Controllability of Fokker-Planck equations and the planning problem for mean field games

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Outlines of the talk

- Brief description of the Mean Field Games model.
  Coupling viscous Hamilton-Jacobi & Fokker-Planck equations.
- The planning problem: a question of optimal transport & optimal control.
- Construction of the solution as limit of standard mean field games problems (from optimal control to exact controllability).
- Uniqueness of weak solutions
- Comments, work in progress and perspectives.

Mean Field Games

The Mean Field Games model was introduced by Lasry-Lions since 2006. A similar paradygm independently developed by Huang-Caines-Malhamé in terms of stochastic dynamics.

Main goal: *describe games with large numbers (a continuum) of agents whose strategies depend on the distribution of the other agents.*

Typical features of the model:

- players act according to the same principles (they are indistinguishable and have the same optimization criteria).

- players have individually a minor (infinitesimal) influence, but their strategy takes into account the mass of co-players.

Roughly: players are particles but have strategies (more sophisticated than particle physics or similar economics models); however, they only consider the statistical state of the mass of co-players (less sophisticated than agents of N-players games)

Goal: introduce a macroscopic description through a mean field approach as the number of players $N \to \infty$.

The simplest form of the continuum limit is a coupled system of PDEs

$$\begin{cases} (1) \qquad -u_t - \Delta u + H(x, Du) = F(x, m) & \text{in } (0, T) \times \Omega \\ (2) \qquad m_t - \Delta m - \mathrm{div}(m\, H_p(x, Du)) = 0 & \text{in } (0, T) \times \Omega\,, \end{cases}$$

where H_p stands for $\frac{\partial H(x,p)}{\partial p}$.

- (1) is the Bellman equation for the agents' value function u.
- (2) is the Kolmogorov-Fokker-Planck equation for the state of the agents. $m(t)$ is the probability density of the state of players at time t.

Roughly, each agent controls the dynamics of a N-d Brownian motion

$$dX_t = \beta_t dt + \sqrt{2} dB_t \,,$$

in order to minimize, among controls β_t, some cost:

$$\inf J(\beta) := \mathbb{E}\left\{ \int_0^T [L(X_s, \beta_s) + F(X_s, m(s))]ds + G(X_T, m(T)) \right\}$$

where $m(t)$ is the probability measure in $\mathbb{R}^N$ induced by the law of the process X_t.

The associated Hamilton-Jacobi-Bellman equation is

$$-u_t - \Delta u + H(x, Du) = F(x, m(t))$$

where $H = \sup_\beta [-\beta \cdot p - L(x, \beta)]$.

If u solves the Bellman equation it gives

• the best value $\inf_\beta J(\beta) = \int u(x,0) dm_0(x)$,
where m_0 is the probability distribution of X_0.

• the optimal feedback $\beta_t^* = b(t, X_t)$, where $b(t,x) = -H_p(x, Du(t,x))$

Recall: given a drift-diffusion process

$$dX_t = b(t, X_t)dt + \sqrt{2}dB_t$$

the probability measure $m(t)$ (distribution law of X_t) satisfies

$$m_t - \Delta m + \text{ div } (bm) = 0$$

in a weak sense

$$\int_\Omega \varphi(t,x)m(t,x)dxdt + \int_0^t \int_\Omega m(\tau,x)L^*\varphi \, dxd\tau = \int_\Omega \varphi(0)\, m_0$$
$$\forall \varphi \in C^2 \,, \; \forall t > 0$$

where $L^* := \partial_t - \Delta + b \cdot D$ and $m(0) =$ initial distribution of X_0.
[just use Ito's formula since $\left[\mathbb{E} \int_0^T \varphi(X_t)dt\right] = \int_\Omega \int_0^T \varphi(x)m(t,x)\, dt$

Hence, the evolution of the state of the agents is governed by their optimal decisions $b_t^* = -H_p(\cdot, Du(\cdot))$:

$$m_t - \Delta m - \text{ div } (m\, H_p(x, Du)) = 0$$

This is the Mean Field Games system (with horizon T):

$$\begin{cases} (1) \qquad -u_t - \Delta u + H(x, Du) = F(x, m) & \text{in } (0,T) \times \Omega \\ (2) \qquad m_t - \Delta m - \operatorname{div}(m\, H_p(x, Du)) = 0 & \text{in } (0,T) \times \Omega\,, \end{cases}$$

usually complemented with initial-terminal conditions:

-$m(0) = m_0$ (initial distribution of the agents)

-$u(T) = G(x, m(T))$ (final pay-off)

$+$ boundary conditions (here for simplicity periodic setting)

Main novelties are:

- the backward-forward structure.

  Existence is related to *economic equilibrium with rational anticipations*.

- the interaction in the strategy process: the coupling $F(x, m)$

Rmk: This is not the most general structure.

Cost criterion $L(X_t, \alpha_t, m(t)) \to H(x, m, Du)$.

Some known results on MFG system:

- Different type of existence results (smooth solutions, or weak solutions, depending on the growth of the coupling term $F(x, m)$)
- Uniqueness conditions: $m \mapsto F(x, m)$ increasing together with $p \mapsto H(x, p)$ convex, imply uniqueness,stability ([Lasry-Lions]).
- Numerical approximations ([Achdou-Capuzzo Dolcetta]). Analysis of discrete time, finite states models ([Gomes-Mohr-Souza]).
- Behavior in the long horizon: $T \to \infty$ and convergence to an ergodic invariant measure ([Cardaliaguet-Lasry-Lions-Porretta]).
- Derivation of the system as limit of Nash equilibria of N-players games, as $N \to \infty$ (where m is replaced by the empirical density $\frac{1}{N-1} \sum_{j \neq i} \delta_{X_s^j}$)). Rates of convergence.

Several particular models of MFG, often 1-d, were considered for specific applied problems (Lachapelle, Turinici, Carlier, Tembine, Gueant, Huang, Caines, Malhame, Markovich,....)

- Interpretation in terms of optimal control problems (primal-dual state system arising from optimality conditions).

Consider the state equation as a controlled FP:

$$m_t = \Delta m + \operatorname{div}(\alpha\, m)\,, \qquad m(0) = m_0$$

and optimize w.r.t. α

$$\inf_{\alpha} \int_0^T \int_\Omega \{L(x,\alpha)m + \Phi(m(s))\}\, dt + \mathcal{G}(m(T))$$

where $\Phi'(m) = F(m)$ and $\mathcal{G}'(m) = G(m)$. Optimality first order condition gives

$$\alpha_{opt} = -H_p(x, \nabla u(t,x))$$

where u is the solution of the adjoint equation.

(Note: monotonicity of the coupling functions $F(m)$ and $G(m)$ gives convexity for the optimization problem)

The planning problem

Pb: Can we prescribe the terminal distribution of the agents $m(T)$?

Suggested by P.L. Lions, this is an optimal planning problem:
can we drive in time T the density of the agents from the initial configuration m_0 to a target configuration m_1 in a way which is optimal for the agents' cost ?

In this case, the MFG system

$$\begin{cases} -u_t - \Delta u + H(x, Du) = F(x, m) & \text{in } (0, T) \times \Omega \\ m_t - \Delta m - \mathrm{div}(m\, H_p(x, Du)) = 0 & \text{in } (0, T) \times \Omega\,, \end{cases}$$

is complemented with initial and terminal conditions for the density m:

$$m(0) = m_0\,, \qquad m(T) = m_1\,.$$

The final pay-off $u(T)$ is not prescribed here; it is a degree of freedom to be used to reach the target.
For simplicity, assume periodic setting (neglect boundary conditions)

This can be seen as an optimal transport problem for the probability densities of the process law

$$\begin{cases} dX_t = \alpha(X_t)dt + \sqrt{2}\, dB_t \\ m_0 = \mathcal{L}(X_0)\,, \quad m_1 = \mathcal{L}(X_T) \end{cases}$$

minimizing

$$\mathbb{E}\left[\int_0^T L(X_t, \alpha(X_t))\, dt + \int_0^T F(X_t, m(X_t))\, dt\right]$$

among admissible feedback $\alpha(\cdot)$.

Alternatively, the pb. can be seen as an exact controllability problem for the Fokker-Planck equation (+ optimization among admissible drifts α):

Pb: Minimize

$$\min_{\alpha \in L^2(m\, dxdt)} \int_0^T \int_\Omega \{L(x,\alpha)\, m + \Phi(x,m)\}\ dxdt\,,$$

among the α which exactly control the FP equation:

$$\begin{cases} m_t - \Delta m - \text{ div } (\alpha\, m) = 0 \\ m(0) = m_0\,, m(T) = m_1 \end{cases}$$

- The above problem can be seen as a *second order version* of the Monge-Kantorovich mass transfer problem in the formulation of [Benamou-Brennier]

The model case $H(x, Du) = \frac{1}{2}|Du|^2$ was solved by P.L. Lions:

-$F(m)$ nondecreasing and bounded, m_0, m_1 smooth and positive:

$$m_0, m_1 \in C^1(\overline{\Omega}), \qquad m_0, m_1 > 0, \qquad \int_\Omega m_0\, dx = \int_\Omega m_1\, dx = 1$$

Then, there exists a smooth solution (u, m) to

$$\begin{cases} -u_t - \Delta u + \frac{1}{2}|Du|^2 = F(m) & \text{in } (0, T) \times \Omega \\ m_t - \Delta m - \mathrm{div}(m\, Du) = 0 & \text{in } (0, T) \times \Omega\,, \\ m(0) = m_0\,, \quad m(T) = m_1\,, \\ +\text{periodic b.c.} \end{cases}$$

Moreover, the smooth solution is unique (up to adding a constant to u).

Lions' proof relies on a change of unknown, including the Hopf-Cole transform, which reduces the pb. to a semilinear system:

$$\varphi := e^{-\frac{1}{2}\,u}\,; \qquad \psi := m\, e^{\frac{1}{2}\,u}$$

$$\Rightarrow \begin{cases} -\varphi_t - \Delta\varphi + \frac{1}{2}F(x, \varphi\,\psi)\varphi = 0 \\ \psi_t - \Delta\psi + \frac{1}{2}F(x, \varphi\,\psi)\psi = 0 \end{cases}$$

In the uncoupled case $F = F(x)$, this result was also indirectly given by A. Blaquiere in '92. He noticed that the exact controllability of FP:

$$\begin{cases} m_t - \Delta m - \operatorname{div}(\alpha\, m) = 0 \\ m(0) = m_0\,, m(T) = m_1 \end{cases}$$

could be constructed through products: $m = v\, w$, $\alpha = D \log v$, where

$$\begin{cases} v_t - \Delta v + F(x) v = 0 \\ v(0) = v_0\,, \end{cases} \qquad \begin{cases} -w_t - \Delta w + F(x) w = 0 \\ w(T) = w_0\,, \end{cases}$$

provided there exists (v_0, w_0) such that

$$\begin{cases} v_0 w(0) = m_0 \\ v(T) w_0 = m_1 \end{cases} \iff \begin{cases} v_0 (e^{-T\,A} w_0) = m_0 \\ (e^{-T\,A} v_0) w_0 = m_1 \end{cases}$$

The key point was a representation thm by Beurling ('60):
given a positive density $\rho(x, y)$, $\exists$ a product measure $\sigma = \sigma(x) \otimes \sigma(y)$:

$$\nu := \rho(x, y)\, [\sigma(x) \otimes \sigma(y)]$$

has prescribed marginals $\nu(\cdot, y)$ and $\nu(x, \cdot)$

A nonlinear approach for a general result

Goal: find a solution as limit of standard mean field games problems [optimal control $\to$ exact controllability]

We assume that $H(x,p)$ satisfies, for some $\alpha, \beta, \gamma > 0$:

$$H_p(x,p) \cdot p - 2H(x,p) \geq -\gamma$$
$$|H_p(x,p)| \leq \beta\,(1 + |p|)$$
$$H(x,p) \geq \alpha |p|^2 - \gamma$$

Model case: inhomogeneous quadratic-like growths, ex. $H = b(x)\nabla u + c(x)|Du|^2$

We also assume that
-$m \mapsto F(x,m)$ is nondecreasing (with any possible growth at infinity)
-$p \mapsto H(x,p)$ is convex

We still suppose to be in a periodic setting (e.g. dynamics is on a torus)

Theorem (P.)

Let $m_0, m_1 \in C^1(\overline{\Omega})$, $m_0, m_1 > 0$, $\int_\Omega m_0\, dx = \int_\Omega m_1\, dx = 1$.
Under the above assumptions, there exists a weak solution (u, m) *to the planning problem*

$$\begin{cases} -u_t - \Delta u + H(x, Du) = F(x, m) & \text{in } (0, T) \times \Omega \\ m_t - \Delta m - \operatorname{div}(m\, H_p(x, Du)) = 0 & \text{in } (0, T) \times \Omega\,, \\ m(0) = m_0\,, \quad m(T) = m_1 \end{cases}$$

If in addition $H(x, \cdot)$ *is strictly convex, then the weak solution* (u, m) *is unique (up to adding a constant to* u*).*

MB: By weak solutions, we mean distributional solutions + the extra info that $m\,|Du|^2 \in Ł^1$.

This is very natural (in the optimization viewpoint) and crucial for uniqueness.

Comments & remarks:

- Compared with P.-L. Lions' result, we obtain only weak solutions (smoothness ? open problem). Nevertheless, weak solutions are unique!

- Our approach only relies on energy methods (estimates, compactness); we construct the solution from penalized optimal control problems, as it is customary for exact controllability.

- Numerical schemes were studied in [Achdou-Camilli-Capuzzo Dolcetta '12] (consistency, existence for the discretized problem).

  This was our starting motivation: find the arguments (at the continuous level) which may be used for the convergence of numerical approximations.

Existence of solution

We obtain a solution as limit of MFG problems with standard initial-terminal conditions by penalizing the terminal pay-off $u(T)$:

$$\begin{cases} -(u_\varepsilon)_t - \Delta u_\varepsilon + H(x, Du_\varepsilon) = F(x, m_\varepsilon) & \text{in } Q_T \\ (m_\varepsilon)_t - \Delta m_\varepsilon - \mathrm{div}(m_\varepsilon\, H_p(x, Du_\varepsilon)) = 0 & \text{in } Q_T \\ m_\varepsilon(0) = m_0\,, \quad u_\varepsilon(T) = \frac{m_\varepsilon(T) - m_1}{\varepsilon}\,. \end{cases}$$

This is a very natural approach: it corresponds to the penalized optimization

$$\min_\alpha \int_0^T \int_\Omega \{L(x,\alpha)\, m + \Phi(x,m)\}\, dxdt + \frac{1}{2\varepsilon} \int_\Omega |m(T) - m_1|^2\, dx\,,$$

$$\alpha \in L^2(m\, dxdt)\,, \begin{cases} m_t - \Delta m - \text{ div } (\alpha\, m) = 0 \\ m(0) = m_0\,, \end{cases}$$

Bu the limit $\varepsilon \to 0$ is quite delicate...

A priori estimates ? Compactness ?

Part I. Estimates.

We use several tools among which some structural points of MFG system:

1. The structure of Hamiltonian system:

$$\begin{cases} -u_t - \Delta u + H(x, Du) = F(x, m) \\ m_t - \Delta m - \mathrm{div}(m\, H_p(x, Du)) = 0 \end{cases} \quad \Longleftrightarrow \quad \begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial \mathcal{E}}{\partial m} \\ \dfrac{\partial m}{\partial t} = -\dfrac{\partial \mathcal{E}}{\partial u} \end{cases}$$

where

$$\mathcal{E}(u, m) = \int_\Omega [\frac{1}{2} m |\nabla u|^2 + \nabla u \cdot \nabla m - \Phi(x, m)] dx$$

In particular, the quantity $\mathcal{E}(u(t), m(t))$ is constant along the flow.

2. This gives a kind of observability inequality: any solution satisfies

$$\int_\Omega |Du(0)|^2 \, dx \le C \left\{ \int_0^T \int_\Omega m \, |Du|^2 \, dxdt + 1 \right\} \tag{1}$$

where $C = C(T, H, m_0)$.

3. Energy estimates of the system:

$$\int_\Omega (um)_t = -\int_\Omega \{H_p(x,Du)Du - H(x,Du)\}\, m - \int_\Omega F(x,m)m$$
$$\lesssim -\int_\Omega \left\{ m|Du|^2 + \Phi(x,m) \right\}\, dx$$

The right hand side is, roughly, the energy.
Same computation with $m - m_1$ yields, using $m_1 > 0$ and smooth:

$$\int_\Omega (u(m-m_1))_t = -\int_\Omega \{H_p(x,Du)Du - H(x,Du)\}\, m - \int_\Omega F(x,m)m$$
$$-\int_\Omega m_1 H(x,Du) + \int_\Omega DuDm_1\, dx - \int_\Omega F(x,m)m_1 dx$$
$$\lesssim -c\int_\Omega \left\{ m|Du|^2 + m_1|Du|^2 + F(x,m)m \right\}\, dx$$

Therefore

$$\int_0^T \int_\Omega \left\{ (m+1)|Du|^2 + F(x,m)m \right\}\, dx + \int_\Omega u(T)(m(T)-m_1)dx$$
$$\le \int_\Omega u(0)(m_0 - m_1)\, dx + c\,.$$

But last term is estimated by Poincarè-Wirtinger inequality since $m_0 - m_1$ has zero average:

$$\int_\Omega u(0)(m_0 - m_1)\, dx \le c \int_\Omega |Du(0)|\, dx$$

and the observability inequality closes the estimate.

$$\int_0^T \int_\Omega \left\{ (m+1)|Du|^2 + F(x,m)m \right\} + \int_\Omega u(T)(m(T)-m_1) \leq c \int_\Omega |Du(0)|$$

$$\int_\Omega |Du(0)|^2 \leq C \left\{ \int_0^T \int_\Omega m\,|Du|^2 + 1 \right\}$$

Conclusion: if $u(T) = \frac{1}{\varepsilon}(m_\varepsilon(T) - m_1)$, we get

$$\int_\Omega \frac{|m_\varepsilon(T) - m_1|^2}{\varepsilon}\, dx + \int_0^T \int_\Omega (m_\varepsilon + 1)|Du_\varepsilon|^2 + \int_\Omega F(x, m_\varepsilon)m_\varepsilon \leq c$$

and in turn the estimate $\|Du_\varepsilon(0)\|_{L^2} \leq c$. Using back the equation of u_ε we estimate its average, so we end up with

$$u_\varepsilon \quad \text{bounded in } L^2(0,T;H^1(\Omega)), \qquad u_\varepsilon(0) \quad \text{bounded in } H^1(\Omega).$$

Now, from the bound at $t = 0$ we are able to improve the estimate and we obtain

$$\|u_\varepsilon(t)\|_{L^2} \quad \text{bounded, uniformly in } [0,T]$$

and in particular

$$\|u_\varepsilon(T)\|_{L^2} = \frac{1}{\varepsilon}\|m_\varepsilon(T) - m_1\|_{L^2} \leq C$$

Part II. Compactness.

- First, we use the stability of MFG system. (u_1, m_1), (u_2, m_2) solutions:

$$\int_\Omega [(u_1 - u_2)(m_1 - m_2)]_t \, dx$$
$$+ \int_0^T \int_\Omega m_1 \left[H(x, Du_2) - H(x, Du_1) - H_p(Du_1)\,(Du_2 - Du_1)\right] dxdt$$
$$+ \int_0^T \int_\Omega m_2 \left[H(x, Du_1) - H(x, Du_2) - H_p(Du_2)\,(Du_1 - Du_2)\right] dxdt$$
$$+ \int_0^T \int_\Omega [F(x, m_1) - F(x, m_2)]\,[m_1 - m_2] \; dxdt = 0\,.$$

Applying this equality to $(u_\varepsilon, m_\varepsilon)$ and (u_η, m_η), integrating and using the L^2 bounds

$$\int_0^T \int_\Omega m_\varepsilon \{H(x, Du_\eta) - H(x, Du_\varepsilon) - H_p(x, Du_\varepsilon)(Du_\eta - Du_\varepsilon)\} \; dxdt$$
$$+ \int_0^T \int_\Omega m_\eta \{H(x, Du_\varepsilon) - H(x, Du_\eta) - H_p(x, Du_\eta)(Du_\varepsilon - Du_\eta)\} \; dxdt$$
$$+ \int_0^T \int_\Omega (F(x, m_\varepsilon) - F(x, m_\eta))(m_\varepsilon - m_\eta)\, dxdt \le c\,(\varepsilon + \eta)\,.$$

The monotonicity of F and convexity of H, together with some real analysis argument, lead to

$$m_\varepsilon |Du_\varepsilon|^2 \to m|Du|^2 \qquad \text{in } L^1.$$

The convergence

$$m_\varepsilon |Du_\varepsilon|^2 \to m|Du|^2 \qquad \text{in } L^1. \tag{2}$$

is enough for the KFP equation: $m_\varepsilon(t) \to m(t)$ uniformly in L^1, m solves

$$\begin{cases} m_t - \Delta m - \mathrm{div}(m\, H_p(x, Du)) = 0 & \text{in } (0,T)\times\Omega\,, \\ m(0) = m_0\,, \quad m(T) = m_1 \end{cases}$$

But the strong convergence $Du_\varepsilon \to Du$ in L^2 is needed to pass to the limit in the HJB equation.
Main pb: we have (2), but we do not have bounds from below on m_ε.

Our rough idea is: pass to the limit in the set $\{m > 0\}$

$$\int_0^T\int_\Omega H(x, Du_\varepsilon) \sim \int_0^T\int_\Omega H(x, Du_\varepsilon)\phi(m_\varepsilon) \qquad \text{with } \Phi(0) = 0,\ \Phi \text{ close to } 1$$
$$\to \int_0^T\int_\Omega H(x, Du)\phi(m) \sim \int_0^T\int_\Omega H(x, Du)$$

proving that $\{m > 0\}$ has full measure by a (uniform) *log*-estimate:

$$\|\log m\|_{L^1} \le c \qquad (\Rightarrow m > 0 \text{ a.e.})$$

We conclude $H(x, Du_\varepsilon) \to H(x, Du)$ in L^1 and we recover the HJB equation.

About uniqueness (briefly).

Uniqueness of weak solutions is highly non trivial:

1. The typical setting for well-posedness of

$$(FP) \qquad m_t - \Delta m + \operatorname{div}(m\, b) = 0 \qquad (t,x) \in (0,T)\times\Omega\,, \quad \Omega \subset \mathbb{R}^N$$

is

$$b \in L^\infty(0,T; L^N(\Omega))\,, \qquad \text{or} \quad b \in L^{N+2}((0,T)\times\Omega)$$

or in general $b \in L^r(0,T; L^q(\Omega))$ with $\frac{N}{2r} + \frac{1}{q} \le \frac{1}{2}$ ([Aronson-Serrin], [Ladysenskaya-Solonnikov-Uraltseva]).

Pb. MFGames: $b = H_p(x, Du) \simeq |Du|$ is (a priori) outside this class

2.Uniqueness may fail for unbounded solutions of HJB:

$$\exists \;\; u \in L^2(0,T; H^1_0)\,, \;\; u \neq 0 \text{ sol. of } \quad \begin{cases} u_t - \Delta u + |Du|^2 = 0 \\ u(0) = 0 \end{cases}$$

Counterexamples are constructed with log of fundamental solutions.

Conclusion: Distributional solutions are not unique.
BUT: the extra estimate $m\,|Du|^2 \in L^1$ will be enough !!!

Uniqueness stands on the following two main steps:

1. New results on weak solutions of Fokker-Planck equations.

Key-point: we can consider solutions of Fokker-Planck

$$m_t - \Delta m - \text{ div } (bm) = 0$$

such that $m \geq 0, \quad m|b|^2 \in L^1$

In this framework, we can prove:

1. Weak (=distributional) solutions of (FP) are unique in this class (see also [Bogachev-Da Prato-Röckner '11])
2. Weak solutions are renormalized solutions;

   (in the sense of [Di Perna-Lions], [Lions-Murat])

   Moreover, we show that solutions can be regularized and obtained as limit of smooth solutions.

2. A crossed regularity lemma: any two weak solutions (u_1, m_1) and (u_2, m_2) satisfy

$$F(m_i)m_j \in L^1\,, \qquad m_i|Du_j|^2 \in L^1\,, \qquad \forall i,j = 1,2\,.$$

Thanks to the previous key-steps, and using renormalized solutions, one extends the uniqueness argument of [Lasry-Lions] from classical to weak solutions.

Comments, work in progress, open problems

- This energy approach is likely to be extended to more general superlinear growths of the Hamiltonian $H(x, p)$, and certainly to different boundary conditions.
- By contrast, the case of linear growths is unclear: existence should fail at least for small T. Sol. may exist in long time?
- The case that m_0 and m_1 are not strictly positive is open (very interesting !)
- Regularity of solutions is an open problem. Somehow related to the question whether we have lower bounds on m.
- Rigorous connections between FP equation and the stochastic flow (cfr. [Krylov-Röckner '07], [Figalli '08], [Le Bris-Lions]) for a complete stochastic optimal transport result.
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- More general classes of Mean Field Games, ex. congestion models: $H = \frac{|Du|^2}{m^\gamma}$.

- The deterministic case:

$$\begin{cases} -u_t + H(x, Du) = F(x, m) \\ m_t - \mathrm{div}(m\, H_p(x, Du)) = 0 \\ m(0) = m_0\,, \quad m(T) = m_1 \end{cases}$$

-$F = 0 \rightarrow$ optimal transport ([Benamou-Brenier], [Villani],...).

-$F = F(m)$ increasing $\rightarrow$ results by P.L. Lions (totally different method).

General results ? Is there some unifying framework ?

Thanks for the attention !