Mean field games and Nash equilibria in large populations

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Outline

1. Mean field game theory: what is about ?
2. PDE systems in mean field games: a forward-backward coupling of Hamilton-Jacobi and Fokker-Planck equations.
3. Structure conditions in mean field game systems. Sample results of existence, uniqueness, long time behavior.
4. The master equation in mean field games.
5. The case of major/minor players.

Mean field game theory: what is about ?

MFG theory was introduced since 2006 by J-M Lasry and P-L Lions.
A similar model developed independently by [Huang-Caines-Malhamé].

Goal: investigate the concept of Nash equilibria for large populations of rational agents (small, indistinguishable...)

Structural assumptions in the model

(i) (*symmetry*) An infinite number (a continuum) of indistinguishable agents with similar preferences

(ii) (*controlled dynamical states*) Each agent controls his/her own dynamical state (typically, a controlled SDE)

(iii) (*weak interaction*) each single agent has no influence on the others' choices. But the overall distribution of the states has an impact !

Basic idea: export the principle of statistical mechanics to (non cooperative) strategic interactions within rational particles

Model setting: N agents are controlling their own dynamics

$$dX^i_\tau = \alpha^i_\tau d\tau + \sqrt{2} dB^i_\tau$$

where B^i_t are independent Brownian motions, α^i_t are control processes.

• weak coupling $\rightsquigarrow$ cost depend on the empirical density $\mu^N_s = \frac{1}{N} \sum_j \delta_{X^j_s}$

$$\to \inf J^i(\alpha) := \mathbb{E} \int_t^T [L(X^i_s, \alpha^i_s) + F(X^i_s, \mu^N_s) ds + G(X^i_T, \mu^N_T)$$

• Nash equilibria in feedback form satisfy a system of HJ equations (see e.g. [Bensoussan-Frehse]):

$$-\partial_t u^i - \sum_j \Delta_{x_j} u^i + H^i(x^i, \nabla_{x^i} u^i) + \sum_{j \neq i} H^j_p(x^j, \nabla_{x^j} u^j) \cdot \nabla_{x_j} u^i = F$$

Lasry-Lions' idea: as $N \to \infty$

(i) u^i has a "weak" dependence from x^j, $j \neq i \rightsquigarrow \nabla_{x_j} u^i = O(\frac{1}{N})$

(ii) if players are initially i.i.d. with law m_0, the density m_N of all players should factorize: $m_N \simeq \prod_{i=1}^N m(x^i)$ where m is the law associated to the optimal stochastic process.

$\rightarrow$ Limit of Nash equilibria of symmetric N-players games will satisfy, as $N \rightarrow \infty$, a system of PDEs coupling a Hamilton-Jacobi equation for the individual strategies with a Kolmogorov-Fokker-Planck equation for the distribution law

Impact of the theory:

- The mean-field limit system allows for a huge simplification: numerical approximations of PDEs provide cheap computations of the equilibria of complex systems.
- Construction of quasi-Nash equilibria (in feedback form) for N-persons games through the solution of the MFG system
- Applications: finance, market economics (oil producers, carbon markets...), engineering (smart grids...), crowd dynamics, socio-politics (learning, opinion formation etc...)

Macroscopic (mean-field) description

$$\begin{cases} dX_s = \alpha(X_s)ds + \sqrt{2}\ dB_s\,, \\ X_t = x \end{cases}$$

dynamics of each agent

$$u(t,x) = \inf_{\{\alpha(\cdot)\}} \mathbb{E} \int_t^T L(X_s, \alpha_s, \mu_s) + G(X_T, \mu_T)$$

where μ_t is an exogenous fixed distribution law

$\downarrow$ *HJB*

$$-\partial_t u - \Delta u + H(x, \mu, Du) = 0$$

$$\underbrace{\alpha_t^* = -H_p(X_t, \mu_t, Du(t, X_t))}_{\text{optimal policy}},$$

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Kolmogorov

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HJB

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$$\underbrace{\alpha_t^* = -H_p(X_t, \mu_t, Du(t, X_t))}_{\text{optimal policy}},$$

$$-\partial_t m - \Delta m + \operatorname{div}(\alpha\, m) = 0$$

$$m(t) = \mathcal{L}(X_t)$$

Macroscopic (mean-field) description

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Nash equilibrium: $\mathcal{L}(X_t^*) = \mu_t$

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where μ_t is an exogenous fixed distribution law

Kolm.

HJB

$$-\partial_t m - \Delta m + \text{ div } (\alpha\, m) = 0$$
$$m(t) = \mathcal{L}(X_t)$$

$$-\partial_t u - \Delta u + H(x, \mu, Du) = 0$$
$$\underbrace{\alpha_t^* = -H_p(X_t, \mu_t, Du(t, X_t))}_{\text{optimal policy}},$$

Nash equilibrium: $\mathcal{L}(X_t^*) = \mu_t$

$$\partial_t m - \Delta m - \mathrm{div}(m\, H_p(x, m, Du)) = 0 \qquad -\partial_t u - \Delta u + H(x, m, Du) = 0$$

The MFG system of PDEs

The mean field game system in a time horizon T. Model case:

$$\begin{cases} (1) \qquad -\partial_t u - \nu\Delta u + H(t,x,Du) = F(t,x,m) & \text{in } (0,T)\times\Omega \\ (2) \qquad \partial_t m - \nu\Delta m - \mathrm{div}(m\,H_p(t,x,Du)) = 0 & \text{in } (0,T)\times\Omega\,, \end{cases}$$

where H_p stands for $\frac{\partial H(t,x,p)}{\partial p}$.

- (1) is the Bellman equation for the agents' value function u.
- (2) is the Kolmogorov-Fokker-Planck equation for the distribution of agents. $m(t)$ is the probability density of the state of players at time t.

The system is usually complemented with initial-terminal conditions:

-$m(0) = m_0$ (initial distribution of the agents)

-$u(T) = G(x, m(T))$ (final pay-off)

$+$ boundary conditions (here for simplicity assume periodic b.c.)

Rmk: This is not the most general structure.

Cost criterion $L(X_t, \alpha_t, m(t)) \to H(x, m, Du)$.

Comments:

- The coupling of Hamilton-Jacobi and Fokker-Planck appears in many macroeconomics models, so-called heterogeneous agents models, e.g. (see [Achdou-Buera-Lasry-Lions-Moll], [Gueant-Lasry-Lions, Paris-Princeton Lectures]):

(i) the Aiyagari-Bewley-Huggett model for wealth & income distribution ([Achdou-Han-Lasry-Lions-Moll])

(ii) knowledge diffusion, research and growth models ([Luttmer], [Lucas-Moll])

(iii) oil market, mining industries, etc...([Gueant-Lasry-Lions, Chan-Sircar, Achdou-Giraud-Lasry-Lions])

Lots of open problems and interesting questions....

- An increasing huge literature on mean field games goes through the probabilistic approach $\rightsquigarrow$ [Carmona-Delarue] and stands on the theory of forward-backward SDEs (Pontryagin's principle) and McKean-Vlasov equations.

- Forward-backward structure $\to$ connection with optimality systems in control theory and with optimal transport.

Link with optimal control systems

MFG as optimality system (optimal control with Fokker-Planck state eq.).

Ex: Optimize in terms of the field α

$$\partial_t m = \Delta m + \text{ div } (\alpha\, m)\,, \qquad m(0) = m_0$$

$$\longrightarrow \quad \inf_\alpha \int_0^T \int_\Omega \{L(x,\alpha)m + \Phi(m(s))\}\, dt + \mathcal{G}(m(T))$$

where $\Phi'(m) = F(m)$ and $\mathcal{G}'(m) = G(m)$.

First order optimality conditions give the adjoint state u:

$$\begin{cases} Du + L_\alpha(x,\alpha) = 0 \quad (m - q.o.) \\ -\partial_t u - \Delta u - \alpha \cdot Du - L(x,\alpha) = F(m) \end{cases} \Leftrightarrow \quad \begin{array}{l} \alpha_{opt} = -H_p(x, Du(t,x)) \\ -\partial_t u - \Delta u + H(x, Du) = F(m) \end{array}$$

Rmk: $F(m), G(m)$ nondecreasing $\Rightarrow$ convexity of the functional

MFG system: structure conditions

$$\begin{cases} -\partial_t u - \nu\Delta u + H(x, Du) = F(x, m) & \text{in } (0, T) \times \Omega \\ \partial_t m - \nu\Delta m - \mathrm{div}(m\, H_p(x, Du)) = 0 & \text{in } (0, T) \times \Omega\,, \\ m(0) = m_0\,, \qquad u(T) = G(x, m(T)) & \\ + \text{ boundary conditions (otherwise } \Omega = T^N \text{ is the flat torus, or } \Omega = \mathbb{R}^d) & \end{cases}$$

Structure conditions in most PDE results:

- $p \mapsto H(x, p)$ is convex (possibly uniformly) and C^1
- Smoothing coupling:

  $m \mapsto F(\cdot, m)$ and $m \mapsto G(\cdot, m)$ are continuous from $\mathcal{P}(\Omega)$ to $C^2(\Omega)$

  $\rightsquigarrow \quad Ex : F(x, m) = K(x, \cdot) \star m(\cdot) = \int_\Omega K(x, y) dm(y)$
- Key-assumption: F, G nondecreasing $\rightarrow$ uniqueness, stability...!

Monotone couplings $\rightarrow$ uniqueness

Use the adjoint structure in MFG system (as for optimality systems of convex functionals):

$$-\frac{d}{dt}\left[\int_\Omega (u_1-u_2)(m_1-m_2)\right] = \int_\Omega [F(m_1)-F(m_2)]\,(m_1-m_2)$$

$$\underbrace{+\int_\Omega [H(Du_1)-H(Du_2)](m_1-m_2) - [m_1 H_p(Du_1) - m_2 H_p(Du_2)]D(u_1-u_2)}_{\int_\Omega m_1\,[H(Du_2)-H(Du_1)-H_p(Du_1)\,D(u_2-u_1)]\;+\;\int_\Omega m_2\,[H(Du_1)-H(Du_2)-H_p(Du_2)\,D(u_1-u_2)]}$$

$\rightsquigarrow$ H convex + F nondecreasing $\Rightarrow$ all terms are ≥ 0 !!

$$-\frac{d}{dt}\left[\int_\Omega (u_1-u_2)(m_1-m_2)\right] \geq 0$$

If $m_1(0) = m_2(0)$ (= the initial distribution m_0), and if $\int_\Omega (u_1-u_2)(m_1-m_2) \geq 0$ at time T (G nondecreasing), then one gets uniqueness.

Sample results on the MFG system.

1. Existence, uniqueness of smooth (global) solutions ([Lasry-Lions])
   If H satisfies one of the following:

$$(i) \quad |H(x,p)| \le c(1+|p|^2)$$

$$(ii) \quad H_x \cdot p \ge -C(1+|p|^2)$$

   and if the coupling functions F, G are smoothing[1], then there exists a classical solution (u, m) to the MFG system.

   In addition, the solution is unique if F, G are monotone operators and if H is strictly convex.

2. Numerical schemes: consistency and convergence ([Achdou-Capuzzo Dolcetta], [Achdou-P.], [Ferreira-Gomes])

   This is a finite differences scheme (implicit backward/forward, monotone scheme for HJB and defined by duality for KFP)

[1] $m \mapsto F(\cdot, m)$ is continuous from $C^0([0,T]; L^1)$ to $C(\overline{Q}_T)$ with bounded range in $L^\infty((0,T); W^{1,\infty}(\Omega))$
$m \mapsto G(\cdot, m)$ is continuous from L^1 to $C(\overline{\Omega})$ with bounded range in $W^{1,\infty}(\Omega)$

3. Stability in long time ([Cardaliaguet-Lasry-Lions-P.], [Cardaliaguet-P.])

In long horizon $[0, T]$, $T >> 1$, the solution (u^T, m^T) of the MFG system is nearly stationary for most of the time:

$\exists$ a unique stationary solution $\bar\lambda \in \mathbb{R}$ and $(\bar u, \bar m)$ satisfying (here Ω is the flat torus)

$$\begin{cases} \bar\lambda - \Delta \bar u + H(x, D\bar u) = F(x, \bar m)\,, & \int_\Omega \bar u = 0 \\ -\Delta \bar m - \operatorname{div}\, (\bar m H_p(x, D\bar u)) = 0\,, & \int_\Omega \bar m = 1 \end{cases}$$

such that (Du^T, m^T) is exponentially close to $(D\bar u, \bar m)$ for a large time:

$$\|Du^T(t) - D\bar u\|_{C^{0,\alpha}} + \|m^T(t) - \bar m\|_{C^{0,\alpha}} \leq C\left(e^{-\omega(T-t)} + e^{-\omega t}\right)\,,$$

$\to$ stability appears in a large intermediate time $[\delta T, (1-\delta)T]$.

Boundary layers appear at initial and final time, yet for most of the time the strategies are almost stationary

This is indeed a manifestation of the turnpike property - cfr. optimality systems ...-

The turnpike property

An efficient expanding economy should for most of the time be nearly an equilibrium path

The above result should be regarded as a typical *turnpike result*, in the terminology introduced by P. Samuelson in 1949:

if we are planning long-run growth, no matter where we start, and where we desire to end up, it will pay in the intermediate stages to get into a steady growth phase. [2]

[Dorfman-Samuelson-Solow, Linear programming and economic analysis, 1958]

[2] It is exactly like a turnpike paralleled by a network of minor roads. There is a fastest route between any two points; and if the origin and destination are close together and far from the turnpike, the best route may not touch the turnpike. But if origin and destination are far enough apart, it will always pay to get on to the turnpike and cover distance at the best rate of travel, even if this means adding a little mileage at either end.

4. Solutions to the master equation ([Cardaliaguet-Delarue-Lasry-Lions])

(u, m) is the unique solution of the MFG system

$$\begin{cases} -\partial_t u - \nu\Delta u + H(x, Du) = F(x, m) & \text{in } (t_0, T) \times \Omega \\ \partial_t m - \nu\Delta m - \mathrm{div}(m\, H_p(x, Du)) = 0 & \text{in } (t_0, T) \times \Omega\,, \\ m(t_0) = m_0\,, \qquad u(T) = G(x, m(T)) \end{cases}$$

if and only if

$$u(t) = U(t, x, m(t))$$

where $U : Q_T \times \mathcal{P}(T^N))$ solves the the master equation in $Q_T \times \mathcal{P}$:

$$\begin{cases} -\partial_t U - \Delta_x U + H(x, D_x U) - \int \mathrm{div}_y(D_m U(x, m))(y) dm(y) \\ \quad + \int D_m U(x, m)(y) \cdot H_p(y, D_x U(y, m)) dm(y) = F(x, m) \\ U(T, x, m) = G(x, m) \end{cases}$$

$D_m U := \frac{d}{dy}\left(\frac{\delta U}{\delta m}\big|_{(t,x,m)}(y)\right)$ where $\frac{\delta U}{\delta m}$ is the representation of Gateaux derivative of U if $m \in L^2$

- The master equation encodes the MFG system in a unique equation (but infinitely dimensional !). It is usually a key point in proving the convergence of solutions of N-players systems towards solutions of the mean field system (microscopics $\to$ macroscopics)

- The master equation is a nonlinear (degenerate) transport-diffusion equation. Global existence is ensured by the monotonicity structure condition:

$$F(m), G(m) \text{ monotone} \Rightarrow$$
$$\Rightarrow \text{solutions } U \text{ are monotone, global and unique !}$$

- The master equation plays a key-role in case of common noise for the agents (aggregate shocks, in the economists ' words..).

  In that case the usual MFG system would turn into a system of stochastic PDEs, while the master equation allows for a purely PDE approach

  $\rightsquigarrow$ Local (in time) existence for the master equation with common noise can be proved e.g. by splitting methods ([Cardaliaguet-Cirant-P. '19])

Mean field games with major/minor players

The setting: infinitely many small agents interact with a major one. All play closed loop strategies in feedback form.

- X_t^0 = state of the major player ; B_t^0 a given Brownian motion in $\mathbb{R}^{d_0}$

  $$dX_t^0 = \alpha^0(t, X_t^0, m_t)dt + \sqrt{2}dB_t^0 \qquad \underline{\text{dynamics of the major player}}$$

  where $\{m_t\}$ is a stochastic flow of measures in $\mathcal{P}_2(\mathbb{R}^d)$ which is adapted to the filtration generated by $B^0 := \{B_t^0\}$.

- X_t = state of the representative minor player; B_t a given Brownian motion in $\mathbb{R}^d$ (indep. of B_t^0)

  $$dX_t = \alpha(t, X_t, X_t^0, m_t)dt + \sqrt{2}dB_t\,, \qquad \underline{\text{dynamics of each minor player}}$$

- Here α^0, α are (deterministic) functions bounded and locally Lipschitz continuous.
- An initial measure $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ is given for the distribution of minor players ($\mu_0 = \mathcal{L}(X_0)$) and an initial position $x_0^0 \in \mathbb{R}^{d_0}$ is given for the major player

- Optimization of the minor player

$$J(\alpha;[\alpha^0,m_t])=\mathbb{E}\int_0^T L(X_t,X_t^0,\alpha(t,X_t,X_t^0,m_t),m_t)dt+G(X_T,X_T^0,m_T)$$

where (X_t^0,m_t) is a given exogenous stochastic flow.

- Optimization of the major player

$$J^0(\alpha^0;[\alpha]):=\mathbb{E}\left[\int_0^T L^0(X_t^0,\alpha_t^0(t,X_t^0,m_t),m_t)dt+G^0(X_T^0,m_T)\right],$$

where (X_t^0,m_t) is now the flow generated by α and α^0:

$$\begin{cases} dX_t^0=\alpha^0(t,X_t^0,m_t)dt+\sqrt{2}dB_t^0 \\ d_t m_t=\left\{\Delta m_t-\ \text{div}\ (m_t\alpha(t,x,X_t^0,m_t))\right\}dt, \\ m_0=\mu_0,\ X_0^0=x_0^0. \end{cases} \tag{1}$$

In other words, m_t is the conditional law of X_t given X_t^0

$\rightsquigarrow$ X_t satisfies the McKean-Vlasov SDE

$$dX_s=\alpha(s,X_s,X_s^0,\mathcal{L}(X_s/X_s^0))ds+\sqrt{2}dB_s,\qquad \mathcal{L}(X_0)=\mu_0\,.$$

Notice:

- the minor players optimize giving for fixed

  (i) the strategy and position of the major player

  (ii) the mean field of the other players

- Conversely, the major player has an impact on the minor players. When he deviates, he needs to consider the reaction of minor players, hence the change in m_t.

This formulation appears for the first time in [Carmona-Wang] and is somehow different from previous suggested model of major/minor mean field games ([Huang], [Nourian-Caines], [Bensoussa-Chau-Yam]).

Definition (Nash equilibrium in the game)
A pair $(\bar{\alpha}, \bar{\alpha}^0)$ of feedback strategies is an equilibrium if:

1. (consistency) the flow of measures $\bar{m}_t$ corresponds to the optimal distribution of minor players. This means that $(\bar{X}_t^0, \bar{m}_t)$ solve

$$\begin{cases} d\bar{X}_t^0 = \bar{\alpha}^0(t, \bar{X}_t^0, \bar{m}_t)dt + \sqrt{2}dB_t^0, \\ d_t\bar{m}_t = \left\{\Delta\bar{m}_t - \operatorname{div}\left(\bar{m}_t\bar{\alpha}(t, x, \bar{X}_t^0, \bar{m}_t)\right)\right\} dt \end{cases} \tag{2}$$

2. The strategy $\bar{\alpha}$ is optimal for each minor player, given $(\bar{X}_t^0, \bar{m}_t)$:

$$J(\bar{\alpha}; [\bar{\alpha}^0, \bar{m}_t]) \leq J(\alpha; [\bar{\alpha}^0, \bar{m}_t]) \tag{3}$$

   for any other Markovian feedback control $\alpha_t := \alpha(t, X_t, \bar{X}_t^0, \bar{m}_t)$.

3. The strategy $\bar{\alpha}^0$ is optimal for the major player:

$$J^0(\bar{\alpha}^0; [\bar{\alpha}]) \leq J^0(\alpha^0; [\bar{\alpha}]),$$

   for any different feedback law $\alpha^0(t, x, m)$.,

Approach by master equation

Mean field game theory describes this major/minor problem through a system of master equations:

$$
\begin{aligned}
U^0(t, x_0, m) &\simeq \text{ value function of the major player} \\
U(t, x, x_0, m) &\simeq \text{ value function of each minor player}
\end{aligned}
$$

satisfy the system

$$
\begin{cases}
-\partial_t U^0 - \Delta_{x_0} U^0 + H^0(x_0, D_{x_0} U^0, m) - \int_{\mathbb{R}^d} \mathrm{div}_y D_m U^0(t, x_0, m, y) dm(y) \\
\quad + \int_{\mathbb{R}^d} D_m U^0(t, x_0, m, y) \cdot D_p H(y, x_0, D_x U(t, y, x_0, m), m) dm(y) = 0 \\
\qquad\qquad\qquad\qquad\qquad\qquad \text{in } (0, T) \times \mathbb{R}^{d_0} \times \mathcal{P}_2, \\
-\partial_t U - \Delta_x U - \Delta_{x_0} U + H(x, x_0, D_x U, m) - \int_{\mathbb{R}^d} \mathrm{div}_y D_m U(t, x, x_0, m, y) dm(y) \\
\quad + D_{x_0} U \cdot D_p H^0(x_0, D_{x_0} U^0(t, x_0, m), m) \\
\quad + \int_{\mathbb{R}^d} D_m U(t, x, x_0, m, y) \cdot D_p H(y, x_0, D_x U(t, y, x_0, m), m) dm(y) = 0 \\
\qquad\qquad\qquad\qquad\qquad\qquad \text{in } (0, T) \times \mathbb{R}^d \times \mathbb{R}^{d_0} \times \mathcal{P}_2, \\
U^0(T, x_0, m) = G^0(x_0, m), \qquad \text{in } \mathbb{R}^{d_0} \times \mathcal{P}_2, \\
U(T, x, x_0, m) = G(x, x_0, m) \qquad \text{in } \mathbb{R}^d \times \mathbb{R}^{d_0} \times \mathcal{P}_2.
\end{cases}
\tag{4}
$$

[Cardaliaguet-Cirant- P., preprint]

Under standard assumptions (regularity, global Lipschitz growth of the Hamiltonians $H^0(x_0, p, m)$, $H(x, x_0, p, m)$) we prove the following results:

1. (verification) If (U^0, U) is a classical solution to the system of master equations (4), then the pair

$$(\bar{\alpha}(t, x, x_0, m), \bar{\alpha}^0(t, x_0, m)) :=$$
$$= -(D_p H(x, x_0, D_x U(t, x, x_0, m), m), D_p H^0(x_0, D_{x_0} U^0(t, x_0, m), m))$$

is a Nash equilibrium of the game.

2. (local in time existence) There exists T_0 such that the system of master equations (4) admits a classical solution (U^0, U) for $T \leq T_0$.

3. (consistency) Nash equilibria of $N(minor) + 1(major)$-players differential games converge to the solution

Precisely, we consider the system of Nash equilibria for N minor agents:

$$\begin{cases} -\partial_t u^{N,0} - \sum_{j=0}^N \Delta_{x_j} u^{N,0} + H^0(x_0, D_{x_0} u^{N,0}, m_x^N) \\ \qquad + \sum_{j=1}^N D_{x_j} u^{N,0} \cdot D_p H(x_j, x_0, D_{x_j} u^{N,j}, m_x^{N,j}) = 0 \\ -\partial_t u^{N,i} - \sum_{j=0}^N \Delta_{x_j} u^{N,i} + H(x_i, x_0, D_{x_i} u^{N,i}, m_x^{N,i}) \\ \qquad + D_{x_0} u^{N,i} \cdot D_p H^0(x_0, D_{x_0} u^{N,0}, m_x^N) \\ \qquad\qquad + \sum_{j\neq i,\ j\geq 1} D_{x_j} u^{N,i} \cdot D_p H(x_j, x_0, D_{x_j} u^{N,j}, m_x^{N,j}) = 0 \\ u^{N,0}(T,x) = G^0(x_0, m_x^N),\ u^{N,i}(T,x) = G(x_i, x_0, m_x^{N,i}). \end{cases} \tag{5}$$

where $m_x^N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$, $m_x^{N,i} = \frac{1}{N-1}\sum_{j\neq\{0,i\}} \delta_{x_j}$.
Let $(u^{N,i})$ be a classical solution to the Nash system (5) and (U^0, U) be a classical solution to the system (4) of master equations. There is a constant C, independent of N, $\mathbf{x} \in \mathbb{R}^{d_0} \times (\mathbb{R}^d)^N$ and $t \in [0, T]$, such that

$$\left|u^{N,0}(t,\mathbf{x}) - U^0(t, x_0, m_{\mathbf{x}}^N)\right| + \sup_{i=1,\dots,N} \left|u^{N,i}(t,\mathbf{x}) - U(t, x_i, x_0, m_{\mathbf{x}}^{N,i})\right|$$

$$\leq CN^{-1}\left(1 + \frac{1}{N}\sum_{i=1}^N |x_i|\right),$$

where $\mathbf{x} = (x_0, \dots, x_N)$.

Thanks for the attention !