Long time behavior of mean field games

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based on joint works with
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Multi-agents systems $\rightsquigarrow$ mean field games, mean field control problems [Lasry-Lions '06], [Huang-Caines-Malhamé '06], [Bensoussan-Frehse-Yam '13]

- Individual costs depend on the collective behavior
- Individuals' strategy drive the collective behavior

- Rational agents are represented by *controlled dynamical states*, e.g.

$$dX_s = \alpha_s ds + \sqrt{2}\ dB_s$$
$$\to \inf \mathbb{E}\left\{ \int_t^T [L(X_s, \alpha_s) + F(X_s, m(s))]ds + G(X_T, m(T)) \right\}$$

- The collective behavior is represented by (the evolution of) the *distribution law of the individual states*

$\rightsquigarrow$ solutions are equilibria: consistency between the anticipated guess of the collective behavior which is done by the agents and the effective evolution of the population.

Model case of MFG system (with horizon T):

$$\begin{cases} (1) \qquad -\partial_t u - \Delta u + H(x, Du) = F(x, m) & \text{in } (0, T) \times \Omega \\ (2) \qquad \partial_t m - \Delta m - \operatorname{div}(m\, H_p(x, Du)) = 0 & \text{in } (0, T) \times \Omega\,, \\ \qquad\qquad m(0) = m_0\,, \qquad \text{u(T)=G(x, m(T))} \end{cases}$$

where H_p stands for $\frac{\partial H(t,x,p)}{\partial p}$.

- (1) is the (backward) Bellman equation for the agents' value function u.

  $\rightarrow u(T) = G(x, m(T))$ is the final pay-off

- (2) is the (forward) Kolmogorov equation for the distribution of agents.
  $m(t)$ is the probability density of the state of players at time t.

  $\rightarrow m(0) = m_0$ is the initial distribution law.

Existence of solutions via fixed point: monotonicity and/or smoothness of the coupling F, G play a key role

Link with optimal control systems

MFG as optimality system (optimal control with Fokker-Planck state eq.).

Ex: Optimize in terms of the field α

$$\partial_t m = \Delta m + \operatorname{div}(\alpha\, m)\,, \qquad m(0) = m_0$$

$$\longrightarrow \quad \inf_\alpha \int_0^T [\int_\Omega m\, L(x,\alpha) + \mathcal{F}(m(s))]ds + \int_\Omega \mathcal{G}(m(T))$$

where $\mathcal{F}'(m) = F(m)$ and $\mathcal{G}'(m) = G(m)$.

First order optimality conditions give the adjoint state u:

$$\begin{cases} Du + L_\alpha(x,\alpha) = 0 \quad (m-q.o.) \\ -\partial_t u - \Delta u - \alpha\cdot Du - L(x,\alpha) = F(m) \end{cases} \Leftrightarrow \quad \begin{array}{l} \alpha_{opt} = -H_p(x, Du(t,x)) \\ -\partial_t u - \Delta u + H(x,Du) = F(m) \end{array}$$

Rmk: <u>$F(m), G(m)$ nondecreasing $\Rightarrow$ convexity of the functional</u>

Pb: What is the behavior of the MFG system when the horizon $T \to \infty$?

$$\begin{cases} -\partial_t u^T - \Delta u^T + H(x, Du^T) = F(x, m^T), & \text{in } (0, T) \\ \partial_t m^T - \Delta m^T - \text{ div } (m^T H_p(x, Du^T)) = 0, & \text{in } (0, T) \\ m^T(x, 0) = m_0(x), \quad u^T(x, T) = G(x, m(T)). \end{cases}$$

Here we are in the periodic setting: $\Omega = \mathbb{T}^N$ is the flat torus.

Main assumptions:

- $p \mapsto H(x, p)$ is uniformly convex and smooth: $c_0 I \leq H_{pp} \leq C_0 I$
- Smoothing coupling (nonlocal case):

  $m \mapsto F(\cdot, m)$ and $m \mapsto G(\cdot, m)$ are continuous from $\mathcal{P}(\Omega)$ to $C^2(\Omega)$
- $F(m), G(m)$ are nondecreasing

  $\rightsquigarrow \quad Ex : F(x, m) = [K \star m(\cdot)] \star K = \int K(x, z) \int K(z, y) dm(y) dz$

(+ further technical conditions on derivatives of H, F....)

Rmk: other sets of assumptions are possible (e.g. local couplings & Lipschitz Hamiltonians)

Long time behavior

Natural question: shall we find some extension of what is known (both long time and vanishing discount limits) for a single HJ equation ?

[Fathi], [Fathi-Siconolfi], [Davini-Siconolfi], [Ishii], [Barles-Souganidis], [Barles-Ishii-Mitake], [Cagnetti-Gomes-Mitake-Tran], [Gomes], [Davini-Fathi-Iturriaga-Zavidovique], [Ishii-Mitake-Tran],....

Main difficulties compared to the single HJ equation:

- one can not use: comparison arguments, L^∞-contraction, etc...

the source term in the HJ equation varies in time !

- forward-backward structure: some boundary layer could appear at $t = 0$ or $t = T$

Look at the Fokker-Planck equation: in MFG systems, the drift depends on u which depends on m !! $\rightsquigarrow$ Mc-Kean-Vlasov equations

$$\partial_t m - \Delta m - \operatorname{div}\left(m\, B^T(t, x, m)\right) = 0 \qquad \text{in } (0, T)$$

Stationary problem for Mean Field Games [Lasry-Lions '07]

$F(x,\cdot)$ nondecreasing $\Rightarrow$ $\exists$ a unique constant $\bar\lambda$ and unique $(\bar u, \bar m)$:

$$\begin{cases} \bar\lambda - \Delta \bar u + H(x, D\bar u) = F(x, \bar m)\,, & \int_\Omega \bar u = 0 \\ -\Delta \bar m - \operatorname{div}\,(\bar m H_p(x, D\bar u)) = 0\,, & \int_\Omega \bar m = 1 \end{cases}$$

Moreover, $\bar u$, $\bar m$ are smooth, $\bar m > 0$

Expected ergodic behavior of MFG system: $u^T/T \to \bar\lambda$

Qns:

(i) can we say - if and how - that $m^T \to \bar m$?

(ii) what about the convergence of $u^T - \bar\lambda(T-t)$?

(iii) what about the limit of the discounted problem ?

$$\begin{cases} -\partial_t v + \delta v - \Delta v + H(x, Dv) = F(x, \mu) & \text{in } (0,\infty)\times\Omega \\ \partial_t \mu - \Delta\mu - \mathrm{div}(\mu\, H_p(x, Dv)) = 0 & \text{in } (0,\infty)\times\Omega \\ \mu(0) = m_0\,, \quad v \in L^\infty((0,\infty)\times\Omega) \end{cases}$$

Convergence results (Part I):

1. (ergodic behavior) $\frac{u^T(x,0)}{T} \to \bar{\lambda}$

2. (convergence of the large time average)

$$\frac{1}{T}\int_0^T\int_\Omega |Du^T - D\bar{u}|^2 + (F(x,m^T) - F(x,\bar{m}))(m-\bar{m})\,dx \to 0$$

3. (turnpike property): (Du^T, m^T) are exponentially close to $(D\bar{u},\bar{m})$ in the long transient time:

$$\|Du^T(t) - D\bar{u}\|_{C^{0,\alpha}} + \|m^T(t) - \bar{m}\|_{C^{0,\alpha}} \le C\left(e^{-\omega(T-t)} + e^{-\omega t}\right),$$

$\rightarrow$ a stationary behavior will appear in a large intermediate time $[\delta T, (1-\delta)T]$ inside the horizon $(0,T)$.

Rmk: This is typical of (stable) optimal control problems !
(see e.g. [P.-Zuazua '13], [Trelat-Zuazua '15])

The turnpike property

Turnpike property of optimal control problems:

$\rightsquigarrow$ in a long horizon (robust) optimization, optimal controls and states are nearly stationary

Terminology introduced by Nobel Prize P. Samuelson in 1949:

if we are planning long-run growth, no matter where we start, and where we desire to end up, it will pay in the intermediate stages to get into a steady growth phase. [1]

[Dorfman-Samuelson-Solow, Linear programming and economic analysis, 1958]

[1] It is exactly like a turnpike paralleled by a network of minor roads. There is a fastest route between any two points; and if the origin and destination are close together and far from the turnpike, the best route may not touch the turnpike. But if origin and destination are far enough apart, it will always pay to get on to the turnpike and cover distance at the best rate of travel, even if this means adding a little mileage at either end.

Main ingredients:

1. Global convexity and stability of MFG system

Any couple of solutions (u_1, m_1) and (u_2, m_2) satisfies

$$-\frac{d}{dt}\int_\Omega (u_1-u_2)(m_1-m_2)dx \geq$$
$$\int_\Omega \gamma\frac{(m_1+m_2)}{2}|Du_1-Du_2|^2+(F(x,m_1)-F(x,m_2))(m_1-m_2)\ dx$$

Rmk: Since $m_1 - m_2$ has zero mean, the equality is invariant by adding a constant (even time dependent) to u_1 or u_2

Apply the energy equality to (u^T, m^T) and $(\bar{u}, \bar{m})$ between 0 and T....

$$\int_0^T \int_\Omega \frac{(m^T + \bar m)}{2} |Du^T - D\bar u|^2 + (F(x, m^T) - F(x, \bar m))(m^T - \bar m)\, dx$$
$$= -\left[\int_\Omega (u^T - \bar u)(m^T - \bar m) dx\right]_0^T \quad \overset{?}{\leq} \quad C$$

Typically, $u^T(0) \sim CT$. But, if we set $\langle u \rangle := \int u\, dx$, then

$$\int_\Omega u^T(0)(m_0 - \bar m) dx = \int_\Omega (u^T(0) - \langle u^T(0)\rangle)(m_0 - \bar m) dx \leq c \|Du^T(0)\|_{L^2}$$

$\Rightarrow$ it is enough to bound $\|Du^T(0)\|$ independently of T.

Conclusion: global gradient bounds $\to$ average convergence

$$\frac{1}{T}\int_0^T \int_\Omega |Du^T - D\bar u|^2 + (F(x, m^T) - F(x, \bar m))(m^T - \bar m)\, dx \leq \frac{C}{T} \to 0$$

Here: smooth couplings + uniformly convex Hamiltonian
$\Rightarrow$ semiconcavity estimates (global in time) $\Rightarrow$ uniform gradient bound

2. **Exponential rate of stability**: a recipe in 4 steps

Step 1: look at the linearized system around the stationary couple $(\bar u, \bar m)$:

$$\begin{cases} -v^T_t - \Delta v^T + H_p(x, D\bar u)Dv^T = \hat F'(\bar m)\mu^T & t \in (0,T) \\ \mu^T_t - \Delta \mu^T - \operatorname{div}(\mu^T H_p(x, D\bar u)) = \operatorname{div}(\bar m H_{pp}(x, D\bar u)Dv^T) & t \in (0,T) \\ \mu^T(0) = \mu_0\,, \quad v^T(T) = 0 \end{cases}$$

This defines a feedback operator as

$$\mathcal{E}(T)\mu_0 := v^T(0) - \int v^T(0)\,dx$$

and, by time shifting, we have $v^T(t) = \mathcal{E}(T-t)\mu^T(t)$

- $\mathcal{E}(T)$ is bounded and converges, as $T \to \infty$

$$\mathcal{E}(T)\mu_0 \overset{T\to\infty}{\rightarrow} \quad \hat E\mu_0 := \hat v(0) - \int \hat v(0)\,dx\,,$$

where $\hat E$ is the corresponding infinite horizon feedback

Simply, $(\mu^T, v^T) \to (\hat\mu, \hat v)$ sol. in infinite horizon

$$\begin{cases} -\hat v_t - \Delta \hat v + H_p(x, D\bar u) D\hat v = \hat F'(\bar m)\hat\mu & t \in (0,\infty) \\ \hat\mu_t - \Delta\hat\mu - \operatorname{div}(\hat\mu H_p(x, D\bar u)) = \operatorname{div}(\bar m H_{pp}(x, D\bar u) D\hat v) & t \in (0,\infty) \\ \hat\mu(0) = \mu_0\,, \quad \hat v(t) \quad \text{bounded} \end{cases}$$

and

$$\hat E \mu_0 := \hat v(0) - \int \hat v(0)\, dx$$

By time shifting, $\hat v(t) = \hat E \hat\mu(t)$.

- The infinite horizon problem defines a linear semigroup

$$\underbrace{\hat\mu_t - \Delta\hat\mu - \operatorname{div}(\hat\mu H_p(x, D\bar u)) = \operatorname{div}(\bar m H_{pp}(x, D\bar u) D\hat E\hat\mu)}_{\hat\mu_t + L\hat\mu = 0}$$

  which has exponential decay ω.

  Notice: $(\hat E\mu, \mu) := (\hat v, \hat\mu)$ is a Lyapunov functional for the system

  Moreover, the feedback itself exponentially converges

$$\|\mathcal{E}(T) - \hat E\| \leq c e^{-\omega T}$$

The linearized system can therefore be decoupled as

$$\begin{cases} v^T(t) = \mathcal{E}(T-t)\mu^T(t) & t \in (0,T) \\ \mu_t^T - \Delta\mu^T - \operatorname{div}\,(\mu^T H_p(x, D\bar{u})) = \operatorname{div}\,(\bar{m} H_{pp}(x, D\bar{u}) D v^T) & t \in (0,T) \end{cases}$$

$$\rightsquigarrow \qquad \mu_t^T + L\mu_t = \underbrace{\operatorname{div}\,\left(\bar{m} H_{pp}(x, D\bar{u}) D \left(\mathcal{E}(T-t) - \hat{E}\right)\mu^T\right)}_{\simeq O\left(e^{-\omega(T-t)}\right)}$$

where L is an exponential decaying semigroup: $\|e^{-Lt}\| \leq e^{-\omega t}$

It is natural to conclude that

$$\|\mu^T\| \lesssim C\left(e^{-\omega t} + e^{-\omega(T-t)}\right)$$

Step 2: The solution to the nonhomogeneous system

$$\begin{cases} -v_t^T - \Delta v^T + H_p(x, D\bar{u})Dv^T = \hat{F}'(\bar{m})\mu^T + f_2 \\ \mu_t^T - \Delta\mu^T - \text{ div } (\mu^T H_p(x, D\bar{u}) = \text{ div } (\bar{m}H_{pp}(x, D\bar{u})Dv^T) + f_1 \\ \mu^T(0) = \mu_0 \,, \quad v^T(T) = g \end{cases}$$

satisfies

$$\|v^T(t) - \mathcal{E}(T-t)\mu^T(t)\| \leq ce^{-\omega(T-t)}\|g\| + c\int_t^T e^{-\omega(s-t)}(\|f_1(s)\| + \|f_2(s)\|)ds$$

Consequence: if f_1, f_2, g are small perturbations, then v^T behaves as the feedback of the linearized system

Step 3: Through a fixed point argument, we can preserve the exponential estimate for the original nonlinear problem:

$$\|m^T(t) - \bar{m}\| + \|\tilde{u}^T - \bar{u}\| \leq C\left(e^{-\omega t} + e^{-\omega(T-t)}\right)$$

provided initial and terminal data are close to stationary.

Step 4: The full result for any initial-terminal condition follows using the average convergence.

What about the convergence of $u^T(x,t) - \bar\lambda(T-t)$?

The exponential estimate brings a crucial information:

$$\|m^T(t) - \bar m\|_\infty \leq C\left(e^{-\omega t} + e^{-\omega(T-t)}\right)$$

Hence the source term is stationary up to an exponentially small term

$$-\partial_t u^T - \Delta u^T + H(x, Du^T) \simeq F(x, \bar m) \pm C\left(e^{-\omega t} + e^{-\omega(T-t)}\right)$$

$$\Rightarrow \qquad \bar u + \bar\lambda(T-t) \pm M \qquad \text{are super/sub solutions}$$

$$\Rightarrow \qquad u^T(x,t) - \bar\lambda(T-t) \qquad \text{is uniformly bounded}$$

Then we can go further in the asymptotic analysis...

Convergence results (Part II)

1. (convergence at any time scale) As $T \to \infty$, we have

$$\begin{cases} u^T(x,t) - \bar\lambda(T-t) \to v \\ m^T \to \mu \end{cases}$$

where (v, μ) solve

$$\begin{cases} -v_t + \bar\lambda - \Delta v + H(x, Dv) = F(x, \mu(t)), & t \in (0, \infty) \\ \mu_t - \Delta\mu - \operatorname{div}(\mu H_p(x, Dv)) = 0, & t \in (0, \infty) \\ \mu(0) = m_0, v \in L^\infty, \end{cases} \qquad (1)$$

Rmk: problem (1) is solvable for a unique μ and a unique v up to a constant ($\Rightarrow Dv$ is unique).

1. (vanishing discount limit) Let (v^δ, μ^δ) be the solution of

$$\begin{cases} -\partial_t v_\delta + \delta v_\delta - \Delta v_\delta + H(x, Dv_\delta) = F(x, \mu_\delta) & \text{in } (0,\infty)\times\Omega \\ \partial_t \mu_\delta - \Delta\mu_\delta - \mathrm{div}(\mu_\delta H_p(x, Dv_\delta)) = 0 & \text{in } (0,\infty)\times\Omega \\ \mu_\delta(0) = m_0\,, \quad v_\delta \quad \text{bounded} \end{cases}$$

As $\delta \to 0$, we have

$$v_\delta - \frac{\bar\lambda}{\delta} \to v \;; \qquad \mu_\delta \to \mu$$

where (v, μ) is the unique solution of

$$\begin{cases} -v_t + \bar\lambda - \Delta v + H(x, Dv) = F(x, \mu(t))\,, & t \in (0,\infty) \\ \mu_t - \Delta\mu - \text{ div } (\mu\, H_p(x, Dv)) = 0\,, & t \in (0,\infty) \\ \mu(0) = m_0\,, \quad v \in L^\infty\,, \quad \lim_{t\to\infty} \int v(t)dx = \theta \end{cases}$$

Selection principle: the constant θ is characterized as the unique ergodic constant of the linearized stationary problem:

$$\begin{cases} \theta + \bar u - \Delta\varphi + H_p(x, D\bar u)D\varphi = F'(\bar m)\rho\,, & \text{in } \Omega \\ -\Delta\rho - \text{ div } (\rho\, H_p(x, D\bar u)) - \text{ div } (\bar m\, H_{pp}(x, D\bar u)D\varphi) = 0\,, & \text{in } \Omega \end{cases}$$

Lifting the problem: look at the master equation. From

$$\begin{cases} -\partial_t u - \Delta u + H(x, Du) = F(x, m) & \text{in } (t_0, T) \times \Omega \\ \partial_t m - \Delta m - \operatorname{div}(m\, H_p(x, Du)) = 0 & \text{in } (t_0, T) \times \Omega \\ u(T) = G(x, m(T)),\ m(t_0) = m_0 \end{cases} \tag{2}$$

one can define a map $U^T : (0, T) \times T^N \times \mathcal{P}(\Omega) \to \mathbb{R}$:

$$U^T(t_0, x, m_0) := u^T(t_0, x; m_0) \qquad \text{solution of (2)}$$

Notice: U^T has the same role as the feedback in the linearized system

By time shifting, one has

$$u^T(t, x) = U^T(t, x, m^T(t)),$$

$$\rightsquigarrow \quad \partial_t m - \Delta m - \operatorname{div}(m\, H_p(x, DU^T(t, x, m(t)))) = 0 \quad \text{in } (0, T)$$

We know ([Cardaliaguet-Delarue-Lasry-Lions]) that U^T satisfies the master equation in $Q_T \times \mathcal{P}$:

$$\begin{cases} -\partial_t U - \Delta_x U + H(x, D_x U) - \int \operatorname{div}_y (D_m U(x,m))(y) dm(y) \\ \quad + \int D_m U(x,m)(y) \cdot H_p(D_x U(y)) dm(y) = F(x,m) \\ U(T,x,m) = G(x,m) \end{cases} \tag{3}$$

Careful: This is a (nonlinear) infinite dimensional equation !!

Here $D_m U(m,y)$ is a suitable derivative in the space of probability measures (see e.g. [Ambrosio-Gigli-Savaré]):

Roughly, $D_m U := \frac{d}{dy}\left(\frac{\delta U}{\delta m}\big|_{(t,x,m)}(y)\right)$ where $\frac{\delta U}{\delta m}$ is the representation of Gateaux derivative of U if $m \in L^2$

Pb: $\lim\limits_{T \to \infty} U^T$?

Long time behavior of the master equation

$$
\begin{cases}
-\partial_t U - \Delta_x U + H(x, D_x U) - \int \mathrm{div}_y (D_m U(x,m))(y) dm(y) \\
\quad + \int D_m U(x,m)(y) \cdot H_p(y, D_x U(y,m)) dm(y) = F(x,m) \\
U(T,x,m) = G(x,m)
\end{cases}
$$

Key properties of U:

- U is monotone with respect to m:

$$
\int_\Omega (U(t,x,m_1) - U(t,x,m_2))\, d(m_1 - m_2)(x) \geq 0 \quad \forall t\,, \forall m_1, m_2\,.
$$

- The values of U are transported through MFG system in any (t_0, t_1):

$$
\begin{cases}
-\partial_t u - \Delta u + H(x, Du) = F(x,m) & \text{in } (t_0, t_1) \\
\partial_t m - \Delta m - \mathrm{div}(m\, H_p(x, Du)) = 0 & \text{in } (t_0, t_1) \\
u(t_1) = U(t_1, x, m(t_1))\,, \; m(t_0) = m_0 &
\end{cases}
$$

$$
\Rightarrow u(t_0, x) = U(t_0, x, m_0)
$$

- U satisfies global Lipschitz estimates in x and m (+ Hölder in time):

$$\|D_x U(t,x,m)\|_\infty + \|D_m U(t,x,m)\|_\infty + \|D_m D_x U(t,x,m)\|_\infty \le K$$

for K independent of T.

Rmk: the regularity of $D_m U$ depends on the linearized MFG system !

$$\begin{cases} \mu_t - \Delta\mu - \operatorname{div}(\mu H_p(x,Du)) = -\operatorname{div}(mH_{pp}(x,Du)Dv) & t \in (0,T) \\ -v_t - \Delta v + H_p(x,Du)Dv = F'(m)\mu & t \in (0,T) \\ \mu(0) = \mu_0\,, \quad v(T) = G'(m(T))\mu(T) \end{cases}$$

$$\Rightarrow \quad v(0,x) = \int_\Omega \frac{\delta U}{\delta m}(0,x,m,y)\mu_0(y)dy$$

Main results: ergodic problem / long time convergence / vanishing discount for the master equation

1. (cell problem) The stationary ergodic master equation

$$\lambda - \Delta_x U(x,m) + H(x, D_x U(x,m)) - \int_y \operatorname{div}\,(D_m U(x,m))dm(y) + \int_y D_m U(x,m).H_p(y, D_x U(x,m))dm(y) = F(x,m)$$

admits a solution if and only if $\lambda = \bar{\lambda}$.
Moreover the solution $\bar{U}$ is unique up to a constant.

2. (long time convergence) There exists a solution $\bar{U}$ of the cell problem such that

$$U^T(t,x,m) - \bar{\lambda}(T-t) \to \bar{U}(x,m) \quad \text{as } T \to \infty.$$

3 (vanishing discount) There exists a unique solution U_δ of the discounted master equation

$$\delta U_\delta - \Delta_x U_\delta(x,m) + H(x, D_x U_\delta(x,m)) - \int_y \operatorname{div}\,(D_m U_\delta(x,m))dm(y) \\ + \int_y D_m U_\delta(x,m).H_p(y, D_x U_\delta(x,m))dm(y) = F(x,m)$$

and one has

$$\delta U_\delta \to \bar\lambda, \qquad U_\delta(x,m) - \frac{\bar\lambda}{\delta} \to U(x,m)$$

for a specific solution U of the cell problem.

Namely, U is the unique solution of the ergodic problem such that $U(x,\bar m) = \bar u(x) + \theta$, where θ is the unique ergodic constant of a linearized stationary problem...

$$\begin{cases} -\partial_t u^T - \Delta u^T + H(x, Du^T) = F(x, m^T) \\ \partial_t m^T - \Delta m^T - \mathrm{div}(m^T H_p(x, Du^T)) = 0 \\ u^T(T) = G(x, m^T(T)), \ m(0) = m_0 \end{cases} \quad \Rightarrow \quad u^T(t,x) = U^T(t, x, m^T(t))$$

and

$$\begin{cases} -\partial_t v_\delta + \delta v_\delta - \Delta v_\delta + H(x, Dv_\delta) = F(x, m_\delta) \\ \partial_t \mu_\delta - \Delta \mu_\delta - \mathrm{div}(\mu_\delta H_p(x, Dv_\delta)) = 0 \\ m(0) = m_0, \quad v \quad \text{bounded} \end{cases} \quad \Rightarrow \quad v_\delta(t,x) = U_\delta(x, \mu_\delta(t))$$

$$\Rightarrow \qquad u^T(t,x) - \bar\lambda(T-t) \overset{T\to\infty}{\to} \bar U(x, m(t))$$

$$v_\delta(t,x) - \frac{\bar\lambda}{\delta} \overset{\delta\to 0}{\to} U(x, m(t))$$

where m is the solution of the nonlinear evolution equation

$$\begin{cases} \partial_t m - \Delta m - \mathrm{div}(m\, B(x,m)) = 0 \\ m(0) = m_0 \end{cases}$$

with $B = H_p(x, DU(x,m)) = H_p(x, D\bar U(x,m))$.
Notice: $\bar m$ is the unique invariant measure of the above equation.

Conclusions:

- For monotone (smooth) couplings F, G, the long time behavior and the vanishing discount limit of MFG systems can be fully characterized.
- In a large intermediate time the solution is nearly stationary with an exponential rate of proximity. This is similar to the turnpike property observed in optimal control problems in long horizons ([P.-Zuazua])
- The exponential rate proved crucial in order to establish that $u^T - \bar{\lambda}(T-t)$ is bounded and converges as $T \to \infty$. The boundary layers at initial-terminal conditions are fully characterized by studying the long time behavior of the master equation and the stationary limit feedback law.
- Extensions: similar results hold for Lipschitz (locally uniformly convex) Hamiltonians and local couplings. The monotonicity condition on the running cost can be slightly relaxed.
- The full analysis for the first order problem is still largely open

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Thanks for the attention !