PROBLEMS

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PROPOSALS

To be considered for publication, solutions should be received by September 1, 2011.

1866. *Proposed by Sadi Abu-Saymeh and Mowaffaq Hajja, Mathematics Department, Yarmouk University, Irbid, Jordan.*

Let ABC be a triangle, and L and M points on $\overline{AB}$ and $\overline{AC}$, respectively, such that $AL = AM$. Let P be the intersection of $\overline{BM}$ and $\overline{CL}$. Prove that $PB = PC$ if and only if $AB = AC$.

1867. *Proposed by Ángel Plaza and César Rodríguez, Department of Mathematics, Universidad de Las Palmas de Gran Canaria, Las Palmas, Spain.*

Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $\int_0^1 f(t)\,dt = 1$ and n a positive integer. Show that

1. there are distinct $c_1, c_2, \ldots, c_n$ in $(0, 1)$ such that

$$f(c_1) + f(c_2) + \cdots + f(c_n) = n,$$

2. there are distinct $c_1, c_2, \ldots, c_n$ in $(0, 1)$ such that

$$\frac{1}{f(c_1)} + \frac{1}{f(c_2)} + \cdots + \frac{1}{f(c_n)} = n.$$

1868. *Proposed by Donald E. Knuth, Stanford University, Stanford, CA.*

Let $n \geq 2$ be an integer. Remove the central $(n-2)^2$ squares from an $(n+2) \times (n+2)$ array of squares. In how many ways can the remaining squares be covered with $4n$ dominoes?

Math. Mag. **84** (2011) 150–157. doi:10.4169/math.mag.84.2.150.

We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals must, in general, be accompanied by solutions and by any bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. Submitted problems should not be under consideration for publication elsewhere.

Solutions should be written in a style appropriate for this MAGAZINE.

Solutions and new proposals should be mailed to Bernardo M. Ábrego, Problems Editor, Department of Mathematics, California State University, Northridge, 18111 Nordhoff St, Northridge, CA 91330-8313, or mailed electronically (ideally as a LaTeX or pdf file) to mathmagproblems@csun.edu. All communications, written or electronic, should include **on each page** the reader's name, full address, and an e-mail address and/or FAX number.

1869. *Proposed by Marian Duncă, Bucharest, Romania.*

Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing and concave-down function such that $f(0) = 0$. Prove that if x, y, and z are real numbers, and a, b, and c are the lengths of the sides of a triangle, then

$$(x-y)(x-z)f(a) + (y-x)(y-z)f(b) + (z-x)(z-y)f(c) \geq 0.$$

1870. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Calculate

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{m(\zeta(n+m)-1)}{(n+m)^2},$$

where ζ denotes the Riemann Zeta function.

Quickies

Answers to the Quickies are on page 156.

Q1009. *Proposed by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Roma, Italy.*

Let $H_n = \sum_{k=1}^{n} 1/n$. Using the fact that $\sum_{k=1}^{\infty} 1/k^2 = \pi^2/6$, calculate $\sum_{k=1}^{\infty} H_k/k^3$.

Q1010. *Proposed by Michael W. Botsko, Saint Vincent College, Latrobe, PA.*

Let $f : [0, 1] \to \mathbb{R}$ be a continuous real valued function with a continuous nonzero derivative on $(0, 1]$. Prove that if $f(0) = 0$, then $\liminf_{x\to 0^+} f(x)/f'(x) = 0$.

Solutions

Every integer in the list divides the sum **April 2010**

1841. *Proposed by H. A. ShahAli, Tehran, Iran.*

Let $n \geq 3$ be a natural number. Prove that there exist n pairwise distinct natural numbers such that each of them divides the sum of the remaining $n-1$ numbers.

I. *Solution by Northwestern University Math Problem Solving Group, Evanston, IL.*

The list of numbers $1, 2, 3 \cdot 2^0, 3 \cdot 2^1, 3 \cdot 2^2, \ldots, 3 \cdot 2^{n-3}$ has the required property. The sum of all those numbers is

$$1 + 2 + 3 \cdot 2^0 + 3 \cdot 2^1 + 3 \cdot 2^2 + \cdots + 3 \cdot 2^{n-3} = 3 + 3 \cdot (2^{n-2} - 1) = 3 \cdot 2^{n-2}.$$

Each number in the list divides the total sum, and that implies the desired condition.

II. *Solution by Michael Goldenberg, The Ingenuity Project, Baltimore Polytechnic Institute, Baltimore, MD; and Mark Kaplan, The Community College of Baltimore County, Baltimore, MD.*

We choose natural numbers m_k given by

$$m_k = \begin{cases} n!\left(\frac{1}{k!} - \frac{1}{(k+1)!}\right) = n! \cdot \frac{k}{(k+1)!} & \text{if } 1 \leq k \leq n-1, \\ n! \cdot \frac{1}{n!} = 1 & \text{if } k = n. \end{cases}$$

If $n \geq 3$, then $m_1 > m_2 > \cdots > m_{n-1} > m_n$. In addition

$$S = \sum_{k=1}^{n} m_k = n!\left(\left(1 - \frac{1}{2!}\right) + \left(\frac{1}{2!} - \frac{1}{3!}\right) + \cdots + \left(\frac{1}{(n-1)!} - \frac{1}{n!}\right) + \frac{1}{n!}\right) = n!,$$

and $S - m_k$ is a multiple of m_k for $1 \leq k \leq n$.

Editor's Note. Harris Kwong and Nicholas Singer (independently) proved that the only solution for $n = 3$ is $(a, 2a, 3a)$. Erwin Just observes that this problem is a direct Corollary of a problem proposed by him. [Problem 1504, this MAGAZINE **70** (1997), 300.] Reiner Martin and Dmitry Fleischman (independently) provide an insight into a way of classifying all possible solutions which can be completed as follows: If $m_1 < m_2 < \cdots < m_n$ satisfy that the sum $S = m_1 + m_2 + \cdots + m_n$ is divisible by all m_k, say $S = m_k \cdot d_k$, then $d_1 > d_2 > \cdots > d_n$ and

$$\sum_{i=1}^{n} \frac{m_k}{S} = \sum_{i=1}^{n} \frac{1}{d_k} = 1.$$

Reciprocally, if the positive integers $d_1 > d_2 > \cdots > d_n$ satisfy that $\sum_{i=1}^{n}(1/d_k) = 1$, then by letting S be the least common multiple of the d_k and $S = m_k \cdot d_k$, it follows that m_k divides S and

$$\sum_{i=1}^{n} \frac{S}{d_k} = \sum_{i=1}^{n} m_k = S.$$

Thus the classification problem is equivalent to finding all possible partitions of 1 into n different fractions with numerator 1 (called Egyptian Fractions). The first solution is obtained from the partition $1 = 1/2 + 1/3 + 1/6$ by recursively dividing by 2 and adding $1/2$ on both sides. In fact the greedy algorithm can complete any partial sum $1/m_1 + 1/m_2 + \cdots + 1/m_k < 1$ to a partition $1 = 1/m_1 + 1/m_2 + \cdots + 1/m_l$ for some $l > m$. However the complete classification is still an open problem. Some references and related open problems can be found in R. K. Guy, *Unsolved Problems in Number Theory*, Springer-Verlag, 1981, pp. 87–93; and in V. Klee and S. Wagon, *Old and New Unsolved Problems in Plane Geometry and Number Theory*, Mathematical Association of America, 1991, pp. 175–177 and 206–208.

Also solved by Con Amore Problem Group (Denmark); Michel Bataille (France); Brian D. Beasley; D. Bednarchak; Gareth Bendall; Jany C. Binz (Switzerland); Lataianu Bogdan (Canada); Paul Budney; Robert Calcaterra; Michael J. Caulfield; Hyeong Min Choe (Korea) and Jong Jin Park (Korea); John Christopher; CMC 328; Tim Cross (United Kingdom); Chip Curtis; Robert L. Doucette; Toni Ernvall (Finland); Dmitry Fleischman; Fullerton College Math Association; Stefania Garasto (Italy); David Getling (Germany); Eugene A. Herman; Chris Hill; Dan Jurca; Peter Hohler (Switzerland); Bianca–Teodora Iordache (Romania); Omran Kouba (Syria); Victor Y. Kutsenok; Harris Kwong; Elias Lampakis (Greece); Kathleen E. Lewis (Republic of the Gambia); Daniel Lucas, Rachel White, and Meghan Loid; Reiner Martin (Germany); Shoeleh Mutameni; Pedro Perez; Ángel Plaza (Spain); Henry Ricardo; R. Keith Roop-Eckart; Daniel M. Rosenblum; Joel Schlosberg; Harry Sedinger; Seton Hall Problem Solving Group; Achilleas Sinefakopoulos (Greece); Nicholas C. Singer; David Stone and John Hawkins; Taylor University Problem Solving Group; Marian Tetiva (Romania); Texas State Problem Solvers Group; Bob Tomper; Michael Vowe (Switzerland); Stanley Xiao (Canada); and the proposer.

Perpendicular hexagon skewers **April 2010**

1842. *Proposed by Bianca-Teodora Iordache, student, National College "Carol I," Craiova, Romania.*

In the interior of a square of side-length 3 there are several regular hexagons whose sum of perimeters is equal to 42 (the hexagons may overlap). Prove that there are two perpendicular lines such that each one of them intersects at least five of the hexagons.

Solution by CMC 328, Carleton College, Northfield, MN.

We first claim that when we project a regular hexagon of side length a onto a line its shortest possible projection is $a\sqrt{3}$. To see this, observe that we can inscribe a circle of radius $a\sqrt{3}/2$ within the hexagon, and the projection of the hexagon is greater than or equal to the inscribed circle's projection.

Now let us project all the hexagons onto an edge of the square. Since the sum of all the hexagons' perimeters is 42, the sum of all of their side-lengths is 7. Hence, their projection length on one edge of the square is at least $7\sqrt{3} \approx 12.124$. Since all of these projections are onto a segment of length 3, and $3(4) < 7\sqrt{3}$, there must be some region in the segment covered by at least five of the projections. Pick a point in this region and draw a line through this point perpendicular to the edge; this line must intersect at least five hexagons. By carrying out this construction for two perpendicular edges of the square, we get the desired two perpendicular lines.

Also solved by Robert Calcaterra, David Getling (Germany), Victor Y. Kutsenok, Charles Martin, and the proposer.

Permutations with specified left-to-right maxima **April 2010**

1843. *Proposed by José Heber Nieto, Universidad del Zulia, Maracaibo, Venezuela.*

For every positive integer n, let S_n denote the set of permutations of the set $N_n = \{1, 2, \ldots, n\}$. For every $1 \leq j \leq n$, the permutation $\sigma \in S_n$ has a *left to right maximum* (LRM) at position j, if $\sigma(i) < \sigma(j)$ whenever $i < j$. Note that all $\sigma \in S_n$ have a LRM at position 1. Let M be a subset of N_n. Prove that the number of permutations in S_n with LRMs at exactly the positions in M is equal to

$$\prod_{k \in N_n \setminus M} (k-1),$$

where an empty product is equal to 1.

Solution by Robert Calcaterra, University of Wisconsin-Platteville, Platteville, WI.

If $1 \notin M$, the assertion is clearly true so we may assume that $1 \in M$. Let α be the permutation in S_n having its LRMs at exactly the positions in M. We determine the number of ways to choose α. Let $P(x, y)$ be the number of permutations of y elements selected from a set of x elements, which is known to be $x!/(x-y)!$; and let $m_1, m_2, \ldots, m_k = 1$ be the elements of M in descending order. Observe that n must occupy position m_1 in α. Then there are $P(n-1, n-m_1)$ ways to choose the elements of N_n that occupy positions $m_1 + 1$ to n in α. Of the elements of N_n that have not yet been assigned a position in α, the largest one must be assigned to position m_2. Consequently, we may now choose the elements of N_n that occupy positions $m_2 + 1$ to $m_1 - 1$ in $P(m_1 - 2, m_1 - m_2 - 1)$ different ways. Repeating this argument, there are

$$\prod_{j=1}^{k} P(m_{j-1} - 2, m_{j-1} - m_j - 1)$$

ways to choose α, where $m_0 = n + 1$. Since $(m_j - 2)!/(m_j - 1)! = 1/(m_j - 1)$ for $0 < j < k$, this product may be reduced to

$$(n-1)! \Big/ \prod_{j=1}^{k-1} (m_j - 1).$$

This expression is equivalent to the product stated in the problem.

Also solved by Con Amore Problem Group (Denmark), Chip Curtis, Robert L. Doucette, Joe McKenna (Ghana), Joel Schlosberg, John H. Smith, Marian Tetiva, Stanley Xiao (Canada), and the proposer. There was one incorrect submission.

A geometric inequality for the secants of a triangle — April 2010

1844. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu," Bîrlad, Romania.*

Let ABC be a triangle with $a = BC$, $b = AC$, and $c = AB$. Prove that

$$\frac{a^2+b^2+c^2}{2\cdot \text{Area}(ABC)} \geq \sec\frac{A}{2} + \sec\frac{B}{2} + \sec\frac{C}{2}.$$

Solution by Felipe Pérez (student), Facultad de Física, P. Universidad Católica de Chile, Santiago, Chile.

Let $s = (a+b+c)/2$ be the semiperimeter of the triangle ABC. Using the Half-angle Formula and the Law of Cosines gives

$$\cos^2\left(\frac{A}{2}\right) = \frac{1}{2}(\cos A + 1) = \frac{1}{2}\left(\frac{b^2+c^2-a^2+2bc}{2bc}\right) = \frac{s(s-a)}{bc}.$$

Thus

$$\sec\frac{A}{2} = \sqrt{\frac{bc}{s(s-a)}}, \quad \sec\frac{B}{2} = \sqrt{\frac{ac}{s(s-b)}}, \quad \text{and} \quad \sec\frac{C}{2} = \sqrt{\frac{ab}{s(s-c)}}.$$

Then by Heron's Formula for the area of $\triangle ABC$,

$$\begin{aligned}
&\sec\frac{A}{2} + \sec\frac{B}{2} + \sec\frac{C}{2}\\
&\quad = \sqrt{\frac{bc}{s(s-a)}} + \sqrt{\frac{ac}{s(s-b)}} + \sqrt{\frac{ab}{s(s-c)}}\\
&\quad = \frac{\sqrt{b(s-c)\cdot c(s-b)} + \sqrt{a(s-c)\cdot c(s-a)} + \sqrt{a(s-b)\cdot b(s-a)}}{\sqrt{s(s-a)(s-b)(s-c)}}\\
&\quad = \frac{\sqrt{b(s-c)\cdot c(s-b)} + \sqrt{a(s-c)\cdot c(s-a)} + \sqrt{a(s-b)\cdot b(s-a)}}{\text{Area}(ABC)}.
\end{aligned}$$

Using the Arithmetic Mean–Geometric Mean Inequality (the positiveness of each factor is justified by triangle inequality) gives

$$\sqrt{b(s-c)\cdot c(s-b)} \leq \frac{b(s-c)+c(s-b)}{2},$$

and equivalent inequalities for the other two summands. Finally,

$$\begin{aligned}
\sec\frac{A}{2} + \sec\frac{B}{2} + \sec\frac{C}{2} &\leq \frac{1}{2\cdot \text{Area}(ABC)}\big(s(2a+2b+2c) - (2ab+2ac+2bc)\big)\\
&\leq \frac{1}{2\cdot \text{Area}(ABC)}(a^2+b^2+c^2).
\end{aligned}$$

The equality holds if and only if $a = b = c$.

Also solved by George Apostolopoulos (Greece); Dionne Bailey, Elsie Campbell, and Charles Diminnie; Michel Bataille (France); Scott H. Brown; Minh Can; Tim Cross (United Kingdom); Chip Curtis; Marian Dincă; Robert L. Doucette; John N. Fitch; A. Bathi Kasturiarachi; Omran Kouba (Syria); Elias Lampakis (Greece); Kee-Wai Lau (China); Shoeleh Mutameni; Pedro Perez; Henry Ricardo; Achilleas Sinefakopoulos (Greece); Michael Vowe (Switzerland); Haohao Wang and Jerzy Woydylo; John Zerger; and the proposer.

Integrating a square-fractional-reciprocal function **April 2010**

1845. *Proposed by Albert F. S. Wong, Temasek Polytechnic, Singapore.*

Evaluate

$$\int_0^1 \left\{\frac{1}{x}\right\}^2 dx,$$

where $\{\alpha\} = \alpha - \lfloor \alpha \rfloor$ denotes the fractional part of α.

Solution by Allen Stenger, Alamogordo, NM.

Make the change of variable $x = 1/t$ to get

$$\int_0^1 \left\{\frac{1}{x}\right\}^2 dx = \int_1^\infty \frac{\{t\}^2}{t^2} = \sum_{k=1}^\infty \int_k^{k+1} \frac{(t-k)^2}{t^2} dt.$$

Then expand the integrands to get

$$\begin{aligned}\int_k^{k+1} \frac{(t-k)^2}{t^2} dt &= \int_k^{k+1} \left(1 - \frac{2k}{t} + \frac{k^2}{t^2}\right) dt \\ &= 1 - 2k\ln(k+1) + 2k\ln k + \frac{k^2}{k(k+1)} \\ &= 2 + 2\ln(k+1) - \big(2(k+1)\ln(k+1) - 2k\ln k\big) - \frac{1}{k+1}.\end{aligned}$$

Adding these terms from $k = 1$ to $n - 1$, noting the telescoping sum, and rearranging gives

$$\begin{aligned}\sum_{k=1}^{n-1} \int_k^{k+1} \frac{(t-k)^2}{t^2} dt &= 2n - 2 + 2\ln(n!) - 2n\ln n - \sum_{k=1}^{n-1} \frac{1}{k+1} \\ &= 2\left(\ln(n!) - \left(n + \frac{1}{2}\right)\ln n + n\right) - \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n\right) - 1 \\ &= 2\ln\left(\frac{n!}{n^{n+1/2}e^{-n}}\right) - \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n\right) - 1.\end{aligned}$$

Stirling's formula implies that the first term goes to $\ln(2\pi)$ as $n \to \infty$. From the definition of Euler's constant γ the second term goes to $-\gamma$, so the final result is

$$\int_0^1 \left\{\frac{1}{x}\right\}^2 dx = \ln(2\pi) - \gamma - 1 \approx 0.260661.$$

Editor's Note. Some readers pointed out that the problem of calculating the Riemann sums of this integral appeared as Problem 11206, *Amer. Math. Monthly* **114** (2007), 928–929. Ovidiu Furdui mentions that evaluating $\int_0^1 \{k/x\}^2\, dx$ for a positive integer k was published as Problem U27, *Mathematical Reflections* **6** (2006).

Paolo Perfetti, Dmitry Fleischman, and Joel Schlosberg (independently) obtained $1+2\sum_{r=2}^{\infty}(-1)^{r+1}\zeta(r)/(r+1)$ as the answer for this problem. Ovidiu Furdui considered the more general problem of finding $\int_0^1\{1/x\}^k\,dx$ for integer $k\geq 1$. He showed that the answer in this case is $\sum_{r=1}^{\infty}(\zeta(r+1)-1)/\binom{k+r}{r}$.

Also solved by Armstrong Problem Solvers, Michel Bataille (France), Dennis K. Beck, Lataianu Bogdan (Canada), Paul Budney, Robert Calcaterra, Hongwei Chen, John Christopher, Chip Curtis, Richard Daquila, Paul Deiermann, Robert L. Doucette, Dmitry Fleischman, Jet Foncannon, Ovidiu Furdui (Romania), Michael Goldenberg and Mark Kaplan, G.R.A.20 Problem Solving Group (Italy), J. A. Grzesik, Timothy Hall, Gerald A. Heuer, Dan Jurca, Kamil Karayilan (Turkey), Omran Kouba (Syria), Harris Kwong, Elias Lampakis (Greece), David P. Lang, Longxiang Li (China) and Luyuan Yu (China), Masao Mabuchi (Japan), Charles Martin, Reiner Martin (Germany), Kim McInturff, Matthew McMullen, Peter McPolin (Northern Ireland), Paolo Perfetti (Italy), Ángel Plaza (Spain), R. Keith Roop-Eckart, Ossama A. Saleh and Terry J. Walters, Joel Schlosberg, Edward Schmeichel, Seton Hall Problem Solving Group, Nicholas C. Singer, David Stone and John Hawkins, Marian Tetiva (Romania), Bob Tomper, Jan Verster (Canada), Francisco Vial (Chile), Michael Vowe (Switzerland), Stan Wagon, Haohao Wang and Jerzy Woydylo, Vernez Wilson and Farley Mawyer, John Zacharias, and the proposer. There were two incorrect submissions.

Answers

Solutions to the Quickies from page 151.

A1009. The answer is $\pi^4/72$. For n and k positive integers,

$$\frac{1}{n(k+n)}=\frac{1}{k}\left(\frac{1}{n}-\frac{1}{(k+n)}\right).$$

Thus

$$\frac{1}{n^2(k+n)^2}=\frac{1}{k^2n^2}+\frac{1}{k^2(k+n)^2}-\frac{2}{k^3}\left(\frac{1}{n}-\frac{1}{k+n}\right).$$

It follows by symmetry that

$$\begin{aligned}\frac{\pi^4}{36}&=\sum_{k=1}^{\infty}\frac{1}{k^2}\sum_{n=1}^{\infty}\frac{1}{n^2}=\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{k^2n^2}\\&=\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{1}{n^2(k+n)^2}-\frac{1}{k^2(k+n)^2}+\frac{2}{k^3}\left(\frac{1}{n}-\frac{1}{k+n}\right)\right)\\&=\sum_{k=1}^{\infty}\frac{2}{k^3}\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{k+n}\right)=\sum_{k=1}^{\infty}\frac{2}{k^3}\sum_{n=1}^{k}\frac{1}{n}=\sum_{k=1}^{\infty}\frac{2}{k^3}H_k.\end{aligned}$$

The result follows after dividing by 2 both sides of the equality.

A1010. Because f' satisfies the Intermediate Value Property, f' is either always positive or always negative on (0, 1]. Replacing if necessary f by $-f$, we can assume f' is positive on (0, 1]. Then f is also positive on (0, 1] and thus

$$\liminf_{x\to 0^+}\frac{f(x)}{f'(x)}\geq 0.$$

Suppose $\liminf_{x\to 0^+} f(x)/f'(x)>0$ and let A be a positive number such that $A<\liminf_{x\to 0^+} f(x)/f'(x)$. Then there exists δ, $0<\delta<1$, such that $f(x)/f'(x)>A$ for

$0 < x < \delta$. Therefore $f'(x)/f(x) < 1/A$ for $0 < x < \delta$ and thus

$$\ln\left(\frac{f(\delta)}{f(x)}\right) = \int_x^\delta \frac{f'(t)}{f(t)}dt \leq \int_x^\delta \frac{1}{A}dt = \frac{1}{A}(\delta - x).$$

It follows that $f(x) \geq f(\delta)e^{(x-\delta)/A}$ for $0 < x < \delta$. Taking limits we get

$$f(0) = \lim_{x\to 0^+} f(x) \geq f(\delta)e^{-\delta/A} > 0.$$

This is a contradiction, therefore $\liminf_{x\to 0^+} f(x)/f'(x) = 0$.