Linear Algebra and Geometry. Written test of october 30, 2013.

1. Let $\mathcal{B} = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ be an orthogonal basis of V_3 such that $\| \mathbf{u} \| = 1$, $\| \mathbf{v} \| = 2$ and $\| \mathbf{w} \| = 3$. (a) Compute the cosine of the angle between $\mathbf{u} - \mathbf{v} + \mathbf{w}$ and $2\mathbf{u} - \mathbf{v} - 2\mathbf{w}$.
(b) Find a basis of $\big(L(\mathbf{u} - \mathbf{v} + \mathbf{w})\big)^{\perp}$ (express the vectors of the required basis as linear combinations of the vectors of $\mathcal{B}$).

Solution. We start by recalling a well known (and easy) fact from the theory: if $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ is an *orthogonal* basis, $\mathbf{u} = \sum_{i=1}^n x_i \mathbf{u}_i$ and $\mathbf{v} = \sum_{i=1}^n y_i \mathbf{u}_i$ then

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} x_i y_i \mathbf{u}_i \cdot \mathbf{u}_i$$

(a) $(\mathbf{u}-\mathbf{v}+\mathbf{w})\cdot(\mathbf{u}-\mathbf{v}+\mathbf{w}) = 1+4+9 = 14$, $(2\mathbf{u}-\mathbf{v}-2\mathbf{w})\cdot(2\mathbf{u}-\mathbf{v}-2\mathbf{w}) = 4+4+36 = 44$, $(\mathbf{u}-\mathbf{v}+\mathbf{w})\cdot(2\mathbf{u}-\mathbf{v}-2\mathbf{w}) = 2+4-18 = -12$. Therefore the cosine of the angle between $\mathbf{u}-\mathbf{v}+\mathbf{w}$ and $2\mathbf{u}-\mathbf{v}-2\mathbf{w}$ is $-12/(\sqrt{14}\sqrt{44})$.
(b) $\big(L(\mathbf{u} - \mathbf{v} + \mathbf{w})\big)^{\perp}$ is the subspace whose elements are the vectors perpendicular to $\mathbf{u}-\mathbf{v}+\mathbf{w}$, namely the vectors $x\mathbf{u}+y\mathbf{v}+z\mathbf{w}$ such that $x - 4y + 9z = 0$. Solving, we have $(x, y, z) = y(4, 1, 0) + z(-9, 0, 1)$. Therefore $\big(L(\mathbf{u} - \mathbf{v} + \mathbf{w})\big)^{\perp} = L(4\mathbf{u} + \mathbf{v}, -9\mathbf{u} + \mathbf{w})$.

2. Let $\mathbf{r}(t)$ be a space motion such that $\mathbf{r}''(t) = \lambda(t)\mathbf{r}(t)$, where $\lambda(t)$ is a nowhere zero scalar function. Assume that $\mathbf{r}'(t_0) = (1, 2, -1)$ and $\mathbf{r}''(t_0) = (2, 1, 3)$. Is there a plane containing $\mathbf{r}(t)$ for all t? If the answer is yes, find the equation of the plane.

Solution. All this follows easily from the theory. I recall the reasoning: we have that $\mathbf{r} \times \mathbf{r}'' = (\mathbf{r} \times \mathbf{r}')'$ and this, by hypothesis, is constantly zero. Therefore $\mathbf{r}(t) \times \mathbf{r}'(t) \equiv \mathbf{u}$, where $\mathbf{u}$ is a constant vector. Hence $\mathbf{r}(t)\cdot\mathbf{u} \equiv 0$, that is $\mathbf{r}(t)$ belongs to the plane (containing the origin) $X \cdot \mathbf{u} = 0$. To find $\mathbf{u}$ we note that $\mathbf{u} = \mathbf{r}(t_0) \times \mathbf{r}''(t_0)$, which is a non-zero scalar multiple of $(2, 1, 3) \times (1, 2, -1) = (-7, 5, 3)$. In conclusion, the answer is yes and the plane is: $-7x + 5y + 3z = 0$.

3. Let $T : V_2 \to V_2$ be the linear transformation such that $T((1, -2)) = (1, 1)$ and $T((-1, 1)) = (2, 0)$. Find all $(x, y) \in V_2$ such that $T((x, y)) = (-2, 3)$.

Solution. Clearly T is bijective, hence invertible. Therefore a (x, y) such that $T((x, y)) = (-2, 3)$ exists and it is unique, namely $(x, y) = T^{-1}((-2, 3))$. One can compute it in different (very similar) ways. One making full use of the theory is as follows. Let $\mathcal{B} = \{(1, -2), (-1, 1)\}$. We know that

$$m^{\mathcal{B}}_{\mathcal{E}}(T) = A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}.$$

We find

$$A^{-1} = -\frac{1}{2}\begin{pmatrix} 0 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1/2 & -1/2 \end{pmatrix}$$

and we know that

$$A^{-1} = m_{\mathcal{B}}^{\mathcal{E}}(T^{-1})$$

This implies that $A^{-1}\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ is the vector of components of $T^{-1}((-2,3))$ with respect to the basis $\mathcal{B}$. Since $A^{-1}\begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -5/2 \end{pmatrix}$ we have that

$$T^{-1}((-2,3)) = 3(1,-2) - (5/2)(-1,1) = (11/2, -17/2)$$

In conclusion, there is a unique $(x,y) \in V_2$ such that $T((x,y)) = (-2,3)$, namely $(x,y) = (11/2, -17/2)$.

4. Let U be the linear space of real polynomials of degree ≤ 3. For $P, Q \in U$ define:

$$(P,Q) = P(0)Q(0) + P(1)Q(1) + P(2)Q(2)$$

and

$$< P, Q >= P(0)Q(0) + P'(0)Q'(0) + P(-1)Q(-1) + P(1)Q(1)$$

Which of the above formulas defines as inner product on U? For such inner product(s) compute the projection of the polynomial t^3 on the linear subspace W whose elements are the polynomials of degree ≤ 2.

Solution. The first one does not define an inner product because the positivity property is not satisfied. To see this we note that $(P,P) = P(0)^2 + P(1)^2 + P(2)^2$ and, for example, $P(t) = t(t-1)(t-2)$ is such that $(P,P) = 0$, but P is non-zero.

The second one defines an inner product. Indeed the various linearity properties are satisfied (easy to check) and also the positivity. In fact $< P, P >= P(0)^2 + P'(0)^2 + P(1)^2 + P(-1)^2$ which is non-negative. Moreover, if $< P, P >= 0$ then $P(0) = P'(0) = P(1) = P(-1) = 0$. This means that 0, 1 and 2 are zeroes of P, and 0 is at least a double zero. Thus P has 4 zeroes (with two of them coinciding). Since the degree of P is at most 3, this implies that P must be identically zero.

To answer the last question, we first need to find an orthogonal basis of W. To do this we apply the Gram-Schmidt procedure to the basis $\{1, t, t^2\}$ of W. It is easy to see that 1 and t are already orthogonal, as t and t^2. Therefore the required orthogonal basis of W will be $\{1, t, P(t)\}$ with $P(t) = t^2 - (< t^2, 1 > / < 1, 1 >)1 = t^2 - (2/3)1 = t^2 - (2/3)$. Finally, the required projection is the sum of the projections on the one-dimensional linear spaces spanned by the elements of the orthogonal basis, namely

$$\frac{< t^3, 1 >}{< 1, 1 >}1 + \frac{< t^3, t >}{< t, t >}t + \frac{< t^3, t^2 - 2/3 >}{< t^2 - 2/3, t^2 - 2/3 >}(t^2 - 2/3) = 0 + (2/3)t + 0 = (2/3)t$$

5. Let V be the linear space of all real convergent sequences $\{a_n\}$. Define $T : V \to V$ as follows: if $lim_{n\to\infty} a_n = \bar{a}$ then $T\big(\{a_n\}\big) = \{b_n\}$, where $b_n = \bar{a} - a_n$ for $n \geq 1$.

Find all eigenvalues of T and the corresponding eigenspaces. Specify which eigenspaces are infinite dimensional. Compute the dimension of the finite dimensional eigenspaces.

Solution. (This is Exercise 10 os Section 4.4 of Apostol, Vol. II. It was solved in class.) To find the eigenvalues, one cannot use the characteristic polynomial (we are in infinite dimension). We rather argue directly as follows: let $\lambda \in \mathbf{R}$ such that $T(\{a_n)\}) = \{\bar{a} - a_n\} = \{\lambda a_n\}$. Taking the limit we get: $\bar{a} - \bar{a} = 0 = \lambda \bar{a}$. Therefore either $\lambda = 0$ or $\bar{a} = 0$. In the former case we get $a - a_n \equiv 0$. Therefore $a_n = \bar{a}$ for each $n \geq 1$, that is the sequence is constant $= \bar{a}$ for each $n \geq 1$. We have just found out that $\lambda = 0$ is an eigenvalue and that the corresponding eigenspace $E(0)$ is the linear subspace whose elements are the *constant* sequences. This is finite dimensional, and in fact has dimension equal to 1, since every constant sequence is a scalar multiple of the constant sequence $a_n \equiv 1$.

It remains to analyze the case $\bar{a} = 0$. In this case we get $\{-a_n\} = \lambda\{a_n\}$, hence $\lambda = -1$. We have just found out that also -1 is an eigenvalue, and that the corresponding eigenspace $E(-1)$ is the linear subspace consisting of all sequences converging to 0. This is infinite dimensional (for example, it contains all sequences for the form $a_n = \begin{cases} 0 & \text{if } n \neq n_0 \\ 1 & \text{if } n = n_0 \end{cases}$ for each $n_0 \geq 1$, which are linearly independent and infinitely many).

By the previous analysis there are no other eigenvalues.

In conclusion: the eigenvalues are 0 and -1. The eigenspace of 0 is one-dimensional, while the eigenspace of -1 is infinite dimensional.