LINEAR ALGEBRA AND GEOMETRY

SUPPLEMENTARY EXERCISES, 3 (Planes, area, volume...)

1. Find the line containing $(1, -1, 1)$, parallel to the plane $M : x - 3y = 2$, and perpendicular to $A = (2, 1, 1)$.

2. Let $M = \{(1, 2, -1)+s(1, 2, 2)+t(1, -1, 0)\}$ and $U = \{(0, 2, 3)+s(1, 1, 1)+t(-1, 2, 3)\}$. Find a parametric equation of the line $M \cap U$.

3. For each of the following pairs of lines L and S in $\mathcal{V}_3$ compute the intersection of L and R (note: the intersection can be empty) and answer to the following question: is there a plane containing both L and R?
(a) $L = \{(1, 0, -1) + t(1, 1, 1)\}$ $R = \{(2, 0, 2) + t(1, 2 - 1)\}$
(b) $L = \{(1, 0, 0) + t(1, 1, 1)\}$ $R = \{(2, 0, 2) + t(1, 2 - 1)\}$
(c) $L = \{(1, 0, 0) + t(1, 1, 1)\}$ $R = \{(2, 0, 2) + t(1, 1, 1)\}$

4. Let L and M be two lines in $\mathcal{V}_n$. Prove that there is a plane of $\mathcal{V}_n$ containing both L and R if and only if L and R meet (that is, their intersection is a point) or they are parallel.

5. Let $L = \{(1, 1) + t(1, -2)\}$ and let $Q = (1, 0)$. (a) Compute $d(Q, L)$ and find the point $\bar{P}$ of L which is nearest to Q. (b) Find all points $S \in L$ such that the triangle whose vertices are $Q, \bar{P}$ and S is isosceles (this means that two edges have same length). Compute the area of these triangles.

6. Let $A = (1, -2, 2)$ and $B = (1, 2, 0)$. Find a vector $C \in L(A, B)$ such that the angle between A and C is $\pi/4$ and the area of the parallelogram determined by A and C is 1.

7. Let $A = (1, 1, 1)$ and $B = (1, -1, 1)$. (a) Find a orthonormal basis C, D, E of $\mathcal{V}_3$ such that C is parallel to A and D belongs to $L(A, B)$. (b) Find the vector $V \in \mathcal{V}_3$ such that $C \cdot V = \sqrt{3}$, $D \cdot V = \sqrt{2}$, $E \cdot V = \sqrt{2}$.

8. Let L be the line $\{(1, 0, 1) + t(1, 0, 2)\}$, and let M be the plane $\{(0, 1, 0) + t(1, 1, -1) + s(1, 0, 1)\}$. Moreover let $Q = (1, 0, 0)$.
(a) Find all points X belonging to L such that $d(X, M) = 1$.
(b) Find all points X belonging to L such that $d(X, Q) = 1$.

9 . Let L be the intersection of the two planes $M_1 : x - y + z = 1$ and $M_2 : y - 2z = -1$. Let moreover Π be the plane $\{(1, 2, 1) + t(1, -1, -) + s(-1, 0, 2)\}$. Find (if any) a plane containing L and perpendicular to Π. Is such plane unique?

10. Let $A = (1, 2, -2)$ and $P = (1, -5, -2)$. Find two planes orthogonal to A such that their distance from Q is 6.

11. Let $M = \{(3, 1, 1) + t(-2, 1, 0) + s(2, 1, 2)\}$. Find the plane parallel to M such that their distance from $(1, 1, 1)$ is 2. For each of such planes, find the nearest point to $(1, 1, 1)$.

12. Let $M_1 : x - 3y + 2z = 0$ and $M_2 : 2x - 3y + z = 3$. Let moreover $A = (1, 0, -1)$.
(a) Find the plane M containing the line $L = M_1 \cap M_2$ and parallel to A.
(b) Let $Q = (0, -1, 1)$. Find the line R which is perpendicular to M and contains Q.
(c) Find $R \cap M$.

13. let $P = (0, -1, 0)$, $Q = (1, 0, -1)$, $R = (1, -1, 1)$.
(a) Find the area of the triangle whose vertices are P, Q and R.
(b) Let $S = (1, -1, -2)$. Compute the volume of the parallepiped determined by the vectors $Q - P$, $R - P$ and $S - P$.
(c) Find the points X, collinear with P and S such that the volume of the parallepiped determined by the vectors $Q - P$, $R - P$ and $X - P$ is 2.

14. For t varying in $\mathbf{R}$, let $A_t = (t + 2, 1, 1)$, $B_t = (1, t + 2, 1)$, $C_t = (1, 1, t + 2)$.
(a) Find all $t \in \mathbf{R}$ such that A_t, B_t, C_t form an independent set. For such t compute (as a function of t) the volume of the parallelepiped determined by A_t, B_t, C_t. In particular, compute such volume for $t = -2$.
(b) Compute, for each $h \in \mathbf{R}$, the solutions of the system (depending on h): $\begin{cases} hx + y + z = 0 \\ x + hy + z = 0 \\ x + y + hz = 0 \end{cases}$.

15. For t, a varying in $\mathbf{R}$, let us consider the system $\begin{cases} x + y + z = 1 \\ 2x + ty + z = -1 \\ 6x + 7y + 3z = a \end{cases}$. Find for which values of t and a there is a unique solution, no solution, infinitely many solutions.

16. Let $A = (2, -1, 2)$ and $B = (1, 0, -1)$ (note that $A \cdot B = 0$). Describe and find parametric equations for the subset of vectors $X \in \mathcal{V}_3$ which are orthogonal to B and such that the angle between A and X is $\pi/4$.