$e^{i\pi}=-1$
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József Wildt International Mathematical Competition

The Edition XXXth , 2020 [1]

The solution of the problems W.1 - W.60 must be mailed before 26. October 2020, to Mihály Bencze, str. Hărmanului 6, 505600 Săcele - Négyfalu, Jud. Braşov, Romania,
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W1. Consider the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{b^2}=1$$

(a and $b>0$) and the ellipse E which is the intersection of the ellipsoid with the plane of equation

$$mx+ny+pz=0$$

where the point $P=[m,n,p]$ is a random point from the unit sphere ($m^2+n^2+p^2=1$). Consider the random variable A_E the area of the ellipse E. If the point P is chosen with uniform distribution with respect to the area on the unit sphere, what is the expectation of A_E ?

Eugen J. Ionaşcu

W2. Let $(a_n)_{n\geq 1}$ be a sequence of nonnegative real numbers which converges to $a\in\mathbb{R}$.
a). Calculate

$$\lim_{n\to\infty}\sqrt[n]{\int_0^1 (1+a_nx^n)^n\,\mathrm{d}x}.$$

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b). Calculate

$$\lim_{n\to\infty} \sqrt[n]{\int_0^1 \left(1+\frac{a_1x+a_3x^3+\cdots+a_{2n-1}x^{2n-1}}{n}\right)^n \mathrm{d}x}.$$

Ovidiu Furdui and Alina Sîntămărian

W3. Let $n \geq 2$ be an integer. Calculate

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin^{2n-1} x+\cos^{2n-1} x} \mathrm{d}x.$$

Ovidiu Furdui and Alina Sîntămărian

W4. Let $(a_n)_{n\geq 1}$ be a positive real sequence such that

$$\lim_{n\to\infty} \frac{a_n}{n} = a \in R_+^* \text{ and } \lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right)^n = b \in R_+^*$$

Compute

$$\lim_{n\to\infty} (a_{n+1} - a_n)$$

D.M. Bătineţu-Giurgiu and Neculai Stanciu

W5. Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be positive real sequences such that

$$\lim_{n\to\infty} \frac{a_{n+1}-a_n}{n} = a \in R_+^* \text{ and } \lim_{n\to\infty} \frac{b_{n+1}}{nb_n} = b \in R_+^*$$

Compute

$$\lim_{n\to\infty} \left(\frac{a_{n+1}}{\sqrt[n+1]{b_{n+1}}} - \frac{a_n}{\sqrt[n]{b_n}}\right)$$

D.M. Bătineţu-Giurgiu and Neculai Stanciu

W6. Determine the function $f : (0, \pi) \to R$ which satisfy

$$f'(x) = \frac{\cos 2020x}{\sin x}$$

for any real $x \in (0, \pi)$.

D.M. Bătineţu-Giurgiu and Neculai Stanciu

W7. If $a, b > 0$ then:

$$\left(\frac{a+b}{2} - \frac{2ab}{a+b}\right)\arctan(\sqrt{ab}) + \left(\frac{2ab}{a+b} - \sqrt{ab}\right)\arctan\left(\frac{a+b}{2}\right) +$$

$$+\left(\sqrt{ab} - \frac{a+b}{2}\right)\arctan\left(\sqrt{\frac{a^2+b^2}{2}}\right) \geq 0$$

Daniel Sitaru

W8. If $a, b > 0$ then:

$$\left(\frac{a+b}{2} - \frac{2ab}{a+b}\right)\arctan\left(\frac{\sqrt{2ab} - \sqrt{a^2+b^2}}{\sqrt{2} + \sqrt{ab}(a^2+b^2)}\right) +$$

$$+\left(\sqrt{\frac{a^2+b^2}{2}} - \sqrt{ab}\right)\arctan\left(\frac{(a-b)^2}{2+2ab}\right) \geq 0$$

Daniel Sitaru

W9. In any triangle ABC the following relationship holds:

$$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} \geq 93312r^6$$

D.M. Bătineţu-Giurgiu and Daniel Sitaru

W10. Let be $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}, a_n, b_n \in \mathbb{R}_+^* = (0, \infty)$ such that $\lim_{n\to\infty} a_n = a \in \mathbb{R}_+^*$ and $(b_n)_{n\geq 1}$ is a bounded sequence. If $(x_n)_{n\geq 1}, x_n = \prod_{k=1}^n (ka_h + b_h)$ find:

$$\lim_{n\to\infty} \left(\sqrt[n+1]{x_{n+1}} - \sqrt[n]{x_n} \right)$$

D.M. Bătineţu-Giurgiu and Daniel Sitaru

W11. If $a, b, c \in \mathbb{N} \setminus \{0, 1, 2, 3\}$ then:

$$b^2 \cdot \sqrt[a]{a} + c^2 \cdot \sqrt[b]{b} + a^2 \cdot \sqrt[c]{c} \geq 48\sqrt{2}$$

Daniel Sitaru

W12. If $m, n, p, q \in \mathbb{N}; m, n, p, q \geq 4$ then:

$$4^n(4^n+1) + 4^m(4^m+1) + 4^p(4^p+1) + 4^q(4^q+1) \geq 4mnpq(mnpq+1)$$

Daniel Sitaru

W13. Count the number N of all sets $A := \{x_1, x_2, x_3, x_4\}$ of non-negative integers satisfying

$$x_1 + x_2 + x_3 + x_4 = 36$$

in at least four different ways.

Eugen J. Ionaşcu

W14. Let $\{F_n\}_{n\geq 1}$ be the Fibonacci sequence defined by $F_1 = F_2 = 1$ and for all $n \geq 3$,

$$F_n = F_{n-1} + F_{n-2}$$

Prove that among the first 10 000 000 000 000 002 terms of the sequence there is one term that ends up with 8 zeroes.

José Luis Díaz-Barrero

W15. Show that the number

$$4 \sin \frac{\pi}{34} \left(\sin \frac{3\pi}{34} + \sin \frac{7\pi}{34} + \sin \frac{11\pi}{34} + \sin \frac{15\pi}{34} \right)$$

is an integer and determine it.

José Luis Díaz-Barrero

W16. Prove that:

$$\left[10^{n+3} \cdot \sqrt{\overline{\underbrace{11...1}_{2n \text{ times}}}} \right] = \overline{\underbrace{33...3}_{2n \text{ times}}166}$$

for any $n \in N^*$.

Ovidiu Pop

W17. Let $(K, +, \cdot)$ be a field with the property $-x = x^{-1}$, $\forall x \in K$, $x \neq 0$. Prove that:

$$(K, +, \cdot) \simeq (Z_2, +, \cdot)$$

Ovidiu Pop

W18. Let $D := \{(x, y) \mid x, y \in \mathbb{R}_+,\ x \neq y \text{ and } x^y = y^x\}$.
(Obvious that $x \neq 1$ and $y \neq 1$).
And let $\alpha \leq \beta$ be positive real numbers. Find

$$\inf_{(x,y)\in D} x^\alpha y^\beta.$$

Arkady Alt

W19. Prove inequality

$$\prod_{k=2}^{n}\left(1+\frac{k^{p-1}}{1^p+2^p+...+k^p}\right)<e^{(p-1)/2}$$

Arkady Alt

W20. Let $p \in (0, 1)$ and $a > 0$ be real numbers. Determine asymptotic behavior of the sequence $\{a_n\}_{n=1}^{\infty}$ defined recursively

$$a_1 = a, a_{n+1} = \frac{a_n}{1+a_n^p}, n \in \mathbb{N}$$

(a_n asymptotically equivalent to function $\varphi(n)$ if $\lim\limits_{n\to\infty} \dfrac{a_n}{\varphi(n)} = 1$).

Arkady Alt

W21. Evaluate

$$\lim_{n\to\infty}\left(\frac{1+\frac{1}{3}+\cdots+\frac{1}{2n+1}}{\ln\sqrt{n}}\right)^{\ln\sqrt{n}}.$$

Ángel Plaza

W22. Prove that

$$\operatorname{Re}\left[\operatorname{Li}_2\left(\frac{1-i\sqrt{3}}{2}\right)+\operatorname{Li}_2\left(\frac{\sqrt{3}-i}{2\sqrt{3}}\right)\right]=\frac{7\pi^2}{72}-\frac{\ln^2 3}{8}$$

where as usual

$$\operatorname{Li}_2(z) = -\int_0^z \frac{\ln(1-t)}{t}dt,$$

$z \in \mathbb{C}\backslash[1, \infty)$

Paolo Perfetti

W23. Prove that

$$\int_{\pi/6}^{\pi/3} \frac{u}{du} \sin u = \frac{8}{3}\sum_{k=0}^{\infty} \frac{(-1)^k}{3^k(2k+1)^2} + \frac{\pi \ln 3}{3\sqrt{3}} - \frac{4C}{3} + \frac{\pi}{6}\ln(2+\sqrt{3}) +$$

$$-\mathrm{Im}\left[\frac{2}{\sqrt{3}}\mathrm{Li}_2\left(\frac{1-i\sqrt{3}}{2}\right) - \frac{2}{\sqrt{3}}\mathrm{Li}_2\left(\frac{\sqrt{3}-i}{2\sqrt{3}}\right)\right]$$

where as usual

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\ln(1-t)}{t}dt,$$

$z \in \mathbb{C}\backslash[1,\infty)$ and $C = \sum\limits_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$ is the Catalan constant

Paolo Perfetti

W24. Let $M = \{3,4,5,6,7,8,9,10,11,12,13,15,17,19,21,23\}$ be. Prove that for any $a_i > 0$, $i = \overline{1,n}$, $n \in M$, inequality occurs:

$$\frac{a_1^2}{(a_2+a_3)^4} + \frac{a_2^2}{(a_3+a_4)^4} + \dots + \frac{a_{n-1}^2}{(a_n+a_1)^4} + \frac{a_n^2}{(a_1+a_2)^4} \geq \frac{n^3}{16s^2},$$

where $s = \sum\limits_{i=1}^{n} a_i$.

Marius Olteanu

W25. In the Crelle [ABCD] tetrahedron, we note with $A', B', C', A'', B'', C''$ the tangent points of the hexatangent sphere $\varphi(J,\rho)$, associated with the thetrahedron, with the edges $|BC|, |CA|, |AB|, |DA|$, $|DB|, |DC|$. Show that inequalities occur:

a).

$$2\sqrt{3}R \geq 6\rho \geq A'A'' + B'B'' + C'C'' \geq 6\sqrt{3}r,$$

b).

$$4R^2 \geq 12\rho^2 \geq (A'A'')^2 + (B'B'')^2 + (C'C'')^2 \geq 36r^2$$

c).

$$\frac{8R^3}{3\sqrt{3}} \geq 8\rho^3 \geq A'A'' \cdot B'B'' \cdot C'C'' \geq 24\sqrt{3}r^3$$

where r, R is the length of the radius of the sphere inscribed and respectively circumscribed to the tetrahedron.

Marius Olteanu

W26. Let P_n denote the n-th Pell number defined by $P_{n+1} = 2P_n + P_{n-1}$, $P_0 = 0$, $P_1 = 1$. Further, let T_n denote the n-th triangular number, that is $T_n = \binom{n+1}{2}$. Show that

$$\sum_{n=0}^{\infty} 4T_n \cdot \frac{P_n}{3^{n+2}} = P_3 + P_4.$$

Ángel Plaza

W27. Let

$$P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$$

where $a_0, \ldots, a_n$ are integers. Show that if P takes the value 2020 for four distinct integral values of x, then P cannot take the value 2001 for any integral value of x.

Ángel Plaza

W28. For positive integers $j \leq n$, prove that

$$\sum_{k=j}^{n} \binom{2n}{2k}\binom{k}{j} = \frac{n\ 4^{n-j}}{j}\binom{2n-j-1}{j-1}. \qquad (1)$$

Ángel Plaza

W29. For $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $r > 1$. If $x_{00}, y_{00} > 0$, and reals $x_{ij}, y_{ij}, i = 1, 2, \ldots, n, j = 1, 2, \ldots, m$, then

$$\left(\frac{\left(\sum_{j=1}^{m}\sum_{i=1}^{n}(x_{ij}+y_{ij})^r\right)^{1/r}}{(x_{00}+y_{00})^{1/q}}\right)^p \leq$$

$$\le \left(\frac{\left(\sum_{j=1}^{m} \sum_{i=1}^{n} x_{ij}^{r}\right)^{1/r}}{x_{00}^{1/q}} \right)^{p} + \left(\frac{\left(\sum_{j=1}^{m} \sum_{i=1}^{n} y_{ij}^{r}\right)^{1/r}}{y_{00}^{1/q}} \right)^{p}, \tag{1}$$

with equality if and only if either $x_{ij} = y_{ij} = 0$ for $i = 1, \dots, n, j = 1, \dots, m$ or $x_{ij} = \alpha y_{ij}$ for $i = 0, 1, \dots, n, j = 0, 1, \dots, m$, and some $\alpha > 0$.

Chang-Jian Zhao

W30. For $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $r > 1$. If $u(x, y), v(x, y) > 0$, and $f(x, y), g(x, y)$ are continuous functions on $[a, b] \times [c, d]$, then

$$\left(\frac{\left(\int_a^b \int_c^d (f(x,y) + g(x,y))^r dxdy\right)^{1/r}}{(u(x,y) + v(x,y))^{1/q}} \right)^{p} \le$$

$$\le \left(\frac{\left(\int_a^b \int_c^d f(x,y)^r dxdy\right)^{1/r}}{u(x,y)^{1/q}} \right)^{p} + \left(\frac{\left(\int_a^b \int_c^d g(x,y)^r dxdy\right)^{1/r}}{v(x,y)^{1/q}} \right)^{p}, \tag{2}$$

with equality if and only if either

$$\left(\|f(x,y)\|_r^r, \|g(x,y)\|_r^r\right) = \alpha\left(\|u(x,y)\|_r^r, \|v(x,y)\|_r^r\right)$$

for some $\alpha > 0$ or $\|f(x,y)\|_r^r = \|g(x,y)\|_r^r = 0$

Chang-Jian Zhao

W31. P real polynomial degree $n \geq 1$ such that

$$P\left(0\right), P\left(1\right), P\left(4\right), P\left(9\right), ..., P\left(n^2\right)$$

are in Z. Prove that $\forall a \in Z, P\left(a^2\right) \in Z$.

Moubinool Omarjee

W32. ompute the quadruple integral

$$A = \frac{1}{\pi^2} \int\int\int_{[0,1]^2 \times [-\pi;\pi]^2} ab\sqrt{a^2 + b^2 - 2ab\cos\left(x - y\right)}dadbdxdy$$

Moubinool Omarjee

W33. Let $p \in N, f : [0,1] \to (0,\infty)$, continuous function and

$$a_n = \int_0^1 x^p \cdot \sqrt[n]{f(x)} dx, n \in N, n \geq 2.$$

Demonstrate that:

a). $\lim\limits_{n\to\infty} a_n = \frac{1}{p+1}$ b). $\lim\limits_{n\to\infty} ((p+1)a_n)^n = e^{(p+1)\int_0^1 x^p \ln f(x) dx}$

Nicolae Papacu

W34. If $a, b, c \in (0,\infty)$, then prove it:
1). $\frac{a^3+b^2c+bc^2}{bc} + \frac{b^3+c^2a+ac^2}{ac} + \frac{c^3+a^2b+ab^2}{ab} \geq 3(a+b+c)$
2). $\frac{bc}{a^3+b^2c+bc^2} + \frac{ac}{b^3+c^2a+ac^2} + \frac{ab}{c^3+a^2b+ab^2} \leq \frac{1}{3}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$

Nicolae Papacu

W35. In all triangle ABC holds:

$$(b^n + c^p) \tan^{n+p} \frac{A}{2} + (c^n + a^p) \tan^{n+p} \frac{B}{2} +$$

$$+ (a^n + b^p) \tan^{n+p} \frac{C}{2} \geq 6 \cdot \sqrt{\left(\frac{4r^2}{R\sqrt{3}}\right)^{n+p}}$$

where $n, p \in (0,\infty)$.

Nicolae Papacu

W36. For all $x \in \left(0, \frac{\pi}{4}\right)$ prove

$$\frac{\left(\sin^2 x\right)^{\sin^2 x} + \left(\tan^2 x\right)^{\tan^2 x}}{\left(\sin^2 x\right)^{\tan^2 x} + \left(\tan^2 x\right)^{\sin^2 x}} < \frac{\sin x}{4 \sin x - 3x}$$

Pirkulyiev Rovsen

W37. For all $x > 0$ prove

$$\frac{\sin^2 x - x}{\ln \left(\frac{\sin^2 x}{x}\right)^{\sqrt{x}}} + \frac{\cos^2 x - x}{\ln \left(\frac{\cos^2 x}{x}\right)^{\sqrt{x}}} > |\sin x| + |\cos x|$$

Pirkulyiev Rovsen

W38. Let $(a_n)_n$ be a sequence, given by the recurrence:

$$ma_{n+1} + (m-2)\,a_n - a_{n-1} = 0$$

where $m \in R$ is a parameter and the first two terms of $(a_n)_n$ are fixed known real numbers. Find $m \in R$, so that

$$\lim_{n\to\infty} a_n = 0$$

Laurenţiu Modan

W39. Prove that:
i).

$$\sum_{k=1}^{n-1} (1+\ln k) \le n^2 - n + 1$$

ii).

$$\sum_{k=1}^{n-1} \sqrt{\ln k} \le \frac{n^2-n+1}{2}$$

Laurenţiu Modan

W40. If $0 \le x_k < k$, for any $k \in \{1, 2, ..., n\}$, $m \in R_{\ge 2}$, then prove that

$$\frac{1}{\sqrt[m]{(1-x_1)(2-x_2)...(n-x_n)}} + \frac{1}{\sqrt[m]{(1+x_1)(2+x_2)...(n+x_n)}} \ge \frac{2}{\sqrt[m]{n!}}$$

Dorin Mărghidanu

W41. If $m, n \in N_{\ge 2}$, find the best constant $k \in R$, for which

$$\sum_{j=2}^{n} \sum_{i=2}^{m} \frac{1}{i^j} < k$$

Dorin Mărghidanu

W42. If a,b,c are non-negative real numbers, such that $a+b+c=3m$, $(m\geq 1)$ then

$$\left(a^a+b^a+c^a\right)\left(a^b+b^b+c^b\right)\left(a^c+b^c+c^c\right)\geq 27m^{3m}$$

Dorin Mărghidanu

W43. Let $f_1, f_2, ..., f_n$ nonnegative and concave functions. Then

$$(f_1 f_2)^{\frac{2^n-1}{n\cdot 2^n}}\left(\frac{\prod\limits_{k=1}^{n}\left(\sqrt[2^k]{f_1}+\sqrt[2^k]{f_2}\right)}{f_1+f_2}\right)^{\frac{1}{n}}$$

is concave.

Mihály Bencze and Marius Drăgan

W44. We consider a function $f:R\to R$ such that

$$f(x+y)+f(xy-1)=f(x)f(y)+f(x)+f(y)+1$$

for each $x,y\in R:$
i). Calculate $f(0)$ and $f(-1)$
ii). Prove that f is an even function
iii). Give an example of such a function
iv). Find a monoton functions with above property

Mihály Bencze and Marius Drăgan

W45. Let a_1, a_2, a_3, a_4 be a strictly positive numbers. Then is true the following inequality:

$$4\left(a_1a_2^n+a_2a_3^n+a_3a_4^n+a_4a_1^n\right)^n\leq\left(a_1^n+a_2^n+a_3^n+a_4^n\right)^{n+1}$$

for each $n\in N$.

Mihály Bencze and Marius Drăgan

W46. Let $x_1, x_2, \ldots, x_n\geq 0$, $\alpha,\beta>0$, $\beta\geq\alpha$, $t\in R$, such that $x_1^{x_2^t}\cdot x_2^{x_3^t}\ldots x_n^{x_1^t}=1$. Then

$$x_1^\beta x_2^t+x_2^\beta x_3^t+\cdots+x_n^\beta x_1^t\geq x_1^\alpha x_2^t+x_2^\alpha x_3^t+\cdots+x_n^\alpha x_1^t.$$

Marius Drăgan

W47. Let $x, y, z > 0$ such that

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=\frac{91}{10}$$

Compute

$$\left[(x^3+y^3+z^3)\left(\frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3}\right)\right]$$

where $[\cdot]$ represent the integer part.

Marian Cucoaneş and Marius Drăgan

W48. Let ABC a triangle such that

$$S^2 = 2R^2 + 8Rr + 3r^2$$

Then $\frac{R}{r} = 2$ or $\frac{R}{r} \geq \sqrt{2}+1$.

Marian Cucoaneş and Marius Drăgan

W49. Let $a, b, c > 0$ so that $a+b+c=1$. Then

$$(a+2ab+2ac+bc)^a(b+2bc+2ba+ca)^b(c+2ca+2cb+ab)^c \leq 1.$$

Marius Drăgan

W50. Let $f:[0,1] \to R$ one differentiable function, white f' continuous one $[0,1]$ and $|f'(x)| \leq 1$, $(\forall)\, x \in [0,1]$. If

$$2\left|\int_0^1 f(x)\,dx\right| \leq 1$$

show that:

$$(n+2)\left|\int_0^1 x^n f(x)\,dx\right| \leq 1, \quad (\forall)\, x \geq 1$$

Florin Stănescu and Şerban Cioculescu

W51. Consider the sequence of real numbers $(a_n)_{n\geq 1}$ such that

$$\lim_{n\to\infty}\frac{1}{n^r}\sum_{k=1}^{n}\frac{a_k}{k}=l\in R, r\in N^*$$

Show that:

$$\lim_{n\to\infty}\left(\frac{\sum\limits_{p=n+1}^{2n}\sum\limits_{k=1}^{p}\sum\limits_{i=1}^{k}\frac{a_i}{p\cdot i}}{n^{r+1}}\right)=l\left(\frac{2^{r+1}-1}{r\left(r+1\right)}-\frac{2^r}{\left(r+1\right)^2}\right)$$

Florin Stănescu and Şerban Cioculescu

W52. If $f\in C^{(3)}\left([0,1]\right)$, such that $f\left(0\right)=f'\left(0\right)=f\left(1\right)=0$ and $\left|f'''\left(x\right)\right|\leq 1$, $\left(\forall\right)x\in[0,1]$, show that:
a).

$$\left|f\left(x\right)\right|\leq\frac{x\left(1-x\right)}{\sqrt{3}}\cdot\left(\int\limits_0^x\frac{f\left(t\right)}{t\left(1-t\right)}dt\right)^{\frac{1}{2}},\ \left(\forall\right)x\in[0,1]$$

b).

$$\left|f'\left(x\right)\right|\leq\frac{1-2x}{\sqrt{3}}\cdot\left(\int\limits_0^x\frac{\left|f\left(t\right)\right|dt}{t\left(1-t\right)}dt\right)^{\frac{1}{2}}+\frac{x\left(1-x\right)}{6},\ \left(\forall\right)x\in\left[0,\frac{1}{2}\right]$$

c).

$$\int\limits_0^1\left(1-x\right)^2\cdot\frac{\left|f\left(x\right)\right|}{x}dx\geq 9\cdot\int\limits_0^1\left(\frac{f\left(x\right)}{x}\right)^2dx$$

Florin Stănescu and Şerban Cioculescu

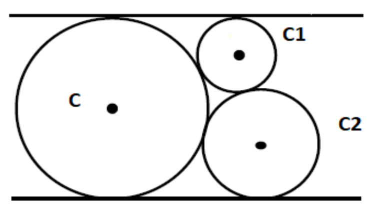
W53. Define the sequence $(w_n)_{n\geq 0}$ by the recurrence relation

$$w_{n+2}=2w_{n+1}+3w_n,\quad w_0=1,w_1=i,\quad n=0,1,\ldots.$$

(1) Find the general formula for w_n and compute the first 9 terms.
(2) Show that $|\Re w_n-\Im w_n)|=1$ for all $n\geq 1$.

Ovidiu Bagdasar

W54. Consider two parallel lines a and b.
The circles C, C_1 are tangent to each other and to the line a.
The circles C, C_2 are tangent to each other and to the line b.
The circles C_1, C_2 are tangent to each other, have radii $R_1 = 9$, $R_2 = 16$.



What is the radius R of the circle C?

Ovidiu Bagdasar

W55. Prove that the equation

$$1320x^3 = (y_1 + y_2 + y_3 + y_4)(z_1 + z_2 + z_3 + z_4)(t_1 + t_2 + t_3 + t_4 + t_5)$$

have infinitely many solutions in the set of Fibonacci numbers.

Mihály Bencze

W56. If $p_k > 0, a_k \geq 2\,(k = 1, 2, ..., n)$ and

$$S_n = \sum_{k=1}^{n} a_k, A_n = \prod_{cyclic} a_1^{p_2+p_3+...+p_n}, B_n = \prod_{k=1}^{n} a_k^{p_k},$$

then

$$\sum_{k=1}^{n} p_k \log_{S_n - a_k} a_k \geq \left(\sum_{k=1}^{n} p_k\right) \log_{A_n} B_n$$

Mihály Bencze

W57. In all triangle ABC holds

$$\sum \sin^2 \frac{A}{2} \cos^2 A \geq \frac{3\left(s^2 - (2R + r)^2\right)}{8R^2}$$

Mihály Bencze and Marius Drăgan

W58. In all triangle ABC holds

$$\sum \sqrt{\frac{a\left(h_a-2r\right)}{\left(3a+b+c\right)\left(h_a+2r\right)}} \leq \frac{3}{4}$$

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W59. If $a_k > 0(k = 1, 2, ..., n)$ then

$$\sum_{cyclic} \left(\frac{(a_1 + a_2 + ... + a_{n-1})^2}{a_n} + \frac{a_n^2}{a_1}\right) \geq \frac{n^2}{2} \sum_{k=1}^{n} a_k$$

Mihály Bencze

W60. Compute

$$\int \frac{(\sin x + \cos x)(4 - 2\sin 2x - \sin^2 2x)e^x dx}{\sin^3 2x}$$

where $x \in \left(0, \frac{\pi}{2}\right)$

Mihály Bencze