Linear Algebra and Geometry. Exam feb. 18, 2015.

1. Let $\mathbf{u} = (1,1,1,-1)$, $\mathbf{v} = (1,1,1,1)$, $\mathbf{w} = (1,0,0,0)$.
(a) Find $\mathbf{v}'$ and $\mathbf{w}' \in \mathcal{V}_4$ such that: $\mathbf{v}'$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$, $\mathbf{w}'$ is a linear combination of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, and $\{\mathbf{u},\mathbf{v}',\mathbf{w}'\}$ is an orthogonal set.
(b) Write $(0,1,0,0)$ as the sum of a linear combination of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ and of a vector orthogonal to $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.

Hint

(a) The request can be summarized as: find $\mathbf{v}' \in L(\mathbf{u},\mathbf{v})$ and $\mathbf{w}' \in L(\mathbf{u},\mathbf{v},\mathbf{w})$ such that $\{\mathbf{u},\mathbf{v}',\mathbf{w}'\}$ is an orthogonal set. Therefore what is required is just the usual Gram=Schmidt orthogonalization of the vectors $\mathbf{u},\mathbf{v},\mathbf{w}$.

(b) By the *Orthogonal Decomposition Theorem* there is a unique solution to this problem:

$$(0,1,0,0) = pr_{L(\mathbf{u},\mathbf{v},\mathbf{w})}((0,1,0,0)) + pr_{L(\mathbf{u},\mathbf{v},\mathbf{w})^\perp}((0,1,0,0))$$

To find the two projections, one first observes that it fact it is enough to compute one one of them: again by the *Orthogonal Decomposition Theorem* one projection determines the other one (for example: $pr_{L(\mathbf{u},\mathbf{v},\mathbf{w})}((0,1,0,0)) = (0,1,0,0) - pr_{L(\mathbf{u},\mathbf{v},\mathbf{w})^\perp}((0,1,0,0))$).

To compute one projection one can proceed in two different ways:

(i) Compute $pr_{L(\mathbf{u},\mathbf{v},\mathbf{w})}((0,1,0,0))$. For this one uses point (a): since $\{\mathbf{u},\mathbf{v}',\mathbf{w}'\}$ is an orthogonal basis of $L(\mathbf{u},\mathbf{v},\mathbf{w})$ then

$$pr_{L(\mathbf{u},\mathbf{v},\mathbf{w})}((0,1,0,0)) = pr_{L(\mathbf{u})}((0,1,0,0)) + pr_{L(\mathbf{v}')}((0,1,0,0)) + pr_{L(\mathbf{w})'}((0,1,0,0)) =$$

$$= \frac{(0,1,0,0)\cdot\mathbf{u}}{(\mathbf{u},\mathbf{u})}\mathbf{u} + \frac{(0,1,0,0)\cdot\mathbf{v}'}{(\mathbf{v}',\mathbf{v}')}\mathbf{v}' + \frac{(0,1,0,0)\cdot\mathbf{w}'}{(\mathbf{w}',\mathbf{w}')}\mathbf{w}'$$

(ii) Compute $pr_{L(\mathbf{u},\mathbf{v},\mathbf{w})^\perp}((0,1,0,0))$. For this one first computes a basis of $L(\mathbf{u},\mathbf{v},\mathbf{w})^\perp$ (which is one-dimensional). For this one just needs to solve the system (a very easy one!)

$$\begin{cases} (x,y,z,t)\cdot\mathbf{u} &= 0\\ (x,y,z,t)\cdot\mathbf{v} &= 0\\ (x,y,z,t)\cdot\mathbf{w} &= 0\end{cases}$$

. The space of solutions is $L(\mathbf{u},\mathbf{v},\mathbf{w})^\perp$ and it will be of the form $L(\mathbf{h})$. Then

$$pr_{L(\mathbf{u},\mathbf{v},\mathbf{w})^\perp}((0,1,0,0)) = \frac{(0,1,0,0)\cdot\mathbf{h}}{(\mathbf{h},\mathbf{h})}\mathbf{h}$$

2. Let us consider the planes $M = \{(1,0,2) + t(1,2,1) + s(0,1,-2)\}$ and $N = \{(1,3,0) + t(1,1,1) + s(1,0,-1)\}$. Describe and find equations for the set of points $P \in \mathcal{V}_3$ such that $d(P,M) = d(P,N)$.

Solution

Letting $P = (x,y,z)$, we are interested in the set defined by the equation

$$d((x,y,z),M) = d((x,y,z),N) \tag{1}$$

To find an explicit expression for the left and right hand side, we compute the cartesian equations of the two planes. They turn out to be

$$M \,:\, 5x - 2y - z = 4 \qquad \text{and} \qquad N:\, x - 2y + z = 1$$

Therefore

$$d((x,y,z),M) = \frac{|5x-2y-z-4|}{\sqrt{6}} \qquad \text{and} \qquad \frac{d((x,y,z),N) = |x-2y+z-1|}{\sqrt{30}}$$

Therefore (1) becomes

$$\frac{|5x-2y-z-4|}{\sqrt{6}} = \frac{|x-2y+z-1|}{\sqrt{30}}$$

which is equivalent to

$$\frac{5x-2y-z-4}{\sqrt{6}} = \frac{x-2y+z-1}{\sqrt{30}} \qquad \text{or} \qquad \frac{5x-2y-z-4}{\sqrt{6}} = -\frac{x-2y+z-1}{\sqrt{30}}$$

which is the equation of the union of two planes.

3. Let V be the linear space of all real polynomials of degree ≤ 2. Find which of the following functions $T : V \to V$ are linear transformations. If the answer is affirmative find $N(T)$ and $T(V)$

(a) $T(P(x)) = xP'(x) - P(x) - 1$;
(b) $T(P(x)) = xP'(x) - P(x+1)$;
(c) $T(P(x)) = (x+1)P'(x) - P(x)$.

Solution

(a) T is not linear. For example, letting $O(x)$ the polynomial which is identically zero (namely $O(x) \equiv 0$) we find $T(O(x)) \equiv -1$. Therefore T is non-linear, because we know that in general, $T(O) = O$ if T is linear (because $T(O) = T(\mathbf{u} - \mathbf{u}) = T(\mathbf{u}) - T(\mathbf{u}) = O$).

(b) T is linear. Indeed:

$$T(P_1(x)+P_2(x)) = T((P_1+P_2)(x)) = x(P_1'+P_2')(x)) -)P_1+P_2)(x+1) =$$

$$= xP_1'(x) - P_1(x+1) + xP_2'(x) - P_2(x+1) =$$

$$= T(P_1(x)) + T(P_2(x))$$

and

$$T(\lambda P(x)) = x(\lambda P'(x)) - \lambda P(x+1) = \lambda(xP'(x) - P(x+1)) =$$

$$= \lambda T(P(x))$$

Calculation of $N(T)$. $N(T)$ is the space of polynomials $P(x) = a + bx + cx^2$ such that $xP'(x) = P(x+1)$, that is

$$bx + 2cx^2 = a + b + c + (b + 2c)x + cx^2$$

that is $c = 0$ and $a = -b$. Therefore

$$N(T) = \{a(x-1) \mid a \in \mathbf{R}\} = L(x-1)$$

Moreover, by the *nullity plus rank Theorem* $T(V)$ is two-dimensional, spanned by $\{T(1), T(x), T)x^2)\}$. Since (for example) $T(1)$ and $T(x^2)$ are independent, it follows that

$$T(V) = L(T(1), T(x^2)) = L(1, x^2 - 2x - 1).$$

(c) Similarly: - T is linear (DO THAT!);
- $N(T) = L(x+1)$ (DO THAT!)
- $T(V) = L(1, x^2 + 2x + 1)$ (DO THAT!)

4. Let $\mathbf{u} = (1,0,1)$, $\mathbf{v} = (-1,1,1)$, and, for t varying in $\mathbf{R}$, $\mathbf{w}_t = (2,1,t)$.
Let also $\mathbf{a} = (2,3,1)$, $\mathbf{b} = (1,-3,1)$, and, for s varying in $\mathbf{R}$, $\mathbf{c}_s = (1,s,0)$.
Find all $t, s \in \mathbf{R}$ such that there is a unique linear transformation $T_{t,s} : \mathcal{V}_3 \to \mathcal{V}_3$ such that $T(\mathbf{u}) = \mathbf{a}$, $T(\mathbf{v}) = \mathbf{b}$ and $T(\mathbf{w_t}) = \mathbf{c}_s$.

Solution

By the *Theorem on linear transformations with prescribed values* , the transformation exists and it is unique if and only if both $\{\mathbf{u}, \mathbf{v}, \mathbf{w}_t\}$ and $\{bfa, \mathbf{b}, \mathbf{c}_s\}$ are bases of $\mathcal{V}_3$, that is are independent sets. An easy computation (for example with determinants) shows that this happens if and only if $t \neq 4$ and $s \neq 6$.

5. Find all values of $t \in \mathbf{R}$ such that the matrix $A_t = \begin{pmatrix} 1 & 1 & t \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ is diagonalizable. For such values of t find a basis $\mathcal{B}$ of $\mathcal{V}_3$ whose elements are eigenvectors of A_t.

Solution

It is clear that the eigenvalues are 1 (double) and 3 (simple). Therefore the eigenvalues are known. Let us compute the eigenspaces:

$$E(3) = N(3I_3 - A_t) = N(\begin{pmatrix} 2 & -1 & -t \\ 0 & 0 & -1 \\ 0 & 0 & 2 \end{pmatrix}) = L((1,2,0))$$

(easy computation).

$$E(1) = N(1I_3 - A_t) = N(\begin{pmatrix} 0 & -1 & -t \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix})$$

If the two non-zero rows are independent, that is $t \neq -1/2$, then the space of solutions of the homogeneous system (the eigenspace) has dimension one. Therefore, together with an eigenvector of 3, we can find at most two independent eigenvectors, hence the matrix A_t is *not* diagonalizable.
Instead, if $t = -1/2$, $\quad E(1) = L((1.0.0),(0,1,2))$. Therefore for $t = -1/2$ we have that A_t is diagonalizable and $\{(1,2,0),(1,0,0),(0,1,2)\}$ is a basis of eigenvectors.