L.A.G. July 18 2014

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1) Let $\underline{u} = (1,2,2)$ and $\underline{v} = (3,1,2)$.

(a) Find an orthogonal basis of V_3, say $\mathcal{B} = \{\underline{a}, \underline{b}, \underline{c}\}$, such that $L(\underline{a}, \underline{b}) = L(\underline{u}, \underline{v})$

(b) Find an orthogonal basis of V_3, say $\mathcal{C} = \{\underline{d}, \underline{e}, \underline{f}\}$, such that the angle between $\underline{u}$ and $\underline{d}$ is $\pi/6$ and $L(\underline{d}, \underline{e}) = L(\underline{u}, \underline{v})$

2) Let $P = (-1,1,1)$ $Q = (2,2,5)$ $\underline{v} = (3,-2,4)$

Find the points R contained in the line $\{P + t\underline{v} \mid t \in \mathbb{R}\}$ such that the triangle whose vertices are P, Q and R has area equal to 30.

3) Let us consider the plane curve of polar equation

$$\rho = 3(1-\cos\theta) \qquad 0 \leq \theta \leq 2\pi$$

(a) Compute the length of the curve

(b) Compute the curvature for $\theta = \pi/2$

4) Let U be the linear space of real polynomials of degree ≤ 2.
Let $T : U \to U$ be the linear transformation defined as follows:
$$T(P(x)) = P'(x)(x-1) - 2P(x)$$

(a) Find a basis of $N(T)$ and a basis of $T(U)$

(b) What is the matrix representing T with respect to the basis $\mathcal{E} = \{1, x, x^2\}$?

5) Let $\mathcal{C}$ be the conic section defined by the cartesian equation

$$x^2 + 4xy - 2y^2 - 10x + 13 = 0$$

Find the coordinates of the center and the equations of the symmetry axes ~~(Sketch the)~~ (in the ~~usual~~ (x,y)-coordinates).

~~Sketch~~ Draw a sketch of the conic section $\mathcal{C}$ (in the (x,y)-plane).

SOLUTION

Ex. 1 (a) $\underline{a} = \underline{u}$ $\quad \underline{b} = \underline{v} - \left(\frac{\underline{v}\cdot\underline{u}}{\underline{u}\cdot\underline{u}}\right)\underline{u} = (3,1,2) - (1,2,2) = (2,-1,0)$

$$\underline{c} = \underline{u}\times\underline{v} = (2,4,-5)$$

(b) To find the vector $\underline{d}$ we argue as follows:

let $\underline{a}' = \frac{\underline{a}}{\|\underline{a}\|} = \frac{1}{3}(1,2,2)$ and $\underline{b}' = \frac{\underline{b}}{\|\underline{b}\|} = \frac{1}{\sqrt{5}}(2,-1,0)$.

Then $\underline{d} = \cos\frac{\pi}{6}\,\underline{a}' + \sin\frac{\pi}{6}\,\underline{b}'$ forms an angle of $\pi/6$ with $\underline{a}'$, hence also with $\underline{a} = \underline{u}$.

Explicitly $\underline{d} = \frac{\sqrt{3}}{2}\cdot\frac{1}{3}(1,2,2) + \frac{1}{2}\cdot\frac{1}{\sqrt{5}}(2,-1,0)$

Next we find $\underline{e}$. We take

$$\underline{e} = \frac{1}{2}\cdot\frac{1}{3}(1,2,2) - \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{5}}(2,-1,0).$$

Concerning $\underline{f}$, we can take $\underline{f} = \underline{c} = \underline{u}\times\underline{v} = (2,4,-5)$.

Ex. 2 Let $R_t = P + t\,\underline{v}$

The area of the triangle whose vertices are P, Q, R_t is $\|(Q-P)\times(R_t-P)\|$.

we have: $Q-P = (3,1,4)$ $\quad R_t - P = t\underline{v} = t(3,-2,4)$.

$$\|(Q-P)\times(R_t-P)\| = \|t(12,0,-9)\| = \|3t(4,0,-3)\| = 3|t|5$$

Hence $\cancel{3|t|\sqrt{}}$ $\frac{15|t|}{2} = 30$ $\quad |t| = 4 \quad t = \pm 4$.

The two points are: $R_1 = P + 4\underline{v} = (-1,1,1) + 4(3,-2,4)$

$R_2 = P - 4\underline{v} = (-1,1,1) - 4(3,-2,4)$

3| See file of exercises (SOLUTIONS) of week 7.

4| (a) Let $P(x) = ax^2+bx+c$.

We have that $T(P(x)) = (2ax+b)(x-1)-2(ax^2+bx+c) =$

$= (-2a-b)x - b - 2c$

Hence $T(P(x)) = 0$ if and only if $\begin{cases} 2a+b=0 \\ b+2c=0 \end{cases}$

That is: $b = -2a = -2c$ and $a=c$

Therefore the polynomials in the null-space $N(T)$

are those of form $ax^2-2ax+a = a(x^2-2x+1)$

Hence $\dim N(T)=1$ and $\{x^2-2x+1\}$ is a basis.

The polynomials of $T(U)$ are those of form

$(-2a-b)x - b - 2c$ for all $a,b,c \in \mathbb{R}$.

Therefore. $T(U) = L(x, 1)$.

(b) The components of $P(x)$ with respect to $\mathcal{E}$ are

(a, b, c).

The components of $T(P(x))$ with respect to $\mathcal{E}$ are

$(0, -2a-b, -b-2c)$. We have

$$\begin{pmatrix} 0 \\ -2a-b \\ -b-2c \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ -2 & -1 & 0 \\ 0 & -1 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Therefore $m_{\mathcal{B}}^{\mathcal{B}}(T) = \begin{pmatrix} 0 & 0 & 0 \\ -2 & -1 & 0 \\ 0 & -1 & -2 \end{pmatrix}$

5| The quadratic part of the equation is the real quadratic form:

$$Q(x,y) = x^2+4xy-2y^2$$

The associated matrix is $A = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$.

Eigenvalues of A: $\lambda_1 = 2 \quad \lambda_2 = -3$

$E_A(2) = L((2,1)) \qquad E_A(-3) = L((-1,2))$

Dividing by the norms of the generators we get the orthonormal basis of V_2, whose elements are eigenvectors of A: $B = \{\frac{1}{\sqrt5}(2,1), \frac{1}{\sqrt5}(-1,2)\}$.

and $m_{\mathcal{E}}^{B}(id) = \frac{1}{\sqrt5}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$ (which has determinant $= 1$)

Let x', y' ~~be the coordinates~~ such that $(x,y) = x'\frac{1}{\sqrt5}(2,1) + y'\frac{1}{\sqrt5}(-1,2)$

We have $\frac{1}{\sqrt5}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ that is $\begin{cases} x = \frac{2}{\sqrt5}x' - \frac{1}{\sqrt5}y' \\ y = \frac{1}{\sqrt5}x' + \frac{2}{\sqrt5}y' \end{cases}$

Substituting we get

$$f(x,y) = x^2+4xy-2y^2-10x+13 = 2(x')^2-3(y')^2-10\left(\frac{2}{\sqrt5}x' - \frac{1}{\sqrt5}y'\right)+13$$

Now we take care of terms of degree one, so to find the (x',y')-coordinates of the center

$$f(x,y) = 2\left((x')^2-2\sqrt5 x'\right) - 3\left((y')^2 - \frac{2\sqrt5}{3}y'\right)+13 =$$

$$= 2\left((x'-\sqrt5)^2 - 5\right) - 3\left(\left(y'-\frac{\sqrt5}{3}\right)^2 - \frac{5}{9}\right) + 13 =$$

$$= 2(x'-\sqrt5)^2 - 3\left(y'-\frac{\sqrt5}{3}\right)^2 + \frac{14}{3} = 0.$$

In conclusion: $-\frac{3}{7}(x'-\sqrt5)^2 + \frac{9}{14}\left(y'-\frac{\sqrt5}{3}\right)^2 = 1.$

Center: $\frac{1}{\sqrt5}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} \sqrt5 \\ \sqrt5/3 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 1/3 \end{pmatrix} = \begin{pmatrix} 7/3 \\ 1/3 \end{pmatrix}$

axes: $\{(7/3, 1/3) + t(2,1)\}$

$\{(7/3, 1/3) + t(-1,2)\}$.