Natural growth and beyond

Alessio Porretta
Università di Roma *Tor Vergata*

Espalia
Roma, 17/06/2015, 14th Anniversary

Elliptic equations with first order terms:

$$\begin{cases} -\Delta u + H(x, Du) = f(x) & \text{in} \ \ \Omega \\ u = 0 \qquad \text{on } \partial\Omega \end{cases}$$

- What is natural and what is unnatural growth ?
- Where is the border between natural and unnatural ?
- Role of gradient bounds and maximal solutions
- Additive eigenvalue and Dirichlet problem

What is the natural growth ? Why is it natural ?

Elliptic equations with first order terms:

$$\begin{cases} -\Delta u + H(x, Du) = f(x) & \text{in } \Omega \\ u = 0 \qquad \text{on } \partial\Omega \end{cases}$$

Natural growth (so called...) :

$$|H(x, Du)| \leq c(1 + |Du|^2)$$

Qn: Is it natural and why ?

Possible answers:

- Euler's equations in Calculus of Variations with quadratic energy:

$$\min \int_\Omega [a(x, u)\frac{|Du|^2}{2} - fu]dx \quad \to \quad -\mathrm{div}(a(x, u)Du) + \frac{1}{2}a'(x, u)|Du|^2 = f$$

- This class of equations is invariant through chain rule

$$-\Delta u + H(x, u, Du) = f(x) \quad \stackrel{v=\phi(u)}{\to} \quad -\Delta v + \tilde{H}(x, v, Dv) = f(x)$$

NB: Invariance holds for bounded solutions

What is natural ?

Under natural growth conditions:

- Smooth data $\Rightarrow$ bounded solutions are smooth
- General solvability (bounded and unbounded data, nonlinear operators, etc..)

  (since [Boccardo-Murat-Puel]...)
- Uniqueness of bounded weak solutions (since [Barles-Murat]...)

What is un-natural ?

Model case: superquadratic Hamiltonian:

$$-\Delta u + \lambda\, u + |Du|^q = f(x) \quad \text{with } q > 2$$

[Capuzzo Dolcetta-Leoni-P. '10]:
viscosity solutions framework (fully nonlinear)
$\rightarrow$ extends to $F(x, D^2 u) + \lambda\, u + |Du|^q \le f$,
F degenerate elliptic, $F \ge -\Lambda \|D^2 u\|$

Similar with p-Laplacian and $q > p$

(see also [Barles '10], [Barles-Koike-Ley-Topp '14] [Barles-Topp '15] for further extensions in viscosity solutions theory, connection to state constraint, nonlocal diffusions etc...)

[Dall'Aglio-P. '14]:
distributional solutions framework (divergence form)
$\rightarrow$ extends to $-\mathrm{div}(a(x, Du)) + \lambda\, u + |Du|^q \le f$,
a degenerate elliptic, $|a| \le \Lambda |Du|$

Similar with p-Laplacian and $q > p$

A list of un-natural properties:

$$-\Delta u + \lambda\, u + |Du|^q = f(x) \quad \text{with } q > 2$$

- Sub solutions are Hölder continuous

  [CD-L-P]: f bounded $\Rightarrow$ USC bounded viscosity subsolutions are $\frac{q-2}{q-1}$-Hölder. Proof by comparison:

$$u(x) \le u(y) + k\left(\frac{|x-y|}{d(x)^{1-\alpha}} + L\,|x-y|^{\alpha}\right)$$

  [D-P]: $f \in L^m, m > \frac{N}{q} \Rightarrow$ distributional subsolutions are α-Hölder with $\alpha = \min(1 - \frac{N}{m\,q}, 1 - \frac{1}{q-1})$. Proof by local Morrey estimate:

$$\int_{B_r} |\nabla u|^q\, dx \le K\, r^{N-\gamma}\,,$$

  where $\gamma = \max(\frac{N}{m}, q')$

[...]

$$-\Delta u + \lambda\, u + |Du|^q = f(x) \quad \text{with } q > 2$$

- Interior Hölder regularity extends up to the boundary (independently of boundary data !)
- Global universal bounds for u^+:

$$\|u^+\|_{L^\infty(\Omega)} \le M$$

where $M = M(\Omega, \frac{1}{\lambda}, \|f\|_{L^m(\Omega)})$, $m > N/q$.

$\rightarrow$ positive sub solutions have a uniform L^∞- bound, independently of boundary values(cfr. [Lasry-Lions '89])

- Loss of boundary data

$\rightarrow$ relaxed formulation of boundary conditions, viscosity solutions theory.

- Viscosity Vs distributional solutions:
  A selection criterion is necessary as in first order problems !!

  Ex: $u(x) = c_0(|x|^{\frac{q-2}{q-1}} - 1)$ satisfies, for a suitable choice of c_0

$$\begin{cases} -\Delta u + |\nabla u|^q = 0 & \text{in } \Omega \\ u \in W_0^{1,q}(\Omega) \cap C(\overline{\Omega}) \end{cases}$$

  Note: u is a distributional solution but not a viscosity solution !!
  Yet u is bounded, Hölder continuous etc...

  Typical first order problem: the L^∞-bound does not bring enough information...

  Uniqueness results for viscosity solutions:
  [Barles-Rouy-Souganidis '99], [Barles-Da Lio '04], [Barles '10]

The picture is more clear when looking at $W^{1,\infty}$- solutions.

[Alarcon-Garcia Melian-Quaas '14]:

$$\begin{cases} \lambda u - \Delta u + h(|Du|) = \mu f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with smooth f, $f < 0$, where $\lambda, \mu \geq 0$ are parameters.

(a) $$\int^\infty \frac{1}{h(s)} ds = \infty \quad \Rightarrow \quad \exists \text{ sol. for every } \lambda \geq 0, \text{ every } \mu.$$

(ex: h sublinear)

(b) $$\int^\infty \frac{s}{h(s)} ds = \infty \quad \Rightarrow \quad \exists \text{ sol. for every } \lambda > 0, \text{ every } \mu.$$

(ex: h subquadratic)

BUT: If $\lambda = 0$ or if $\int^\infty \frac{s}{h(s)} ds < \infty$, then there is a critical μ^*:

$$\exists \text{ sol. for every } \mu < \mu^* < \infty, \ \nexists \text{ sol for } \mu > \mu^*.$$

To understand clearly those thresholds, one need to go back to the classics:

[P-L. Lions '80]: If h is convex, there exists a $W^{1,\infty}$ solution to

$$\begin{cases} \lambda u - \Delta u + h(|Du|) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

if and only if there exists a $W^{1,\infty}$ sub solution ψ:

$$\exists \text{ sol. of (1)} \iff \exists \begin{cases} \lambda \psi - \Delta \psi + h(|D\psi|) \le f & \text{in } \Omega \\ \psi \in W_0^{1,\infty}(\Omega) \end{cases}$$

NB: h convex implies $h \ge h(0) + Dh(0) \cdot Du \Rightarrow$ a smooth super solution always exists.
(for right-wing parties, just reverse all signs: convex into concave, etc....)

Key point: gradient estimates !! $\to$ Bernstein's method.

The case of p-Laplacian, $p > 2$: gradient estimates

[Leonori-P.]
We extend Serrin-Lions classical gradient estimates to p-Laplace eq:

$$\lambda u - \Delta_p u + H(x, Du) = 0 \quad \text{in } \Omega\,,$$

where $\Delta_p u = \mathrm{div}(|Du|^{p-2}Du)$, $p > 2$. Typical results of Bernstein's method:

1. Global gradient bounds

Under structure conditions similar to [Lions '80], we get:

$$\sup_{\overline{\Omega}} |Du|^2 \leq c + \sup_{\partial\Omega} |Du|^2\,,$$

where $c = c(\|u\|_{L^\infty})$.

2. Interior bounds If H is suitably coercive (e.g. $H \geq h(|Du|^{p-1}) - c(x)$, with $\int^\infty \frac{d\tau}{h(\tau)} < \infty$) we get local bounds: for any $\omega \subset\subset \Omega$,

$$\sup_{x\in\omega} |Du_\varepsilon| \leq c_\omega$$

where c_ω only depends on $\lambda\|u^-\|_{L^\infty}$.

Consequence of gradient bounds:

1. Gradient estimates at the boundary $\Rightarrow$ existence of solutions (Lipschitz!): this is Lions's principle (extended to p-Laplacian)

$$\exists \text{ solution} \quad \Longleftrightarrow \quad \exists \begin{cases} \lambda\psi - \Delta_p \psi + h(|D\psi|^{p-1}) \leq f & \text{in} \quad \Omega \\ \psi \in W_0^{1,\infty}(\Omega) \end{cases}$$

$\rightarrow$ all you need is...a barrier !!

2. Interior gradient bounds hold whenever H is "sufficiently" superlinear:

$$\int^{\infty} \frac{1}{h(s)} ds < \infty$$

This is the threshold of super linearity.

Ex: $h(s) = s^q$, $q > 1$, but also $h(s) = s \log^q s$, $q > 1$.

Looking for barriers $\to$ two thresholds appear:
(i) (threshold of super linearity)
If $\int^{\infty} \frac{1}{h(s)} ds < \infty \quad \Rightarrow \quad$ exists U such that

$$\begin{cases} -\Delta p\, U + h(|DU|^{p-1}) & \text{is bounded} \\ \frac{\partial U}{\partial \nu} = +\infty & \text{at the boundary} \end{cases}$$

(ii) (threshold of natural growth)
(ii- a) If $\int^{\infty} \frac{s^{\frac{1}{p-1}}}{h(s)} ds = \infty \quad \Rightarrow \quad U \overset{x \to \partial\Omega}{\to} \infty$
(the barrier can be chosen as large as desired). In this case a bound for the L^∞-norm implies a control of the gradient norm.
$\to$ solutions exist iff bounded sub solutions exist !

(ii- b) If $\int^{\infty} \frac{s^{\frac{1}{p-1}}}{h(s)} ds < \infty \quad \Rightarrow \quad U$ has a finite maximal L^∞ bound
This is the super-natural growth: Solutions have universal upper bounds, loss of boundary conditions due to gradient singularity (Hölder-type behavior)

All depends on local behavior of **Maximal solutions**. Model case:

$$\begin{cases} \lambda U - \Delta p\, U + h(|DU|^{p-1}) = f & \text{is bounded} \\ \frac{\partial U}{\partial \nu} = +\infty & \text{at the boundary} \end{cases}$$

$$\int^{\infty} \frac{1}{h(s)} ds < \infty$$

(local gradient bounds)

$$\int^{\infty} \frac{\tau^{\frac{1}{p-1}}}{h(\tau)} d\tau = \infty \rightarrow \begin{cases} \text{Maximal sol} : U \overset{x\to\partial\Omega}{\rightarrow} +\infty \\ \text{Dirichlet pb:} \\ \exists L^{\infty}\text{- subsol.} \Rightarrow \exists \text{sol.} \end{cases}$$

$$\int^{\infty} \frac{\tau^{\frac{1}{p-1}}}{h(\tau)} d\tau < \infty \rightarrow \begin{cases} \text{Maximal sol} : U \text{ is bounded} \\ \text{Dirichlet pb:} \\ \exists W^{1,\infty}\text{- subsol.} \Rightarrow \exists \text{sol.} \end{cases}$$

The Dirichlet problem

Back to the Dirichlet problem

$$\begin{cases} \lambda u - \Delta_p u + H(x, Du) = 0 & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2}$$

with

$$H \gtrsim h(|Du|^{p-1}) - c(x)\,.$$

+some structure conditions for gradient bounds. Assume

$$\int^{\infty} \frac{\tau^{\frac{1}{p-1}}}{h(\tau)}\, d\tau = \infty \tag{3}$$

i.e. we are below the natural growth threshold. Then we have seen:

(i) for any $\lambda > 0$, (2) admits a solution $u \in W_0^{1,\infty}(\Omega)$

(ii) when $\lambda = 0$, (2) admits a solution if and only if there exists a bounded sub solution ψ such that $\psi = 0$ on $\partial\Omega$.

Pb: What really happens when $\lambda = 0$? And when $\lambda \to 0$?

The additive eigenvalue

The existence of solutions when $\lambda = 0$ can be described in terms of a nonlinear additive eigenvalue related to the maximal solutions.

Model case:

$$\begin{cases} -\Delta_p u + |Du|^q = f(x) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4}$$

with $p - 1 < q \leq p$.

Theorem (Leonori-P.)

1. *There exists a unique constant $c_0 = c_0(f, \Omega)$ such that the problem*

$$\begin{cases} c_0 - \Delta p\, V + |DV|^q = f & \textit{is bounded} \\ V \overset{x \to \partial\Omega}{\to} +\infty & [\iff \frac{\partial V}{\partial \nu} = +\infty \quad \textit{at the boundary}] \end{cases}$$

admits a solution.

2. *We have*

$c_0 > 0 \ \Rightarrow$ *the Dirichlet problem (4) admits a solution*

$c_0 < 0 \ \Rightarrow$ *the Dirichlet problem (4) does not admit any solution*

- If $c_0(f,\Omega)=0$ we conjecture that no solution exists to the Dirichlet problem. In that case we have a full characterization: solutions to the Dirichlet problem exist if and only if $c_0(f,\Omega)>0$.
This is true if $p=2$ [P. '10] and if $\mathrm{osc}(f)$ is not too large [Leonori-P.].

- If $q=p$:

$$\begin{cases} c_0 - \Delta p\, V + |DV|^p = f \quad \text{is bounded} \\ V \overset{x\to\partial\Omega}{\to} +\infty \qquad [\iff \frac{\partial V}{\partial \nu} = +\infty \quad \text{at the boundary}] \end{cases}$$

then c_0 is a classical eigenvalue: $c_0 = \lambda_1\left(-\Delta_p(\cdot) + f(x)(\cdot)^{p-1}\right)$ and $e^{-V/(p-1)}$ is the first eigenfunction of $-\Delta_p(\cdot)+f(x)(\cdot)^{p-1}$.

- The additive constant c_0 and the corresponding eigenfunction V describe the long time behavior of the evolution problem whenever there are no stationary solutions ($c_0<0$). For the case $p=2$, see [Barles-P.-Tabet Tchamba '10] $\to$ c_0 is an *ergodic constant*

- In the case $p=2$ the additive eigenvalue c_0 satisfies Faber-Krahn inequalities [Ferone-Giarrusso-Messano-Posteraro '14].

Summary

$$\begin{tikzcd}
\text{Maximal solutions} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, phantom, "\textit{GRADIENT\quad BOUNDS}"] & \text{Additive eigenvalue} \arrow[d, leftrightarrow] \\
\text{Long time behavior} \arrow[r, leftrightarrow] & \text{Stationary Dirichlet pb.}
\end{tikzcd}$$

References.

1. I. Capuzzo Dolcetta, F. Leoni, A. Porretta, *Hölder estimates for degenerate elliptic equations with coercive Hamiltonians*, Trans. Amer. Math. Soc. **362** (2010).
2. A. Porretta, *The "ergodic limit" for a viscous Hamilton-Jacobi equation with Dirichlet conditions*, Rend. Lincei **21** (2010).
3. G. Barles, A. Porretta, T. Tabet Tchamba, *On the Large Time Behavior of Solutions of the Dirichlet problem for Subquadratic Viscous Hamilton-Jacobi Equations*, J. Math. Pures Appl. **94** (2010).
4. A. Dall'Aglio, A. Porretta, *Local and global regularity of weak solutions of elliptic equations with superquadratic Hamiltonian*, Trans. Amer. Math. Soc. 367 (2015).
5. T. Leonori, A. Porretta, *Large solutions and gradient bounds for quasilinear elliptic equations*, CPDE (2016).