# SOME INEQUALITIES

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(Last updated: September 29, 2004)

**Introduction.** These are a few useful inequalities. Most of them are presented in two versions: in sum form and in integral form. More generally they can be viewed as inequalities involving vectors, the sum version applies to vectors in $\mathbb{R}^n$ and the integral version applies to spaces of functions.

First a few notations and definitions.

*Absolute value.* The absolute value of $x$ is represented $|x|$.

*Norm.* Boldface letters line $\mathbf{u}$ and $\mathbf{v}$ represent vectors. Their scalar product is represented $\mathbf{u}\cdot\mathbf{v}$. In $\mathbb{R}^n$ the scalar product of $\mathbf{u} = (a_1, \dots, a_n)$ and $\mathbf{v} = (b_1, \dots, b_n)$ is

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} a_i b_i \,.$$

For functions $f, g : [a, b] \to \mathbb{R}$ their scalar product is

$$\int_a^b f(x)g(x)\,dx \,.$$

The $p$-norm of $\mathbf{u}$ is represented $\|\mathbf{u}\|_p$. If $\mathbf{u} = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$, its $p$-norm is:

$$\|\mathbf{u}\|_p = \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} .$$

For functions $f : [a, b] \to \mathbb{R}$ the $p$-norm is defined:

$$\|f\|_p = \left(\int_a^b |f(x)|^p\,dx\right)^{1/p} .$$

For $p = 2$ the norm is called Euclidean.

*Convexity.* A function $f : (a, b) \to \mathbb{R}$ is said to be *convex* if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for every $x, y \in (a, b)$, $0 \leq \lambda \leq 1$. Graphically, the condition is that for $x < t < y$ the point $(t, f(t))$ should lie below or on the line connecting the points $(x, f(x))$ and $(y, f(y))$.

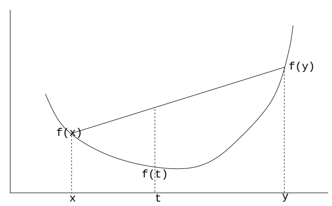


FIGURE 1. Convex function.

## INEQUALITIES

1. **Arithmetic-Geometric Mean Inequality.** (Consequence of convexity of $e^x$ and Jensen's inequality.) The geometric mean of positive numbers is not greater than their arithmetic mean, i.e., if $a_1, a_2, \dots, a_n > 0$, then

$$\left(\prod_{i=1}^{n} a_i\right)^{1/n} \leq \frac{1}{n}\sum_{i=1}^{n} a_i \, .$$

Equality happens only for $a_1 = \cdots = a_n$. (See also the power means inequality.)

2. **Arithmetic-Harmonic Mean Inequality.** The harmonic mean of positive numbers is not greater than their arithmetic mean, i.e., if $a_1, a_2, \dots, a_n > 0$, then

$$\frac{n}{\sum_{i=1}^{n} 1/a_i} \leq \frac{1}{n}\sum_{i=1}^{n} a_i \, .$$

Equality happens only for $a_1 = \cdots = a_n$.

This is a particular case of the Power Means Inequality.

3. **Cauchy.** (Hölder for $p = q = 2$.)

$$|\mathbf{u} \cdot \mathbf{v}| \le \|\mathbf{u}\|_2 \|\mathbf{v}\|_2 \,.$$

$$\Big|\sum_{i=1}^{n} a_i b_i\Big|^2 \le \Big|\sum_{i=1}^{n} a_i^2\Big| \Big|\sum_{i=1}^{n} b_i^2\Big| \,.$$

$$\left|\int_a^b f(x) g(x)\, dx\right|^2 \le \left(\int_a^b |f(x)|^2 \, dx\right) \left(\int_a^b |g(x)|^2 \, dx\right) \,.$$

4. **Chebyshev.** Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be sequences of real numbers which are monotonic in the same direction (we have $a_1 \le a_2 \le \dots \le a_n$ and $b_1 \le b_2 \le \dots \le b_n$, or we could reverse all inequalities.) Then

$$\frac{1}{n} \sum_{i=1}^{n} a_i b_i \ge \Big(\frac{1}{n} \sum_{i=1}^{n} a_i\Big) \Big(\frac{1}{n} \sum_{i=1}^{n} b_i\Big) \,.$$

Note that LHS − RHS $= \dfrac{1}{2n^2} \displaystyle\sum_{i,j} (a_i - a_j)(b_i - b_j) \ge 0$.

5. **Geometric-Harmonic Mean Inequality.** The harmonic mean of positive numbers is not greater than their geometric mean, i.e., if $a_1, a_2, \dots, a_n > 0$, then

$$\frac{n}{\sum_{i=1}^{n} 1/a_i} \le \Big(\prod_{i=1}^{n} a_i\Big)^{1/n} \,.$$

Equality happens only for $a_1 = \dots = a_n$.

This is a particular case of the Power Means Inequality.

6. **Hölder.** If $p > 1$ and $1/p + 1/q = 1$ then

$$|\mathbf{u} \cdot \mathbf{v}| \le \|\mathbf{u}\|_p \|\mathbf{v}\|_q \,.$$

$$\Big|\sum_{i=1}^{n} a_i b_i\Big| \le \Big(\sum_{i=1}^{n} |a_i|^p\Big)^{1/p} \Big(\sum_{i=1}^{n} |b_i|^p\Big)^{1/q} \,.$$

$$\left|\int_a^b f(x) g(x)\, dx\right| \le \left(\int_a^b |f(x)|^p \, dx\right)^{1/p} \left(\int_a^b |g(x)|^q \, dx\right)^{1/q} \,.$$

7. **Jensen.** If $\varphi$ is convex on $(a,b)$, $x_1, x_2, \ldots, x_n \in (a,b)$, $\lambda_i \geq 0$ $(i = 1, 2, \ldots, n)$, $\sum_{i=1}^n \lambda_i = 1$, then

$$\varphi\Big(\sum_{i=1}^n \lambda_i x_i\Big) \leq \sum_{i=1}^n \lambda_i \varphi(x_i)\,.$$

8. **Minkowski.** If $p > 1$ then

$$\|\mathbf{u}+\mathbf{v}\|_p \leq \|\mathbf{u}\|_p + \|\mathbf{v}\|_p\,,$$

$$\Big(\sum_{i=1}^n |a_i + b_i|^p\Big)^{1/p} \leq \Big(\sum_{i=1}^n |a_i|^p\Big)^{1/p} + \Big(\sum_{i=1}^n |b_i|^p\Big)^{1/p}\,,$$

$$\left(\int_a^b |f(x) + g(x)|^p\,dx\right)^{1/p} \leq \left(\int_a^b |f(x)|^p\,dx\right)^{1/p} + \left(\int_a^b |g(x)|^p\,dx\right)^{1/p}\,.$$

Equality holds iff $\mathbf{u}$ and $\mathbf{v}$ are proportional.

9. **Norm Monotonicity.** If $a_i > 0$ $(i = 1, 2, \ldots, n)$, $s > t > 0$, then

$$\Big(\sum_{i=1}^n a_i^s\Big)^{1/s} \leq \Big(\sum_{i=1}^n a_i^t\Big)^{1/t}\,,$$

i.e., if $s > t > 0$, then $\|\mathbf{u}\|_s \leq \|\mathbf{u}\|_t$.

10. **Power Means Inequality.** Let $r$ be a non-zero real number. We define the $r$-mean or $r$th power mean of non-negative numbers $a_1, \ldots, a_n$ as follows:

$$M^r(a_1, \ldots, a_n) = \left(\frac{1}{n}\sum_{i=1}^n a_i^r\right)^{1/r}\,.$$

The ordinary arithmetic mean is $M^1$, $M^2$ is the quadratic mean, $M^{-1}$ is the harmonic mean. Furthermore we define the 0-mean to be equal to the geometric mean:

$$M^0(a_1, \ldots, a_n) = \Big(\prod_{i=1}^n a_i\Big)^{1/n}\,.$$

Then for any real numbers $r$, $s$ such that $r < s$, the following inequality holds:

$$M^r(a_1, \ldots, a_n) \leq M^s(a_1, \ldots, a_n)\,.$$

Equality holds if and only if $a_1 = \cdots = a_n$. (See weighted power means inequality).

11. **Schur.** If $x, y, x$ are positive real numbers and $k$ is a real number such that $k \geq 1$, then

$$x^k(x-y)(x-z) + y^k(y-x)(y-z) + z^k(z-x)(z-y) \geq 0\,.$$

For $k = 1$ the inequality becomes

$$x^3 + y^3 + z^3 + 3xyz \geq xy(x+y) + yz(y+z) + zx(z+x)\,.$$

12. **Schwarz.** (Hölder with $p = q = 2$.)

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|_2 \|\mathbf{v}\|_2\,,$$

$$\left|\sum_{i=1}^n a_i b_i\right|^2 \leq \left(\sum_{i=1}^n |a_i|^2\right)\left(\sum_{i=1}^n |b_i|^2\right),$$

$$\left|\int_a^b f(x)g(x)\,dx\right|^2 \leq \left(\int_a^b |f(x)|^2\,dx\right)\left(\int_a^b |g(x)|^2\,dx\right).$$

13. **Weighted Power Means Inequality.** Let $w_1, \dots, w_n$ be positive real numbers such that $w_1 + \cdots + w_n = 1$. Let $r$ be a non-zero real number. We define the $r$th weighted power mean of non-negative numbers $a_1, \dots, a_n$ as follows:

$$M_w^r(a_1, \dots, a_n) = \left(\sum_{i=1}^n w_i a_i^r\right)^{1/r}.$$

As $r \to 0$ the $r$th weighted power mean tends to:

$$M_w^0(a_1, \dots, a_n) = \left(\prod_{i=1}^n a_i^{w_i}\right).$$

which we call 0th weighted power mean. If $w_i = 1/n$ we get the ordinary $r$th power means.

Then for any real numbers $r$, $s$ such that $r < s$, the following inequality holds:

$$M_w^r(a_1, \dots, a_n) \leq M_w^s(a_1, \dots, a_n)\,.$$

(If $r, s \neq 0$ note convexity of $x^{s/r}$ and recall Jensen's inequality.)