Syzygies of abelian varieties

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Let A be an ample line bundle on an abelian variety X (over an algebraically closed field). A theorem of Koizumi ([Ko],[S]), developing Mumford's ideas and results in [M1], states that if $m \geq 3$ the line bundle $L = A^{\otimes m}$ embeds X in projective space as a projectively normal variety. Moreover, a celebrated theorem of Mumford ([M2]), slightly refined by Kempf ([K4]), asserts that the homogeneous ideal of X is generated by quadrics as soon as $m \geq 4$. Such results turn out to be particular cases of a statement, conjectured by Rob Lazarsfeld, concerning the minimal resolution of the graded algebra $R_L = \bigoplus_{h=0}^{\infty} H^0(X, L^{\otimes h})$ over the polynomial ring $S_L = \bigoplus_{h=0}^{\infty} Sym^h H^0(X, L)$. The purpose of this paper is to prove Lazarsfeld's conjecture.

To put such matters into perspective, it is useful to review the case of projective curves. A classical theorem of Castelnuovo states that a curve X, embedded in projective space by a complete linear system $|L|$, is projectively normal as soon as $\deg L \geq 2g(X)+1$ and a theorem of Mumford, Fujita and Saint-Donat states that if $\deg L \geq 2g(X)+2$ then the homogeneous ideal of X is generated by quadrics. Green ([G1]) unified, re-interpreted and generalized these results to a statement about syzygies. Specifically, given a (smooth) projective variety X and a very ample line bundle L on X, a minimal resolution of R_L as a graded S_L-module (notation as above) looks like

$$0 \to \cdots \to E_p \to \cdots \to E_1 \to E_0 \to R_L \to 0 \tag{1}$$

where $E_0 = S_L \oplus \bigoplus_j S_L(-a_{0j})$, (where, since X is embedded by complete linear system, $a_{0j} \geq 2$ for any j); $E_1 = \bigoplus_j S_L(-a_{1j})$ (where, since the image of X in $\mathbf{P}(H^0(L)^\vee)$ is not contained in any hyperplane, $a_{1j} \geq 2$ for any j); in general, for $p \geq 1$, $E_p = \bigoplus_j S_L(-a_{pj})$ with $a_{pj} \geq p+1$ for any j. Green introduced the following terminology: L is said to satisfy property N_0 if $E_0 = S_L$. This means that the projective coordinate ring of the variety X embedded in $\mathbf{P}(H^0(L)^\vee)$ coincides with R_L, i.e. that X is projectively normal. Moreover L is said to satisfy condition N_1 if it satisfies N_0 and $a_{1j} = 2$ for any j, i.e. the homogeneous ideal of the embedded variety X is generated by quadrics. Inductively, one says that L satisfies condition N_p if it satisfies condition N_{p-1} and $a_{pj} = p+1$ for any j. So N_2 means that the relations between the quadrics defining X are generated by linear ones and, for arbitrary $p \geq 2$, N_p means that the first $p-1$ maps of the resolution of the homogeneous ideal are matrices with linear entries. In a word N_p means that, up to the p-th step, the resolution (1) is as "regular" as it could possibly be. The aforementioned theorem of Green states that *if X is a curve and if $\deg L \geq 2g(X)+1+p$ then L satisfies N_p*. This result stimulated many interesting questions (see [G1], [G2], [L] and [EL]). One of these is how it extends to higher dimension. For arbitrary varieties there is a general conjecture of Mukai and results of Ein and Lazarsfeld ([EL]). In this article we will be concerned with the case of abelian varieties. Lazarsfeld's conjecture states that powers of ample line bundles on abelian varieties of arbitrary dimension behave exactly as in the case of elliptic curves:

Theorem. ($char(k) = 0$) *Let A be an ample line bundle on an abelian variety X. If $m \geq p + 3$ then $A^{\otimes m}$ satisfies condition N_p.*

Note that for elliptic curves this is Green's theorem and, in arbitrary dimension, the cases $p = 0, 1$ are the aforementioned results of Mumford and others. Note also that if A is a principal polarization the bound of Lazarsfeld's conjecture is the best one can hope for. Furthermore, we prove a generalization of Lazarsfeld's conjecture (Theorem 4.3), about the first p steps of the resolution of $R_{A^{\otimes m}}$ when $m \leq p + 3$ (we refer to Section 4 below for terminology, statement and proof).

It is worth to recall that shortly after the appearance of Lazarsfeld's conjecture, Kempf ([K5]) proved it "up to a factor two": his statement, in the present terminology, is that *$A^{\otimes m}$ satisfies condition N_p as soon as $m \geq \max\{3, 2p+2\}$.* Thus the above Theorem improves Kempf's result already for abelian surfaces and $p = 2$.

The main point of the proof will be a criterion – of independent interest – for the surjectivity of multiplication maps of global sections

$$H^0(X, A^{\otimes m}) \otimes H^0(X, E) \to H^0(X, A^{\otimes m} \otimes E), \qquad (*)$$

where E and A are respectively a vector bundle and an ample line bundle on an abelian variety X. Roughly, the criterion (Theorem 3.1) states that the map $(*)$ is surjective as soon as the higher cohomology of certain twisted (skew) "Pontrjagin products" attached to E and suitable powers of A vanish. On the other hand the relation between maps as $(*)$ and syzygies is well known and goes as follows (in characteristic zero): let us denote M_L the kernel of the evaluation map $H^0(L) \otimes \mathcal{O}_X \to L$. If the multiplication map

$$H^0(L) \otimes H^0(M_L^{\otimes p} \otimes L^{\otimes h}) \to H^0(M_L^{\otimes p} \otimes L^{\otimes h+1}) \qquad (**)$$

is surjective for any $h \geq 1$ then the first p steps of the resolution of R_L are linear. Thus Lazarsfeld's conjecture is proved by showing that for any $m \geq p + 3$ and $h \geq 1$ the pair given by $A^{\otimes m}$ and $E = (M_{A^{\otimes m}})^{\otimes p} \otimes A^{\otimes mh}$ satisfy the cohomological conditions required by Theorem 3.1 to ensure the surjectivity of the multiplication map $(*)$.

Let us describe more closely the contents of the paper: the first three sections are devoted to the proof of the above mentioned Theorem 3.1. More precisely in Section 1 it is shown that, under mild hypotheses, given an ample line bundle L, a vector bundle E and a point x on X, the multiplication map $H^0(T_x^*L) \otimes H^0(E) \to H^0((T_x^*L) \otimes E)$ is identified to the evaluation of global sections at x of another vector bundle, denoted $L\hat{*}E$, the (skew) Pontrjagin product of L and E. This suggests to look for ways of understanding the locus where $L\hat{*}E$ is not globally generated. To this purpose in Section 2 we will show the proof of a lemma (Lemma 2.1), implicitely contained in Kempf's work [K5], which is relevant for finding spaces of global sections generating a given locally free sheaf on an abelian variety. This yields a cohomological criterion for global generation (Theorem 2.3) which is interesting for its own sake. Going back to $L\hat{*}E$, it turns out more convenient to describe rather the locus where $n_X^*(L\hat{*}E)$ is not generated by the invariant sections (we denote $n_X : X \to X$ the multiplication by n and $X_n = \ker n_X$), since, as a consequence of the theorem of the cube, one has a very explicit formula for $n_X^*(L\hat{*}E)$ (Prop.1.3). The proof of Theorem 3.1 is in Section 3, where we combine the results of Section 2 with an analysis of the action of the group X_n on $n_X^*(L\hat{*}E)$. As an application it is also shown how to recover Kempf's method ([K5]) from Theorem 3.1. In Section 4, we apply Theorem 3.1 to Lazarsfeld's conjecture and to the announced generalization (Theorem 4.3).

Although we work in the algebraic setting, we have made some simplifying assumptions on the characteristic of the ground field. Moreover one needs the characteristic zero for condition $(**)$ about syzygies, which is somehow simpler to work with.

This work owes much to the reading of George Kempf's penetrating works on these and related subjects, especially [K1]$\cdots$[K6]. My understanding of [K5] in turn owes a lot to some UCLA lectures and seminar talks given by Rob Lazarsfeld on the subject, back in '88-89. Part of this work was done while the author was visiting Colorado State University at Fort Collins and U.N.A.M. at Morelia (Mexico), supported respectively by a C.N.R.-N.A.T.O. scholarship and by U.N.A.M.. My thanks go to these institutions, and especially to Jeanne Duflot and Rick Miranda, at Fort Collins, and Alexis Garcia-Zamora and Sevin Recillas, at Morelia, for really great ospitality.

1. Pontrjagin products and surjectivity of multiplication maps.

Throughout this paper X will denote an abelian variety over an algebraically closed field k. Given an integer n, we will denote $n_X : X \to X$ the map $x \mapsto nx$ and $X_n = \ker(n_X)$. Whenever X_n will be taken into consideration we will assume that $char(k)$ does not divide n so we will not need to consider the scheme structure of X_n. Finally, given two integers m and n, $mp_1 + np_2 : X \times X \to X$ will mean the map $(x, y) \to mx + ny$. E.g. the group law $(x, y) \mapsto x + y$ will be denoted $p_1 + p_2$ and *not* m_X (which will be the multiplication by the integer m).

Let L and E be respectively an invertible sheaf and a locally free sheaf on the abelian variety X. The main theme will be to find criteria ensuring the surjectivity of the multiplication map

$$H^0(L) \otimes H^0(E) \to H^0(L \otimes E). \tag{1}$$

To this purpose, we will rather undertake a global study of the family of multiplication maps

$$M_x : H^0(T_x^*L) \otimes H^0(E) \to H^0((T_x^*L) \otimes E) \tag{2}$$

for x varying in X. Let us define the *"skew Pontrjagin product"* of L and E as

$$L\hat{*}E := p_{1*}((p_1 + p_2)^*L \otimes p_2^*E). \tag{3}$$

(see Remark 1.7 below for a justification of the terminology and some comments). For sake of simplicity, we will assume troughout that

$$H^i((T_x^*L) \otimes E) = 0 \tag{4}$$

for any $x \in X$ and for any $i > 0$. Then Grauert's theorem yields that $L\hat{*}E$ is locally free and that, at the fibers level, the map $L\hat{*}E(x) \to H^0((T_x^*L) \otimes E))$ is an isomorphism. On the other hand, there is a natural isomorphism

$$\Phi : H^0(L) \otimes H^0(E) \xrightarrow{\sim} H^0(L\hat{*}E) \tag{5}$$

obtained as follows: considering the automorphism $\phi = (p_1+p_2, p_2)$ of $X \times X$ we have that $(p_1+p_2)^*L \otimes p_2^*E = \phi^*(p_1^*L \otimes p_2^*E)$. Therefore we have the isomorphism

$$\phi^* : H^0(L) \otimes H^0(E) \to H^0((p_1+p_2)^*L \otimes p_2^*E). \tag{6}$$

The isomorphism Φ follows by composing with the isomorphism $H^0((p_1+p_2)^*L \otimes p_2^*E) \cong H^0(p_{1*}((p_1+p_2)^*L \otimes p_2^*E)$. (In Kempf's paper [K2] this type of argument is called the "Mumford's shearing trick" and ϕ a "shearing" automorphism).

It follows that the evaluation map of $L\hat{*}E$ at x composed with the isomorphism Φ:

$$H^0(L) \otimes H^0(E) \xrightarrow{\Phi} H^0(L\hat{*}E) \xrightarrow{ev_x} L\hat{*}E(x)$$

coincides with the composed map

$$H^0(L) \otimes H^0(E) \xrightarrow{T_x^* \times id} H^0(T_x^*L) \otimes H^0(E) \xrightarrow{M_x} H^0((T_x^*L) \otimes E))$$

Therefore we have

Proposition 1.1. *Assuming hypothesis (4), the locus $M(L,E)$ of points $x \in X$ such that the multiplication map M_x is not surjective coincides with the locus $B(L\hat{*}E)$ of points where $L\hat{*}E$ is not generated by its global sections.*

Note that, in particular, M_x is surjective for any x (resp. for the general $x \in X$) if and only if $L\hat{*}E$ is globally generated (resp. generically globally generated).

Notation 1.2. Given an $\mathcal{O}_X$-module F and a subspace of global sections $V \subset H^0(F)$, $B(V,F)$ will denote the locus of $x \in X$ such that V does not generate F at x, i.e. the support of the cokernel of the evaluation map $V \otimes \mathcal{O}_X \to F$. When V is the full space $H^0(F)$ such a locus will be simply denoted $B(F)$. □

The next step consists in passing to the finite cover $n_X : X \to X$ for a suitable positive integer n and trying to understand the locus $B(n_X^*H^0(L\hat{*}E), n_X^*(L\hat{*}E))$ rather than $B(L\hat{*}E)$ (obviously $B(L\hat{*}E)$ is the image via n_X of $B(n_X^*H^0(L\hat{*}E), n_X^*(L\hat{*}E)))$.

Proposition 1.3. $n_X^*(L\hat{*}E) \cong (L^{\otimes n}\hat{*}(E \otimes L^{\otimes -n+1})) \otimes n_X^*L \otimes L^{\otimes -n}$.

Before proving the Proposition, let us point out why one should expect a formula like that. By the theorem of the square $T_x^*L \otimes T_y^*L^{\otimes n-1} \cong T_{x+(n-1)y}L^{\otimes n}$. Therefore $T_x^*L \otimes E = T_x^*L \otimes L^{\otimes n-1} \otimes E \otimes L^{\otimes -n+1} \cong (T_{x/n}^*L^{\otimes n}) \otimes E \otimes L^{\otimes -n+1}$, where x/n means any $z \in X$ such that $nz = x$. It is therefore natural to expect that such isomorphism at the H^0 level globalizes to an isomorphism $n_X^*(L\hat{*}E) \cong (L^{\otimes n}\hat{*}(E \otimes L^{\otimes -n+1})) \otimes$ (*a line bundle*). In this perspective the proof is a straightforward consequence of the treatment of [M3] pp.55-59:

Proof of Prop.1.3. By flat base extension we have

$$\begin{aligned} n_X^* p_{1*}((p_1+p_2)^*L \otimes p_2^*E) &\cong p_{1*}((n_X, 1_X)^*(p_1+p_2)^*L \otimes p_2^*E) \\ &= p_{1*}((np_1+p_2)^*L \otimes p_2^*E) \end{aligned} \tag{7}$$

On the other hand

$$(np_1+p_2)^*L \cong (p_1+p_2)^*L^{\otimes n} \otimes p_1^*(n_X^*L \otimes L^{\otimes -n}) \otimes p_2^*L^{\otimes -n+1} \tag{8}$$

For $n=2$, (8) follows plugging $f=g=p_1$ and $h=p_2$ in [M3] Cor.2, p.58 to the Theorem of the Cube. For any n, (8) follows by induction in the same way. Finally, plugging (8) into the last member of (7) and using projection formula we get the statement. $\square$

A first good reason to consider pullbacks of type $n_X^*(L\hat{*}E)$ rather than $L\hat{*}E$ itself is that if L and M are algebraically equivalent to powers of the same ample line bundle then, for suitable n, the vector bundle $n_X^*(L\hat{*}M)$ is trivial up to twist by a translate of a power of A (Prop. 1.6). To see this we will need the following basic remarks

Remark 1.4. Let F,E and α be respectively a two coherent sheaves on X and a line bundle in Pic^0X. Then

$$(F\otimes\alpha)\hat{*}E \cong (F\hat{*}(E\otimes\alpha))\otimes\alpha \tag{9}$$

Indeed if $\alpha\in\mathrm{Pic}^0X$ then $(p_1+p_2)^*\alpha \cong p_1^*\alpha\otimes p_2^*\alpha$ (see e.g. [M3], Ch.8(i), p.74). Plugging this into the definition of $(F\otimes\alpha)\hat{*}E$ one gets (9) by projection formula. $\square$

Remark 1.5. Let F be any coherent sheaf on X. There is a natural isomorphism

$$\Psi : F\hat{*}\mathcal{O}_X \xrightarrow{\sim} H^0(F)\otimes\mathcal{O}_X \tag{10}$$

(needless to say, given two sheaves F and E one can define in the same way $F\hat{*}E := p_{1*}((p_1+p_2)^*F\otimes p_2^*E)))$. To get Ψ one uses the same trick as to get the isomorphism Φ in (6). Let us consider the shearing automorphism $\psi=(p_1,p_2-p_1)$ of $X\times X$. Then $\psi^*(p_1+p_2)^*F = p_2^*F$ and we have the isomorphism $\psi^* : p_{1*}((p_1+p_2)^*F)\to p_{1*}(p_2^*F)$. Then Ψ is obtained by composing ψ^* with the Künneth isomorphism $p_{1*}(p_2^*F)\cong H^0(F)\otimes\mathcal{O}_X$. $\square$

Proposition 1.6. *Let A be an ample line bundle on X, a and b two positive integers and $\alpha\in\mathrm{Pic}^0X$. Then*

$$(a+b)_X^*(A^{\otimes a}\hat{*}(A^{\otimes b}\otimes\alpha)) \cong H^0(A^{\otimes a+b}\otimes\alpha)\otimes(a+b)_X^*(A^{\otimes a})\otimes a_X^*(A^{\otimes -a-b}\otimes\alpha^\vee)$$

Proof. By Prop.1.3 we know that

$$(a+b)_X^*(A^{\otimes a}\hat{*}(A^{\otimes b}\otimes\alpha) \cong (A^{\otimes a(a+b)}\hat{*}(A^{\otimes b-a(a+b-1)}\otimes\alpha)\otimes(a+b)_X^*(A^{\otimes a})\otimes A^{\otimes -a(a+b)} \tag{11}$$

$$a_X^*(A^{\otimes a+b}\hat{*}\alpha) \cong (A^{\otimes a(a+b)}\hat{*}(\alpha\otimes A^{\otimes -(a+b)(a-1)})\otimes a_X^*(A^{\otimes a+b})\otimes A^{\otimes -a(a+b)} \tag{12}$$

The statement follows by plugging (12) into (11) (note that $b-a(a+b-1) = -(a+b)(a-1)$), and using that, by Remarks 1.4 and 1.5, $A^{\otimes a+b}\hat{*}\alpha \cong H^0(A^{\otimes a+b}\otimes\alpha)\otimes\alpha^\vee$. $\square$

Remark 1.7. (Pontrjagin products) Given two sheaves F and E on X, the sheaf $F\hat{*}E$ is isomorphic to $F*(-1)^*_X E$, were "$*$" means the *Pontrjagin product* as defined in [Mu], p.160, i.e. $G*H = (p_1+p_2)_*(p_1^*G\otimes p_2^*H)$. The isomorphism can be seen in the usual way using the shearing automorphism $(p_1+p_2,-p_2)$. Note that $\hat{*}$ is "skew-symmetric", i.e. $F\hat{*}E \cong (-1)^*_X(E\hat{*}F)$. Under this form Prop.1.3 can already be deduced from Mukai's results. In fact, given a sheaf G, let us consider its "Fourier transform" $\hat{G} = q_{2*}(q_1^*G\otimes\mathcal{P})$ (where q_i are the projections on $X\times \mathrm{Pic}^0 X$ and $\mathcal{P}$ is a Poincaré sheaf). We have the formula (deduced from [Mu], 3.10) $L\hat{*}E \cong \phi_L^*(\widehat{L\otimes E})\otimes L$ (where we denote $\phi_L : X\to \mathrm{Pic}^0X$ the polarization associated to L). Then, since $\phi_{L^{\otimes n}} = n\phi_L$, we have $n_X^*(L\hat{*}E)\cong \phi_{L^{\otimes n}}(\widehat{L\otimes E})\otimes n_X^*L$ and Prop.1.3 follows formally.

2. Global generation of vector bundles on abelian varieties, according to Kempf.

In view of Proposition 1.1, it is natural to look for criteria guaranteeing that a given subspace of global sections V of a locally free sheaf F on an abelian variety generates F at every point. An useful result in this direction – somehow implicit in Kempf's work [K5] – is Lemma 2.1 below. The main application will be in Theorem 3.1, but already in this section we will show as a corollary a cohomological criterion for the global generation of a locally free sheaf on an abelian variety – a natural generalization of the classical one for line bundles – which is interesting for its own sake (Theorem 2.3.) Since Kempf's argument is not easy to isolate in [K5] and can be applied in other contexts, we will supply here a direct proof of the result along Kempf's lines.

Lemma 2.1 *Let F be a locally free sheaf on an abelian variety X (over an algebraically closed field k) such that $h^i(F\otimes\alpha)=0$ for any $i>0$ and for any $\alpha\in\mathrm{Pic}^0X$. Then the map of quasi-coherent $\mathcal{O}_X$-modules* $\bigoplus_{\alpha\in \mathrm{Pic}^0 X} H^0(F\otimes\alpha)\otimes\alpha^\vee \stackrel{\phi_\alpha}{\to} F$ *is surjective.* (Here ϕ_α is the evaluation map of $F\otimes\alpha$ tensored by $\alpha^\vee$).

Proof. STEP 1: There is a locally free sheaf $\mathcal{T}$ on $X\times\mathrm{Pic}^0X$ and a map $\phi:\mathcal{T}\to p_X^*F$ such that $\mathcal{T}_{|X\times\alpha}\cong H^0(F\otimes\alpha)\otimes a^\vee$ and the maps $\phi_{|X\times\alpha}:\mathcal{T}_{|X\times\alpha}\to(p_X^*F)_{|X\times\alpha}$ are identified to $\phi_\alpha : H^0(F\otimes\alpha)\otimes\alpha^\vee\to F$.

Proof. To construct such a $\mathcal{T}$, let us denote p_i the three projections on $X\times X\times\mathrm{Pic}^0X$ and p_{ij} the three intermediate projections. Moreover let Δ be the diagonal in $X\times X$ and let $\mathcal{P}$ be a Poincaré line bundle on $X\times\mathrm{Pic}^0X$. Finally, let

$$\mathcal{F} = p_{13}^*(p_1^*F\otimes\mathcal{P})\otimes p_{23}^*\mathcal{P}^\vee \tag{1}$$

and $\Phi:\mathcal{F}\to\mathcal{F}_{|\Delta\times\mathrm{Pic}^0X}\cong p_1^*F_{|\Delta\times\mathrm{Pic}^0X}$ the restriction map (note that $p_{13}^*\mathcal{P}\otimes p_{23}^*\mathcal{P}^\vee$ is trivial when restricted to $\Delta\times\mathrm{Pic}^0X$). Then, by the hypothesis on F and Grauert's theorem, the sheaf $\mathcal{T}=p_{23*}(\mathcal{F})$ and the map $\phi = p_{23*}\Phi$ have the required properties.

STEP 2. The surjectivity of the map $\oplus\phi_\alpha$ of the statement of Lemma 2.1 is equivalent to the surjectivity of the map $R^gp_{X*}(\phi) : R^gp_{X*}(\mathcal{T})\to R^gp_{X*}(p_X^*F)\cong F$. (Here

$g = \dim X$ and we have used projection formula and the relative Serre duality isomorphism $R^g p_{X*}(\mathcal{O}_{X\times \mathrm{Pic}^0 X}) \cong \mathcal{O}_X$).

Proof. Given a sheaf G and a point y let us denote $G(y)$ the fibre i.e. $G\otimes k(y)$. By Step 1, the surjectivity of $\oplus\phi_\alpha$ means that for any $x\in X$ the map $\bigoplus_{\alpha\in \mathrm{Pic}^0 X} \mathcal{T}((x,\alpha)) \stackrel{\phi_\alpha(x)}{\to} F(x)$ is surjective, i.e. that the map $F^\vee(x) \to \prod_{\alpha\in\mathrm{Pic}^0 X} \mathcal{T}^\vee((x,\alpha))$ is injective, i.e. that the map $\phi^\vee_{|x\times \mathrm{Pic}^0 X} : F(x)^\vee \otimes \mathcal{O}_{\mathrm{Pic}^0 X} \cong (p_X^* F^\vee)_{|x\times\mathrm{Pic}^0 X} \to \mathcal{T}^\vee_{|x\times \mathrm{Pic}^0 X}$ is injective at the global sections level. By Serre duality this is equivalent to the surjectivity of $H^g(\phi_{|x\times\mathrm{Pic}^0 X}) : H^g(\mathcal{T}_{|x\times\mathrm{Pic}^0 X}) \to F(x)\otimes H^g(\mathcal{O}_{\mathrm{Pic}^0 X}) \cong F(x)$ for any $x\in X$ and this, again by base change, is the statement of Step 2.

STEP 3. The spectral sequence $R^i p_{X*}\circ R^j p_{23*}(\mathcal{F}) \Rightarrow R^{i+j}p_{2*}(\mathcal{F})$ degenerates giving an isomorphism $R^g p_{X*}(\mathcal{T}) \stackrel{\sim}{\to} R^g p_{2*}(\mathcal{F})$. This is clear since, by hypothesis, $R^i p_{23*}(\mathcal{F})$ is zero for $i>0$.

STEP 4. There is a canonical isomorphism $R^g p_{2*}(\mathcal{F}) \stackrel{\sim}{\to} F$ such that the map $R^g p_{X*}(\phi)$ of Step 2 is the composition of this isomorphism with the one supplied by Step 3. Therefore $R^g p_{X*}(\phi)$ is itself an isomorphism. By Step 2 this proves Theorem 4.1.

Proof. In the first place let us note that, by projection formula,

$$R^i p_{12*}(\mathcal{F}) \cong p_1^* F \otimes R^i p_{12*}(p_{13}^*\mathcal{P}\otimes p_{23}^*\mathcal{P}^\vee) \tag{3}$$

On the other hand, it is well known (see [K1] or [K2] p.53) that

$$R^i p_{12*}(p_{13}^*\mathcal{P}\otimes p_{23}^*\mathcal{P}^\vee) \cong \begin{cases} 0 & \text{if } i<g; \\ \mathcal{O}_\Delta & \text{if } i = g \end{cases} \tag{4}$$

Let us recall the proof of (4) for sake of self-containedness: one considers the difference map $d = p_1 - p_2 : X\times X \to X$. By the see-saw principle it follows immediately that $p_{13}^*\mathcal{P}\otimes p_{23}^*\mathcal{P}^\vee \cong (d, \mathrm{id}_{\mathrm{Pic}^0 X})^*\mathcal{P}$. Therefore we are reduced to compute the $R^i p_{12*}$'s of the right hand side. By flat base extension $R^i p_{12*}((d,\mathrm{id}_{\mathrm{Pic}^0 X})^*\mathcal{P})) \cong d^*(R^i p_{X*}(\mathcal{P}))$. At this point one invokes Mumford's duality result ([M] Ch.13, [K2], Th.3.15) which states that $R^i p_{X*}(\mathcal{P}) = H^g(\mathcal{O}_{\mathrm{Pic}^0 X})\otimes \mathcal{O}_e$ if $i=g$ and it is zero otherwise (e is the identity point of X). This proves (4) (after making the Serre duality identification $H^g(\mathcal{O}_{\mathrm{Pic}^0 X})\cong k$). Then by (3) it follows that

$$R^i p_{12*}(\mathcal{F}) \cong \begin{cases} p_1^* F\otimes \mathcal{O}_\Delta & \text{if } i = g \\ 0 & \text{if } i<g \end{cases} \tag{5}$$

Finally, to get the isomorphism of Step 4 let us consider the other natural spectral sequence, that is $R^i p_{2*}\circ R^j p_{12*}(\mathcal{F}) \Rightarrow R^{i+j}p_{2*}(\mathcal{F})$ ($p_2 = p_2\circ p_{12}$, where we denote p_2 the projection on the second factor on both $X\times X$ and $X\times X\times \mathrm{Pic}^0 X$). By (5) it degenerates to the isomorphism $R^g p_{2*}(\mathcal{F}) \cong p_{2*}(R^g p_{12*}(\mathcal{F})) \cong F$. The second part of the statement of Step 4 follows easily. □

Corollary 2.2. *Let F be as in Theorem 4.1. There is a positive integer N such that given N general line bundles $(\alpha_1, \cdots, \alpha_N) \in (\mathrm{Pic}^0 X)^N$ the map $\bigoplus_{i=1}^N H^0(F \otimes \alpha_i) \otimes \alpha_i^\vee \to F$ is surjective.* (Proof left to the reader).

We conclude this section with the announced result on global generation of locally free sheaves. A basic and elementary result on abelian varieties states that the product of two ample line bundles is base-point free. This can be extended to locally free sheaves as follows:

Theorem 2.3 *Let F be as in Theorem 2.1 and let A be an ample line bundle on X. Then $F \otimes A$ is generated by its global sections.*

Note that, at least over the complex numbers, if F is an ample vector bundle, then the cohomological hypothesis on F is satisfied (by Kodaira vanishing). If the rank of F is one, then such condition is in fact equivalent to ampleness.

Proof. Given a tuple $\bar\alpha = (\alpha_1, \ldots, \alpha_N) \in (\mathrm{Pic}^0 X)^N$ we have the commutative diagram

$$\begin{array}{ccc} \bigoplus_{\alpha_i=1}^{N} H^0(F \otimes \alpha_i) \otimes H^0(A \otimes \alpha_i^\vee) \otimes \mathcal{O}_X & \longrightarrow & H^0(F \otimes A) \otimes \mathcal{O}_X \\ \Big\downarrow {\scriptstyle \oplus(\mathrm{id} \otimes ev_{\alpha_i})} & & \Big\downarrow {\scriptstyle ev} \\ \bigoplus_{\alpha_i=1}^{N} H^0(F \otimes \alpha_i) \otimes A \otimes \alpha_i^\vee & \xrightarrow{\oplus(\phi_{\alpha_i} \otimes A)} & F \otimes A \end{array} \qquad (6)$$

Let us denote B^α the base locus $B(A \otimes \alpha^\vee)$ of the line bundle $A \otimes \alpha^\vee$. Note that, since A is ample, $A \otimes \alpha^\vee$ is always effective, so that B^α is always a proper or empty subvariety of X. Let also $B^{\bar\alpha} = B^{\alpha_1} \cup \cdots \cup B^{\alpha_N}$. By Corollary 2.2 we have that $\oplus(\phi_{\alpha_i} \otimes A)$ is surjective for $\bar\alpha$ general in $(\mathrm{Pic}^0 X)^N$. Thus $B(F \otimes A)$, that is the support of the cokernel of the evaluation map of $F \otimes A$, is contained in $B^{\bar\alpha}$ for $\bar\alpha$ varying in a certain non-empty Zariski-open set of $(\mathrm{Pic}^0 X)^N$. If follows easily that $B(F \otimes A)$ is contained in the intersection of all $B^{\bar\alpha}$ for $\bar\alpha$ varying in $(\mathrm{Pic}^0 X)^N$. But such intersection is clearly empty, since already $\bigcap_{\alpha \in \mathrm{Pic}^0 X} B^\alpha$ is empty (it is the intersection of all translates of the base locus $B(A)$). □

3. Cohomological criterion for the surjectivity of multiplication maps.

The main result of this section is the announced cohomological criterion for the surjectivity of multiplication maps:

Theorem 3.1. *Let A and E and m are respectively an ample line bundle, a locally free sheaf on X and a positive integer such that*

$(*)$ $h^i(T_x^*(A^{\otimes m}) \otimes E) = 0$ for any $x \in X$ and for any $i > 0$.

Let also n be a fixed integer ≥ 2 and assume that $char(k)$ does not divide n. The following two conditions are equivalent

(a) $h^i(n_X^*((A^{\otimes m} \hat{*} E) \otimes A^\vee \otimes \alpha)) = 0$ for any $\alpha \in \mathrm{Pic}^0 X$ and for any $i > 0$;

(b) $h^i(E \otimes A^{\otimes -m(n-1)} \otimes (A^{\otimes mn} \hat{*} A^{\otimes n(mn-m-n)} \otimes \alpha)) = 0$ *for any* $i > 0$ *and for any* $\alpha \in \mathrm{Pic}^0 X$*;*

and, if they hold, the map $M_x : H^0(T_x^* A^{\otimes m}) \otimes H^0(E) \to H^0((T_x^* A^{\otimes m}) \otimes E)$ *is surjective for any* $x \in X$.

Before proving Theorem 3.1, let us make some comments about how to apply it. The point is: given an ample line bundle A and a vector bundle E, how to check the equivalent conditions (a) or (b)? For particular vector bundles, as the ones occurring in calculations concerning syzygies, this will be done in the next section. But there are some cases when the situation is much simpler: assume e.g. $n = m = 2$. In this case the vector bundle of (b) is $E \otimes A^{\otimes -2} \otimes (A^{\otimes 4} \hat{*} \alpha)$ which, by Remarks 1.4 and 1.5, is isomorphic to $H^0(A^{\otimes 4} \hat{*} \alpha^\vee) \otimes E \otimes A^{\otimes -2} \otimes \alpha^\vee$. Therefore Theorem 3.1 yields that *if* $h^i(E \otimes A^{\otimes -2} \otimes \alpha) = 0$ *for any* $i > 0$ *and* $\alpha \in \mathrm{Pic}^0 X$ *and assuming the (usually redundant) hypothesis* $(*)$*, then the multiplication map* $M_x : H^0(T_x^* A^{\otimes m}) \otimes H^0(E) \to H^0((T_x^* A^{\otimes m}) \otimes E)$ *is surjective for any* $x \in X$ *and for any* $m \geq 2$ (the case $m > 2$ is proved in a similar and in fact much easier way). This is essentially the cohomological criterion (implicitely) used by Kempf in [K5] to prove his theorem (mentioned in the Introduction) on Lazarsfeld's conjecture. As a particular case one has the following result of Kempf and Sekiguchi (which is turn includes Koizumi's Theorem): *let* m *and* k *such that* $m \leq k$, $m \geq 2$ *and* $m+k \geq 5$*. Then the multiplication map* $H^0(T_x^* A^{\otimes m}) \otimes H^0(A^{\otimes k}) \to H^0((T_x^* A^{\otimes m}) \otimes A^{\otimes k})$ *is surjective for any* $x \in X$ (see e.g. [K6], Th.8.6(c)).

Proof of Theorem 3.1. Let us start by proving the surjectivity of the maps M_x assuming $(*)$ and (a). Note that hypothesis $(*)$ is the "ground" hypothesis (4) of Section 1, so that all the results obtained there apply to the pair given by $A^{\otimes m}$ and E. Therefore, by Proposition 1.1 and the subsequent remark, it is equivalent to prove that $n_X^*(A^{\otimes m} \hat{*} E)$ is generated by its invariant global sections as soon as hypothesis (a) holds.

Let us write $n_X^*(A^{\otimes m} \hat{*} E) = n_X^*((A^{\otimes m} \hat{*} E) \otimes A^\vee \otimes \beta) \otimes n_X^*(A \otimes \beta^\vee)$ for any $\beta \in \mathrm{Pic}^0 X$. Then we have the multiplication map

$$H^0(n_X^*((A^{\otimes m} \hat{*} E) \otimes A^\vee \otimes \beta)) \otimes H^0(n_X^*(A \otimes \beta^\vee)) \xrightarrow{\rho_\beta} H^0(n_X^*(A^{\otimes m} \hat{*} E)) \tag{1}$$

On the other hand we have the decompositions

$$\begin{aligned} H^0(n_X^*((A^{\otimes m} \hat{*} E) \otimes A^\vee \otimes \beta)) &= \bigoplus_{\eta \in (\mathrm{Pic}^0 X)_n} n_X^* H^0((A^{\otimes m} \hat{*} E) \otimes A^\vee \otimes \beta \otimes \eta) \\ H^0(n_X^*(A \otimes \beta^\vee)) &= \bigoplus_{\epsilon \in (\mathrm{Pic}^0 X)_n} n_X^* H^0(A \otimes \beta^\vee \otimes \epsilon) \end{aligned} \tag{2}$$

Theefore for any $\beta \in \mathrm{Pic}^0 X$ we have that $n_X^* H^0(A^{\otimes m} \hat{*} E)$ – the space of invariant sections of $n_X^*(A^{\otimes m} \hat{*} E)$ – contains the space $\rho_\beta(W_\beta)$, where

$$W_\beta = \bigoplus_{\eta \in (\mathrm{Pic}^0 X)_n} n_X^* H^0((A^{\otimes m} \hat{*} E) \otimes A^\vee \otimes \beta \otimes \eta) \otimes n_X^* H^0(A \otimes \beta^\vee \otimes \eta^\vee) \tag{3}$$

We will see W_β as a space of global sections of the following vector bundle, direct sum of copies of $n_X^*(A\otimes\beta^\vee)$:

$$H^0(n_X^*((A^{\otimes m}\hat{*}E)\otimes A^\vee\otimes\beta))\otimes n_X^*(A\otimes\beta^\vee) \tag{4}$$

and, as such, let us denote B^β its the base locus, i.e., in the Notation 1.2,

$$B^\beta = B(W^\beta\ ,\ H^0(n_X^*((A^{\otimes m}\hat{*}E)\otimes A^\vee\otimes\beta))\otimes n_X^*(A\otimes\beta^\vee)\ ) \tag{5}$$

From the decomposition (2) of $H^0(n_X^*(A^{\otimes m}\hat{*}E)\otimes A^\vee\otimes\beta)$ it follows that

$$B^\beta = \bigcup_{\eta\in(\mathrm{Pic}^0X)_n} B(n_X^*H^0(A\otimes\beta^\vee\otimes\eta^\vee)\ ,\ n_X^*(A\otimes\beta^\vee)) \tag{6}$$

and therefore

$$B^\beta = n_X^{-1}\Big(\bigcup_{\eta\in(\mathrm{Pic}^0X)_n} B(A\otimes\beta^\vee\otimes\eta^\vee)\Big) \tag{7}$$

Given an ample line bundle L on X let us denote $\phi_L : X\to\mathrm{Pic}^0X$ the isogeny associated to L, i.e. $x\mapsto T_x^*L\otimes L^\vee$. The subset $S_\beta=\{\ A\otimes\beta^\vee\otimes\eta^\vee\}_{\eta\in(Pic^0X)_n}$ of Pic$X$, coincides with the subset $\{\,T_x^*A\,\}_{x\in\phi_A^{-1}(\beta^\vee+(\mathrm{Pic}^0X)_n)}$. Hence, given a line bundle, say $T_{\bar{z}}^*A$, in S_β, all the other line bundles in S_β are obtained by translation with the elements of $\phi_A^{-1}(\mathrm{Pic}^0X)_n$ (by the way, this last group is the kernel of $\phi_{A^{\otimes n}}$). In conclusion

$$B^\beta = n_X^{-1}\Big(\bigcup_{x\in\phi_A^{-1}((\mathrm{Pic}^0X)_n)} (x+\bar{z}+B(A)\Big) \tag{8}$$

Now by hypothesis (a) we can apply Lemma 2.1 and Corollary 2.2 to the vector bundle $n_X^*((A^{\otimes m}\hat{*}E)\otimes A^\vee)$: there is a finite integer N such that taking $(\beta_1,\cdots,\beta_N)$ general in $(\mathrm{Pic}^0X)^N$ the map

$$\bigoplus_{j=1}^N H^0(n_X^*((A^{\otimes m}\hat{*}E)\otimes A^\vee\otimes\beta_j))\otimes n_X^*(\beta_j^\vee)\to n_X^*(A^{\otimes m}\hat{*}E\otimes A^\vee) \tag{9}$$

is surjective (of course, to pass from Pic^0X to its cover $n_X^* : \mathrm{Pic}^0X\to\mathrm{Pic}^0X$ is completely harmless). Then the proof goes as the one of Theorem 2.3: let us consider the commutative diagram

$$\begin{array}{ccc} \bigoplus_{j=1}^N W^{\beta_j}\otimes\mathcal{O}_X & \xrightarrow{\oplus\rho_{\beta_j}} & n_X^*H^0(A^{\otimes m}\hat{*}E)\otimes\mathcal{O}_X \\ \Big\downarrow{\scriptstyle\oplus(\mathrm{id}\otimes ev_{\beta_j})} & & \Big\downarrow{\scriptstyle ev} \\ \bigoplus_{j=1}^N H^0(n_X^*((A^{\otimes m}\hat{*}E)\otimes A^\vee\otimes\beta_j))\otimes n_X^*(A\otimes\beta_j^\vee) & \longrightarrow & n_X^*(A^{\otimes m}\hat{*}E) \end{array} \tag{10}$$

Twisting (9) by $n_X^*(A)$ and using Cor.2.2 it follows that the bottom arrow is surjective for general $(\beta_1,\dots,\beta_N)$. Therefore, denoting $B = B(\, n_X^* H^0(A^{\otimes m}\hat{*}E)\,,\, n_X^*(A^{\otimes m}\hat{*}E))$ (see Notation 1.2.), we have that $B \subset \bigcup_{i=1}^N B^{\beta_i}$ for $(\beta_1,\dots,\beta_N)$ general in $(\mathrm{Pic}^0 X)^N$. It follows easily that in fact $B \subset \bigcup_{i=1}^N B^{\beta_i}$ for any $(\beta_1,\dots,\beta_N) \in (\mathrm{Pic}^0 X)^N$. Hence it is sufficient to prove that

$$\bigcap_{(\beta_1,\dots,\beta_N)\in(\mathrm{Pic}^0 X)^N} \Big(\bigcup_{i=1}^N B^{\beta_i} \Big) = \emptyset \tag{11}$$

But this follows immediately from the description (8) of the base loci B^β.

It remains to prove that, assuming $(*)$, (a) and (b) are equivalent. First of all, let us observe that $n_X^*((A^{\otimes m}\hat{*}E)) \otimes A^\vee) \cong (A^{\otimes nm}\hat{*}(E\otimes A^{\otimes -m(n-1)})) \otimes n_X^*(A^{\otimes m-1})) \otimes A^{\otimes -nm}$ (Prop. 1.3) and that $n_X^*(A^{\otimes m-1})$ is algebraically equivalent to $A^{\otimes n^2(m-1)}$. Therefore (a) is equivalent to

$$h^i((A^{\otimes mn}\hat{*}(E\otimes A^{\otimes -m(n-1)})) \otimes A^{\otimes n(mn-m-n)} \otimes \alpha) = 0 \tag{12}$$

for any $i > 0$ and $\alpha \in \mathrm{Pic}^0 X$. Finally, the fact that, assuming $(*)$, (12) and (b) are equivalent is in turn a particular case of the following

Lemma 3.2. *Let h be a positive integer and let A and M be a line bundles on X such that A and $A^{\otimes h}\otimes M$ are ample. Let moreover E be a locally free sheaf such that $h^j(T_x^*(A^{\otimes h})\otimes E) = 0$ for any $j > 0$ and for any $x \in X$. Then $H^i((A^{\otimes h}\hat{*}E)\otimes M) \cong H^i(E\otimes(A^{\otimes h}\hat{*}M))$ for any $i \geq 0$.*

Proof. By projection formula we have that

$$(A^{\otimes h}\hat{*}E)\otimes M \cong p_{1*}(p_1^*(M)\otimes(p_1+p_2)^*(A^{\otimes h})\otimes p_2^*(E))$$
$$E\otimes(A^{\otimes h}\hat{*}M) \cong p_{1*}(p_1^*(E)\otimes(p_1+p_2)^*(A^{\otimes h})\otimes p_2^*(M))$$

Applying the automorphism (p_2,p_1) of $X\times X$ to the right hand side of the bottom line we have

$$E\otimes(A^{\otimes h}\hat{*}M) \cong p_{2*}(p_1^*(M)\otimes(p_1+p_2)^*(A^{\otimes h})\otimes p_2^*(E)). \tag{13}$$

To simplify the notation, let us denote $\mathcal{F} = p_1^*(M)\otimes(p_1+p_2)^*(A^{\otimes h})\otimes p_2^*(E)$. By projection formula, the hypothesis ensures that $R^j p_{1*}(\mathcal{F}) = 0$ for any $j > 0$, while the fact that $A^{\otimes h}\otimes M$ is ample yields that $R^j p_{2*}(\mathcal{F}) = 0$ for any $j > 0$. Hence the statement follows from the isomorphism (13) and the fact that the Leray spectral sequences of p_1 and p_2 degenerate to isomorphisms $H^i({p_j}_*\mathcal{F}) \cong H^i(\mathcal{F})$ for $j = 1,2$. $\square$

4. Proof of Lazarsfeld's conjecture.

Let us begin by rewiewing some background material – well known and largely due to Green – about the relation between conditions about syzyigies, like N_p, to the Koszul complex resolving the residue field k as a module over the symmetric algebra. The main fact (see [G1], [G2] Thm 1.2, [L2], p.511 for details) is that condition N_p is equivalent to the exactness in the middle of the complex

$$\overset{p+1}{\Lambda}\ H^0(L)\otimes H^0(L^{\otimes h}) \to \overset{p}{\Lambda}\ H^0(L)\otimes H^0(L^{\otimes h+1}) \to \overset{p-1}{\Lambda}\ H^0(L)\otimes H^0(L^{\otimes h+2}) \tag{1}$$

for any $h \geq 1$. Vector bundles come into play via the observation that, defined M_L the kernel of the evaluation map of L:

$$0 \to M_L \to H^0(L) \otimes \mathcal{O}_X \to L \to 0, \tag{2}$$

and taking wedge products of (2)

$$0 \to \overset{p+1}{\Lambda}\ M_L \to \overset{p+1}{\Lambda}\ H^0(L) \otimes \mathcal{O}_X \to \overset{p}{\Lambda}\ M_L \otimes L \to 0, \tag{3}$$

the exactness of the complex (1) is equivalent to the surjectivity of the map

$$\overset{p+1}{\Lambda}\ H^0(L) \otimes H^0(L^{\otimes h}) \to H^0(\overset{p}{\Lambda}\ M_L \otimes L^{\otimes h+1}) \tag{4}$$

obtained twisting (3) with $L^{\otimes h}$ and taking H^0 of the third arrow. Therefore, if

$$H^1(\Lambda^{p+1} M_L \otimes L^{\otimes h}) = 0 \tag{5}$$

for any $h \geq 1$ then condition N_p holds (the converse is also true as soon as $H^1(L^{\otimes h}) = 0$, as for abelian vareties). This leads to

Lemma 4.1. *(a) Assume char$(k) = 0$. If $H^1(M_L^{\otimes p+1} \otimes L^{\otimes h}) = 0$ for any $h \geq 1$ then L satisfies condition N_p.*

(b) Assume that $H^1(M_L^{\otimes p} \otimes L^{\otimes h}) = 0$. Then $H^1(M_L^{\otimes p+1} \otimes L^{\otimes h}) = 0$ if and only if the multiplication map $H^0(L) \otimes H^0(M_L^{\otimes p} \otimes L^{\otimes h}) \to H^0(M_L^{\otimes p} \otimes L^{\otimes h+1})$ is surjective.

Proof. (a) follows immediately from (5) since, in characteristic zero, $\Lambda^{p+1} M_L$ is a direct summand of $M_L^{\otimes p+1}$, while (b) follows from the exact sequence

$$0 \to M_L^{\otimes p+1} \otimes L^{\otimes h} \to H^0(L) \otimes M_L^{\otimes p} \otimes L^{\otimes h} \to M_L^{\otimes p} \otimes L^{\otimes h+1} \to 0. \tag{6}$$

Proof of Lazarsfeld's conjecture. We will work over an algebraically closed field of characteristic zero. Since the proof, although very short, may seem somehow obscure, let us give a rough outline first. Let us denote $L = A^{\otimes m}$. As announced, the strategy is to use Lemma 4.1 to reduce the problem to check the surjectivity of the multiplication map $H^0(L) \otimes H^0({M_L}^{\otimes p} \otimes L) \to H^0({M_L}^{\otimes p} \otimes L^{\otimes 2})$ (the case $h = 1$ is the critical one, the cases $h > 1$ are easier). Then we use the criterion of Thm 3.1 with $n = 2$. All we have to do is to check that the hypotheses of Thm 3.1 are satisfied in this case. Condition $(*)$ is very mild while the serious condition is (b), i.e. that the higher cohomology of all sheaves ${M_L}^{\otimes p} \otimes (L^{\otimes 2} \hat{*} (A^{\otimes 2(m-2)} \otimes \alpha))$ vanish. Now, using a sequence like (6) twisted by ${M_L}^{\otimes p-1} \otimes (L^{\otimes 2} \hat{*} A^{\otimes 2(m-2)} \otimes \alpha))$ one sees that the required vanishings follow from the vanishing of the higher cohomology of the sheaves ${M_L}^{\otimes p-1} \otimes (L^{\otimes 2} \hat{*} A^{\otimes 2(m-2)} \otimes \alpha))$ and from the surjectivity of the multiplication map

$$H^0(L) \otimes H^0({M_L}^{\otimes p-1} \otimes (L^{\otimes 2} \hat{*} A^{\otimes 2(m-2)} \otimes \alpha)) \to H^0(L \otimes {M_L}^{\otimes p-1} \otimes (L^{\otimes 2} \hat{*} A^{\otimes 2(m-2)} \otimes \alpha))$$

Applying Theorem 3.1 another time, we are reduced to the vanishing of the higher cohomology of ${M_L}^{\otimes p-1} \otimes L^{\otimes -1} \otimes (L^{\otimes 2} \hat{*} (A^{\otimes 2(m-2)} \otimes \alpha)) \otimes (L^{\otimes 2} \hat{*} (A^{\otimes 2(m-2)} \otimes \beta))$. After repeating this procedure p times one eliminates all M_L's, so that we are reduced to the vanishing of

$$L^{\otimes -p} \otimes \bigotimes_{i=1}^{p+1} (L^{\otimes 2} \hat{*} A^{\otimes 2(m-2)} \otimes \alpha_i). \tag{7}$$

Now, passing to a suitable finite cover, the sheaves $L^{\otimes 2} \hat{*} A^{\otimes 2(m-2)} \otimes \alpha_i$ become the sum of copies of the same line bundle (Prop.1.6), so that to check the vanishing of the higher cohomology of (7) is immediate.

Now let us give the formal proof. Throughout what follows, given $\alpha \in \operatorname{Pic}^0 X$, we will denote

$$F^{(m)}_{\alpha} = A^{\otimes 2m} \hat{*} (A^{\otimes 2(m-2)} \otimes \alpha).$$

Proposition 4.2. *Let A be an ample line bundle on X and let p and r be fixed positive integers. Let also h, k, j, m be integers such that $k \geq 0$ and $0 \leq j \leq p+1$. If (a) $(r+1)m \geq p+3+r$; (b) $k+h \geq r+1$; (c) $h-j \geq r-p$ then*

$$H^i((M_{A^{\otimes m}})^{\otimes j} \otimes A^{\otimes mh} \otimes \beta \otimes \bigotimes_{l=1}^{k} F^{(m)}_{\alpha_l}) = 0 \tag{8}$$

for any $\alpha_1, \ldots, \alpha_k, \beta \in \operatorname{Pic}^0 X$ and for any $i > 0$.

Note that Lazarsfeld's conjecture (the Theorem of the introduction) follows as a special case: take $r = 0$, $i = 1$, $j = p+1$, $k = 0$, and $\beta = \mathcal{O}_X$ in Prop.4.2 and then apply Lemma 4.1(a). □

Proof. The proof will be by induction on j. If $j = 0$ we have to prove that if conditions (a_0) $(r+1)m \geq p+3+r$; (b_0) $k+h \geq r+1$; (c_0) $h \geq r-p$ hold then the sheaves

$$\mathcal{E} = A^{\otimes mh} \otimes \beta \otimes \bigotimes_{l=1}^{k} A^{\otimes 2m} \hat{*} (A^{\otimes 2(m-2)} \otimes \alpha_l), \tag{9}$$

have vanishing higher cohomology for any $\alpha_1, \ldots, \alpha_l, \beta \in \operatorname{Pic}^0 X$. Since $H^i(\mathcal{E})$ is a direct summand of $H^i((4m-4)^*_X \mathcal{E})$, it is enough to prove that $(4m-4)^*_X(\mathcal{E})$ has vanishing higher cohomology. By Prop.1.6 and noting that $(a+b)^*_X(A^{\otimes a}) \otimes a^*_X(A^{\otimes -(a+b)})$

is a line bundle algebraically equivalent to $A^{\otimes(a+b)ab}$, we have that the vector bundle $(4m-4)_X^*\mathcal{E}$ is of the form $V\otimes M\otimes(4m-4)_X^*(A^{\otimes mh}\otimes\beta)$, where V is a vector space (namely $V=\bigotimes_{l=1}^k H^0(A^{\otimes 4m-4}\otimes\alpha_l)$) and M is a line bundle algebraically equivalent to $A^{\otimes k(4m-4)(2m-4)2m}$. Since $(4m-4)_X^*(A^{\otimes mh}\otimes\beta)$ is algebraically equivalent to $A^{\otimes(4m-4)^2mh}$ it is sufficient to prove that the quantity

$$f(m,h)=k(4m-4)(2m-4)2m+(4m-4)^2mh$$

is positive as soon as (a_0),(b_0) and (c_0) hold, which follows from an easy calculation. This proves the case $j=0$. Now let us fix j such that $0<j\leq p+1$ and $\alpha_1,\dots,\alpha_k,\beta\in\mathrm{Pic}^0X$. We have to prove (8) provided that (a), (b) and (c) hold. By the exact sequence

$$0\to(M_{A^{\otimes m}})^{\otimes j}\to H^0(A^{\otimes m})\otimes(M_{A^{\otimes m}})^{\otimes j-1}\to A^{\otimes m}\otimes(M_{A^{\otimes m}})^{\otimes j-1}\to 0 \qquad (10)$$

twisted by $A^{\otimes mh}\otimes\beta\otimes\bigotimes_{l=1}^k F_{\alpha_l}^{(m)}$ we have that the vanishings (8) are implied by
(i) the vanishing of the higher cohomology of $(M_{A^{\otimes m}})^{\otimes j-1}\otimes A^{\otimes mh}\otimes\beta\otimes\bigotimes_{l=1}^k F_{\alpha_l}^{(m)}$;
(ii) the vanishing of the higher cohomology of $(M_{A^{\otimes m}})^{\otimes j-1}\otimes A^{\otimes m(h+1)}\otimes\beta\otimes\bigotimes_{l=1}^k F_{\alpha_l}^{(m)}$;
(iii) the surjectivity of the multiplication map

$$H^0(A^{\otimes m})\otimes H^0(M_{A^{\otimes m}}^{\otimes j-1}\otimes A^{\otimes mh}\otimes\beta\otimes\bigotimes_{l=1}^k F_{\alpha_l}^{(m)})\to H^0(M_{A^{\otimes m}}^{\otimes j-1}\otimes A^{\otimes m(h+1)}\otimes\beta\otimes\bigotimes_{l=1}^k F_{\alpha_l}^{(m)}) \qquad (11)$$

Now (i) and (ii) hold by induction. (iii): by Theorem 3.1 applied with $n=2$, we have that the surjectivity of (11) follows from the following conditions
(iv) the vanishing of the higher cohomology of ${M_L}^{\otimes j-1}\otimes T_x^*A^{\otimes 2m}\otimes A^{\otimes mh}\otimes\beta\otimes\bigotimes_{l=1}^k F_{\alpha_l}^{(m)}$ *for any* $x\in X$ (this is hypothesis $(*)$ of Th. 3.1);
(v) the vanishing of the higher cohomology of ${M_L}^{\otimes j-1}\otimes A^{\otimes m(h-1)}\otimes\beta\otimes F_\alpha\otimes\bigotimes_{l=1}^k F_{\alpha_l}^{(m)}$ *for any* $\alpha\in\mathrm{Pic}^0X$ (this is hypothesis (b) of Th. 3.1).

Again, (iv) clearly holds by induction, so the only serious condition we have to check is (v) which holds by induction too since h decreases by one and k increases by one. ▫

A generalization of Lazarsfeld's conjecture. Let $m<p+3$, so that property N_p may fail for $A^{\otimes m}$ (in fact this is what happens if A is a principal polarization). Our last result supplies an upper bound of "how much" it fails. For example one knows that $A^{\otimes 2}$ is h-normal for $h\geq 3$, i.e., going back to the notation of the introduction, $a_{0j}\leq 2$ for any j. Along the same lines a result of Kempf ([K4], [K6] Thm.6.13(a)) states that the homogeneous ideal of the abelian variety X, embedded by the line bundle $A^{\otimes 3}$, is generated by quadrics of cubics , i.e. $a_{1j}\leq 3$ for any j. So in these cases the failure of N_p is of "at most one". More generally, given an integer $r\geq 0$, one may extend Green's condition as follows: L is said to *satisfy property* N_0^r if $a_{0j}\leq 1+r$ (i.e. the embedded variety X is h-normal for $h\geq 2+r$); L is said to *satisfy property* N_1^r if it satisfies N_0^r and $a_{ij}\leq 2+r$ for any j. Inductively one says that L *satisfies property* N_p^r if it satisfies N_{p-1}^r and $a_{pj}\leq p+1+r$ for any j. Roughly, this means that property N_p fails of at most r (note that N_p becomes N_p^0

in this terminology). It should be also said that Castelnuovo-Mumford's theorem yields that any very ample line bundle L satisfies N_p^{g+1}, where $g = \dim X$. The results of Kempf's work [K5] imply, in this terminology, that *if* $(r+1)m \geq \max(3, 2r+2, 2p+2)$ *then* $A^{\otimes m}$ *satisfies* N_p^r. Again, we improve such a result of a factor two:

Theorem 4.3. *(char(k) = 0) If* $(r+1)m \geq p+3+r$ *then* $A^{\otimes m}$ *satisfies* N_p^r.

Proof. By Prop.4.2 applied to the case $i = 1$, $j = p+1$, $k = 0$ and $\beta = \mathcal{O}_X$ one gets that

$$H^1((M_{A^{\otimes m}})^{\otimes p+1} \otimes A^{\otimes hm}) = 0 \tag{13}$$

for any $h \geq r+1$ as soon as $(r+1)m \geq p+3+m$. The statement follows since, arguing exactly as for Lemma 4.1, (13) yields property N_p^r. □

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