behavior and uniqueness
[Impossible here to recall all contributors, let us mention Loewner, Nirenberg, Bandle, Marcus, Véron, Lazer, McKenna, Lair, Wood, G. Diaz, Letelier, J. López-Gómez, Cirstea, Radulescu, Zhang,...]
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**Goal of this talk**: show that **gradient estimates** lead to such qualitative results. Two examples will be discussed
New interest was raised recently on qualitative properties of solutions: multiplicity, symmetry, blow-up profile, second order terms, curvature effects [Del Pino-Letelier, Aftalion-Reichel, Aftalion-Del Pino-Letelier, Du-Guo, Du-Guo-Zhou, ...]

**Goal of this talk:** show that gradient estimates lead to such qualitative results. Two examples will be discussed

- **Radial symmetry in a ball for semilinear equations** (extension of the Gidas-Ni-Nirenberg result). Joint work with L. Véron
New interest was raised recently on qualitative properties of solutions: multiplicity, symmetry, blow-up profile, second order terms, curvature effects

**Goal of this talk**: show that gradient estimates lead to such qualitative results. Two examples will be discussed

- **Radial symmetry in a ball for semilinear equations** (extension of the Gidas-Ni-Nirenberg result). Joint work with L. Véron
- **Boundary blow-up solutions related to stochastic control problems** (viscous Hamilton-Jacobi equations). Joint work with T. Leonori (PHD at Roma Tor Vergata)
Recall the celebrated Gidas-Ni-Nirenberg result:

Let $g$ be a locally Lipschitz function. Then any $u \in C^2(\Omega)$ which is a positive solution of

$$\begin{cases}
-\Delta u + g(u) = 0 & \text{in } B_R(0), \\
u = 0 & \text{on } \partial B_R(0),
\end{cases}$$

is radially symmetric and decreasing.
Radial symmetry: Gidas-Ni-Nirenberg for large solutions

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**Remark**: Of course the same holds if $u|_{\partial \Omega} = m$ is constant and $u \leq m$ inside $\Omega$.

A natural question is: if $g$ also satisfies the Keller-Osserman condition at infinity, does a similar result holds for boundary blow-up solutions?

(Answer is not trivial: to what extent $u = +\infty$ is constant tangentially?...)
Recall the key points in the GNN approach (as well as in many later symmetry results)

- Hopf boundary lemma
- Moving plane method: compare $u$ with its reflection
Recall the key points in the GNN approach (as well as in many later symmetry results)

- Hopf boundary lemma
- moving plane method: compare $u$ with its reflection

Comparing $u$ with its reflection is not easy when solutions blow-up at the boundary:
- how the difference $u - u_{\lambda}$ behaves near the corner points?
- how can we replace the information of Hopf lemma?
With L. Véron, we adopt the following strategy:

(i) we prove that the Gidas-Ni-Nirenberg argument works for boundary blow-up solutions provided one knows that the normal gradient is dominant:

\[
\begin{align*}
\lim_{|x| \to R} \frac{\partial u}{\partial \nu} &= \infty \\
\frac{\partial u}{\partial \tau} &= o \left( \frac{\partial u}{\partial \nu} \right) \quad \text{as } |x| \to R,
\end{align*}
\]

where \( \frac{\partial u}{\partial \nu} \) is the normal derivative and \( \frac{\partial u}{\partial \tau} \) is any tangential derivative of \( u \).
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In some sense we use (1) as a version of Hopf lemma for boundary blow-up solutions.
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In some sense we use (1) as a version of Hopf lemma for boundary blow-up solutions

(ii) we turn our attention to conditions under which (1) can be proved to hold true.
Theorem (Porretta-Véron, J. Functional Anal. '06)

Let $g$ be a locally Lipschitz continuous function. Assume that

(i) Exists a $a > 0$ such that $g$ is positive and convex on $[a, \infty)$

(ii) $g$ satisfies the Keller-Osserman condition at infinity.

Then any $u \in C^2(\Omega)$ solution of

$$\begin{cases} 
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Rmk: The result allows to characterize all solutions in several situations where uniqueness fails:
Ex: Changing sign \( g \), like \( g(u) = u(u-a)(u-1) \) [Aftalion-Reichel, Aftalion-Del Pino-Letelier ’03]; \( g(u) = u^2 \) [Pohozaev ’61]
Some comments:

- Partial results were previously proved by McKenna-Reichel-Walter [Nolin. Anal. '97] by using *second order expansion* of solutions. However, that approach requires stronger assumptions on $g$: indeed,

proving second order expansion for $u \Rightarrow$ proving first order for $\nabla u$
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- Partial results were previously proved by McKenna-Reichel-Walter [Nolin. Anal. '97] by using second order expansion of solutions. However, that approach requires stronger assumptions on $g$: indeed,

  proving second order expansion for $u \Rightarrow$ proving first order for $\nabla u$

- We use the assumption that $g(s)$ is “convex at infinity” in order to prove the estimates for derivatives. i.e. $\frac{\partial u}{\partial \tau} = o \left( \frac{\partial u}{\partial \nu} \right)$.

This assumption is satisfied by any “reasonable” example of function enjoying the Keller-Osserman condition (recall that K-O condition $\Rightarrow$ superlinearity at infinity). However, the most general result (assuming only K-O condition) is open.
This is a special case of a general problem: in a smooth domain $\Omega$, prove that boundary blow-up solutions of

$$\begin{cases} -\Delta u + H(u, \nabla u) = f(x) \quad \text{in } \Omega , \\ u(x) \to +\infty \quad \text{as} \quad d(x) \to 0 \end{cases}$$

satisfy $\frac{\partial u}{\partial \tau} = o \left( \frac{\partial u}{\partial \nu} \right)$. 

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Many situations can be dealt with using asymptotic estimates and blow-up arguments [Bandle-Essen, Bandle-Marcus, Porretta-Véron]

If one can prove that $u(x) \sim \psi(d(x))$ where $\psi$ satisfies the associated ODE

$$
\begin{cases}
\psi'' = H(\psi, \psi') , \\
\psi(0) = +\infty
\end{cases}
$$

then the strategy is:

scaling and blow–up near a point $x_0 \in \partial \Omega$: $u_\delta = \psi(\delta) u(x_0 + \delta \xi)$

elliptic $W^{2,p}$–estimates on $u_\delta$ $\Rightarrow C^1$–compactness

$\Rightarrow \nabla u \sim \psi'(d(x)) \nabla d(x) = -\psi'(d) \nu$

(Related topics: symmetry/uniqueness results in half spaces)
We consider now the problem

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\begin{aligned}
-\Delta u + u + |\nabla u|^q &= f(x) \quad \text{in } \Omega, \\
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Boundary blow-up in viscous Hamilton-Jacobi equations

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\(\Omega\) is a bounded smooth subset in \(\mathbb{R}^N\), \(f\) is (at least) bounded

\(1 < q \leq 2\)
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  (this range is necessary: no such solutions if $q > 2$ or $q \leq 1$)
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  (this range is necessary: no such solutions if $q > 2$ or $q \leq 1$)

Motivation & origin of this model is a state constraint problem for the Brownian motion


“constraining a Brownian motion in a given domain by controlling its drift”
Given a Brownian motion $B_t$ and the SDE

\[
\begin{cases}
    dX_t = a(X_t) dt + \sqrt{2} dB_t, \\
    X_0 = x \in \Omega,
\end{cases}
\]

find an optimal feedback control $a \in C(\Omega)$ such that $X_t$ does never leave the domain $\Omega$. 
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$$a \in \mathcal{A} = \{ a \in C(\Omega) : X_t \in \Omega, \forall t > 0 \text{ a.s.} \}$$
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**Rmk:** in order to constrain a diffusion one needs **vector fields** $a(x)$ which blow-up at $\partial \Omega$. 
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Rmk: in order to constrain a diffusion one needs vector fields $a(x)$ which blow-up at $\partial \Omega$.

Given the cost functional

\[J(x, a) = E \int_0^\infty \left\{ f(X_t) + \gamma_q |a(X_t)|^{q'} \right\} e^{-t} dt\]

where $q' = \frac{q}{q - 1}$, then the value function

\[u(x) = \inf_{a \in \mathcal{A}} J(x, a),\]

is a solution of (2) if $1 < q \leq 2$ (dynamic programming principle).
Theorem (JM. Lasry-PL. Lions)

Let $1 < q \leq 2$. Then the value function $u$ is the unique solution (in $W^{2,p}_{1\text{oc}}(\Omega)$ for every $p < \infty$) of

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\begin{aligned}
-\Delta u + u + |\nabla u|^q &= f(x) \quad \text{in } \Omega, \\
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and

\[a(x) = -q|\nabla u(x)|^{q-2}\nabla u(x)\]

is the unique optimal control law.
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Moreover $u$ satisfies, as $d(x) \to 0$,

\[
\begin{cases}
u(x) \sim C_q d(x)^{-\frac{2-q}{q-1}} & \text{if } 1 < q < 2, \\
u(x) \sim -\log(d(x)) & \text{if } q = 2,
\end{cases}
\]

where $C_q$ is a universal constant, $(C_q = \frac{(q-1)^{-\frac{2-q}{q-1}}}{2-q})$. 
After [LL], one knows that the constrained dynamics

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\begin{cases}
    dX_t = a(X_t)dt + \sqrt{2} dB_t, \\
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is determined by the unique optimal control

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a(X_t) = -q |\nabla u(X_t)|^{q-2} \nabla u(X_t)
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where \( u \) is the boundary blow-up solution of the viscous Hamilton-Jacobi equation

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Next goal: study the qualitative behavior (near the boundary) of \( \nabla u \) to understand the control mechanism
First order asymptotics of the gradient

As a particular case of results in [Porretta-Véron, Adv. Nonlin. Stud. ’06] we have:

$$\lim_{x \to \partial \Omega} d(x) \frac{1}{q-1} \nabla u(x) = \tilde{c}_q \nu(x)$$

where $\nu(x)$ is the outward unit normal on $\partial \Omega$, and $\tilde{c}_q = (q-1)^{-\frac{1}{q-1}}$. 
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$$\frac{\partial u}{\partial \nu} \sim \frac{\tilde{c}_q}{d(x)^{\frac{1}{q-1}}} \quad \text{and} \quad \frac{\partial u}{\partial \tau} = o \left( \frac{\partial u}{\partial \nu} \right).$$
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As before, this is the scaling of the asymptotics of $u$: set $\alpha = \frac{2-q}{q-1}$

$$\begin{cases} 
\text{if } 1 < q < 2, \quad u \sim C_q d(x)^{-\alpha} \quad \rightarrow \quad \nabla u \sim -\tilde{c}_q C_q \alpha d(x)^{-(\alpha+1)} \nabla d(x) \\
\text{if } q = 2, \quad u \sim -\log(d(x)) \quad \rightarrow \quad \nabla u \sim -\frac{1}{d(x)} \nabla d(x)
\end{cases}$$

(note that $\alpha + 1 = \frac{1}{q-1}$, $\tilde{c}_q = C_q \frac{2-q}{q-1}$ and $\nabla d(x) = -\nu$)
We recover the typical result: the first order behavior of $u$ and $\nabla u$ is independent of $\Omega$ and is described by the associated ODE

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Recently, for the equation $\Delta u = u^p$, [Del Pino-Letelier '02], [Bandle-Marcus '05] showed that the influence of the domain in the blow–up appears in second order terms (with curvature effects). Proof is through sub–super solutions which provide a detailed (second order) expansion of $u$. 
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Natural question for our model is: how the feedback control process depends on the geometry of domain?

To get an answer:

▶ Give a precise description of the blow–up of $\nabla u$
  (role of normal and tangential components, second order terms...)

A. Porretta
Gradient estimates for blow-up solutions
Second order terms: curvature effects

Theorem (Leonori–Porretta SIAM J. Math. Anal. '07)

Let $Ω$ be a smooth bounded open subset of $\mathbb{R}^N$, and let $u$ be the unique solution of (2).
Second order terms: curvature effects


Let $\Omega$ be a smooth bounded open subset of $\mathbb{R}^N$, and let $u$ be the unique solution of (2).
Set $\overline{x}$ the projection of $x$ onto $\partial \Omega$ and by $H(\overline{x})$ the mean curvature of $\partial \Omega$ computed at $\overline{x}$. 
Second order terms: curvature effects


Let $\Omega$ be a smooth bounded open subset of $\mathbb{R}^N$, and let $u$ be the unique solution of (2).

Set $\overline{x}$ the projection of $x$ onto $\partial \Omega$ and by $H(\overline{x})$ the mean curvature of $\partial \Omega$ computed at $\overline{x}$.

Being $\nu$ and $\tau$ the normal and tangent vectors, we have, as $d(x) \to 0$,

$$\frac{\partial u}{\partial \nu} = \frac{\tilde{c}_q}{d(x)^{\frac{1}{q-1}}} \left[ 1 + \frac{(N-1)}{2} H(\overline{x}) d(x) + o(d(x)) \right], \quad \forall 1 < q \leq 2,$$
Second order terms: curvature effects

Let $\Omega$ be a smooth bounded open subset of $\mathbb{R}^N$, and let $u$ be the unique solution of (2).
Set $\overline{x}$ the projection of $x$ onto $\partial \Omega$ and by $H(\overline{x})$ the mean curvature of $\partial \Omega$ computed at $\overline{x}$.
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and

$$\begin{cases}
\frac{\partial u}{\partial \tau} \in L^\infty(\Omega) & \text{if } \frac{3}{2} < q \leq 2, \\
\frac{\partial u}{\partial \tau} = O(|\log d|) & \text{if } q = \frac{3}{2}, \\
\frac{\partial u}{\partial \tau} = O\left(\frac{1}{d^{\frac{3-2q}{q-1}}}\right) & \text{if } 1 < q < \frac{3}{2}.
\end{cases}$$
Corollary (Representation of the optimal control)

Let \( a(x) = -q|\nabla u(x)|^{q-2}\nabla u(x) \) be the optimal control for the state constraint problem.

As \( d(x) \to 0 \), we have: for any \( 1 < q < 2 \)

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a(x) = -\left[ \frac{q'}{d(x)} + \frac{q'(N-1)}{2} H(\vec{x}) \right] \nu(x) + o(1)
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For \( q = 2 \) we have

\[
a(x) = - \left[ \frac{2}{d(x)} + (N-1) H(\varphi) + o(1) \right] \nu(x) + \psi(x) \tau(x)
\]

where \( \psi \in L^\infty(\Omega) \).
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a(x) = - \left[ \frac{2}{d(x)} + (N-1) H(\overline{x}) + o(1) \right] \nu(x) + \psi(x) \tau(x)
\]

where \( \psi \in L^\infty(\Omega) \).

Note in particular:

(i) The control tangentially is zero on \( \partial\Omega \) if \( q \neq 2 \), bounded if \( q = 2 \).
Corollary (Representation of the optimal control)

Let \( a(x) = -q|\nabla u(x)|^{q-2}\nabla u(x) \) be the optimal control for the state constraint problem.

As \( d(x) \to 0 \), we have: for any \( 1 < q < 2 \)

\[
a(x) = -\left[ \frac{q'}{d(x)} + \frac{q'(N-1)}{2} H(\bar{x}) \right] \nu(x) + o(1)
\]

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Note in particular:

(i) The control tangentially is zero on \( \partial \Omega \) if \( q \neq 2 \), bounded if \( q = 2 \).

(ii) On the hypersurfaces parallel to \( \partial \Omega \), the control is maximum where the domain has a maximal mean curvature.
The “constrained dynamics”

Near the boundary, the dynamics looks like

\[
\begin{aligned}
dX_t &= \left[ \frac{q'}{d(x_t)} + \frac{q'(N-1)}{2} H(x_t) \right] \nabla d(x_t) dt + \sqrt{2} dB_t , \\
X_0 &= x \in \Omega ,
\end{aligned}
\]

The control (i.e. the drift) has to be stronger where the domain is more curved.
Method of proof: asymptotic expansion of the gradient

Remarks (with respect to first order asymptotics):

- The second order expansion of the gradient cannot be obtained here just using sub–super solutions nor “rescaling from the expansion of \( u \)

(it may happen that \( u - C_q d(x)^{-\alpha} \) has a non trivial trace on \( \partial \Omega \), the second order behavior of \( u \) cannot be determined)
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Our approach relies instead on a regularity result, and we obtain the previous statements by proving a complete asymptotic expansion for \( \nabla u \) with respect to \( d(x) \):

- introduce a formal asymptotic expansion \( S \)
- prove directly that \( u - S \) is Lipschitz
  (without knowing the boundary value of \( u - S \)): this is possible thanks to a priori estimates and approximation with Neumann-type boundary condition
It sounds similar to a *corrector result*:
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$$ u(x) \sim C_q d(x)^{-\alpha} $$

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Then we introduce as a *corrector*

$$S = d(x)^{-\alpha} \sum_{k=0}^{m_\alpha} \sigma_k(x)d(x)^k$$

and look for a result of the type

$$u - S \text{ is Lipschitz in } \Omega.$$
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Of course one has that $\sigma_0 = C_q$ is known, and $\sigma_k$, $k = 1, \ldots, m$ are smooth functions to be determined.
Indeed, we will prove that there exists a unique choice of the functions $\sigma_k$ such that

$$u - S$$

is Lipschitz

where $S = d(x)^{-\alpha} \sum_{k=0}^{m\alpha} \sigma_k(x)d(x)^k$. 

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Gradient estimates for blow-up solutions
Indeed, we will prove that there exists a unique choice of the functions $\sigma_k$ such that

$$u - S \quad \text{is Lipschitz}$$

where $S = d(x)^{-\alpha} \sum_{k=0}^{m_{\alpha}} \sigma_k(x)d(x)^k$.

The coefficients $\sigma_k$ can be explicitly computed, hence we deduce all singular terms of the expansion, since

$$\nabla u - \nabla S \in L^\infty$$
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The coefficients $\sigma_k$ can be explicitly computed, hence we deduce all singular terms of the expansion, since

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In particular, the computation of $\sigma_k$ gives

$$\sigma_1(x) = \frac{\tilde{c}_q}{1 - \alpha} \frac{\Delta d(x)}{2},$$

hence the mean curvature in second order terms

$$(\Delta d(x))_{\partial \Omega} = -(N - 1)H(x))$$
Key point: Lipschitz estimates on the reduced (“linearized”) equation.
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(a) Take \( S = d(x)^{-\alpha} \sum_{k=0}^{m} \sigma_k(x)d(x)^k \) and look at the equation satisfied by \( z = u - S \)

Using the first order behavior \[ \frac{\partial u}{\partial \tau} = o \left( \frac{\partial u}{\partial \nu} \right) \] and an asymptotic development near the boundary the equation for \( z \) looks like

\[-\Delta z + z - \frac{\alpha+2}{d(x)} \nabla z \nabla d(x) + O(d^\alpha |\nabla z|^2) = f(x) + g(x),\]

\[ g = \Delta S - S - |\nabla S|^q \]
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(b) Using Bernstein's method we get estimates for $|\nabla z|^2$ depending on the regularity of $f$ and $g$. Next two ingredients:
(i) Choose the coefficients $\sigma_k(x)$ of $S$ in a way that $g$ is smooth (this gives a unique choice of the corrector $S$)
(ii) In order to get global Lipschitz estimates in $\Omega$, we approximate $u - S$ with solutions satisfying Neumann boundary conditions.
Comments, extensions, work in progress
The result extends to inhomogeneous diffusions

\[
\begin{cases}
    dX_t = a(X_t)dt + \sqrt{2} \sigma(X_t)dB_t, \\
    X_0 = x \in \Omega,
\end{cases}
\]

with associated HJB equation

\[-\text{tr} (A(x)D^2 u) + \lambda u + |\nabla u|^q = f(x)\]

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where \( A(x) = \sigma(x)\sigma^T(x) \).
Assuming \( A(x) \) elliptic and smooth, one can use the same approach
replacing the distance function \( d(x) \) with the solution of the first
order equation
\[
\begin{cases}
    A(x)\nabla \rho \nabla \rho = \gamma |\nabla \rho|^q \quad \text{in } \Omega \\
    \rho > 0 , \\
    \rho = 0 \quad \text{on } \partial \Omega .
\end{cases}
\]
Things to be done (or in progress)...

- Existence/blow-up of explosive solutions in singular domains
  (link with Wiener criteria for the Brownian motion)
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- **Existence/blow–up of explosive solutions in singular domains**
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- **general diffusions**, possibly non smooth and/or degenerate. ⇒
  approach by viscosity solutions
  (cfr. degenerate state constraint problems [Katsoulakis],
  [Ishii–Loreti], [Barles-Burdeau, Barles-Rouy, B-R-Souganidis]...)

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Gradient estimates for blow-up solutions