Linear Algebra and Geometry. Written test of july 22, 2013.

1. Let $P = (1, -1)$ and let $R = L((3, -4)) = \{t(3, -4) \mid t \in \mathbf{R}\}$. Find all lines S parallel to L such that the distance $D(P, S) = 10$.

Solution. A line parallel to R has cartesian equation of the form $4x + 3y = c$, with $c \in \mathbf{R}$. We have to find the c's such that the distance from P is equal to 10. We know that the distance is $\frac{|(4,3)\cdot(1,-1)-c|}{\|(4,3)\|} = \frac{|1-c|}{5}$. We want $\frac{|1-c|}{5} = 10$ which splits into the two possibilities: $1 - c = 50$, and $c - 1 = 50$. therefore there are two lines as required: $S_1 : 4x + 3y = -49$ and $S_2 : 4x + 3y = 51$.

2. Let $\mathcal{C}$ be a parabola and let F be its focus. let P be any point of $\mathcal{C}$. We denote Θ_P the angle between the tangent line to $\mathcal{C}$ at P and the line passing through P and parallel to the symmetry axis of $\mathcal{C}$. We denote Φ_p the line between the tangent line to $\mathcal{C}$ at P and the line passing trough F and P. What is the right relation between θ_p and ϕ_P?
(a) $\theta_P = \phi_P$; (b) $\theta_P = \pi - \phi_P$; (c) $\theta_P = \pi/2 - \phi_P$; (d) $\theta_P = 2\phi_P$.
Indicate the right one and prove it.

Solution. The right one is $\Theta_P = \phi_P$, see Apostol, Vol. I, Example 4 of Section 14.5 and Exercise 14 of Section 14.7 *. Proof: we can take F as the origin. Let $\mathbf{r}(t)$ be any parametrization of the parabola. By definition of parabola,

$$\| \mathbf{r}(t) \| = d(\mathbf{r}(t), L)$$

where L is the directrix. We have that $d(\mathbf{r}(t), L) = \mathbf{r}(t) \cdot N + \textit{constant}$, where N is a unit vector parallel to the symmetry axis (the distance $d(\mathbf{R}(t), L) = \mathbf{r}(t)\cdot N + d(F, L)$). Instead, writing $\mathbf{r}(t) = \lambda(t)\mathbf{u}(t)$, where $\mathbf{u}(t)$ is a unit vector parallel to $\mathbf{r}(t)$ and $\lambda(t) > 0$, we have that $\| \mathbf{r}(t) \| = \lambda(t)$. In conclusion

$$\lambda(t) = \mathbf{r}(t) \cdot N + \textit{constant}$$

Differentiating, we get

$$\lambda'(t) = \mathbf{r}'(t) \cdot N$$

On the other hand $\mathbf{r}'(t) \cdot \mathbf{u}(t) = \lambda'(t)$ (in fact $\mathbf{r}'(t) = \lambda'(t)\mathbf{u}(t) + \lambda(t)\mathbf{u}'(t)$ and $\mathbf{u}'(t)$ is perpendicular to $\mathbf{u}(t)$, since $\mathbf{u}(t)$ has constant norm equal to 1). In conclusion

$$\mathbf{r}' \cdot \mathbf{u}(t) = \mathbf{r}'(t) \cdot N$$

which means that $\theta(t) = \phi(t)$.

3. Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 1 \\ -3 & 1 \\ -2 & 2 \end{pmatrix}$.

* In fact, to be precise, the angle between two lines is not well defined: it depends on the choice of directions of the lines. In this sense, depending on the orientation, also the relation $\theta_P = \pi - \phi_P$ can be correct (one should make a picture to specify)

(a) Describe (if any) all matrices $X \in \mathcal{M}_{3,3}(\mathbf{R})$ such that $AX = C$.
(b) Describe all matrices $X \in \mathcal{M}_{3,3}$ such that $BX = C$.

Solution. (a) A is invertible ($\det A \neq 0$). Therefore a matrix as required exists and it is unique: $X = A^{-1}C$.
(b) B is not invertible, therefore we cannot apply the same method. We rather use gaussian elimination: denoting $X_1, x_2, X_3 \in V_2$ the row-vectors of the matrix X, they must be the solutions of the system (whose unknown are vectors of V_2):

$$\begin{cases} X_1 - X_2 + X_3 & =(1,1) \\ \quad 2X_2 + X_3 & =(-3,1) \\ X_1 + X_2 + 2X_3 & =(-2,2) \end{cases}$$

We proceed in the same way one finds the inverse matrix with gaussian elimination:

$$\begin{pmatrix} 1 & -1 & 1 & 1 & 1 \\ 0 & 2 & 1 & -3 & 1 \\ 1 & 1 & 2 & -2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 & 1 & 1 \\ 0 & 2 & 1 & -3 & 1 \\ 0 & 2 & 1 & -3 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 & 1 & 1 \\ 0 & 2 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The system is compatible: therefore matrices X as required do exist (and they are infinitely many). To describe them, we note that our system is equivalent to

$$\begin{cases} X_1 - X_2 + X_3 & =(1,1) \\ \quad 2X_2 + X_3 & =(-3,1) \end{cases}$$

Solving we have $X_3 = (-3, 1) - 2X_2$ and $X_1 = (1, 1) + X_2 - X_3 = \ldots = (4, 0) + 3X_2$. Therefore our matrices have rows of the form $X_1 = (4, 0)+3Y$, $X_2 = Y$, $X_3 = (-3, 1)-2Y$, for all $Y \in V_2$. In more compact notation, the matrices X have the form

$$X = \begin{pmatrix} 4 & 0 \\ 0 & 0 \\ -3 & 1 \end{pmatrix} + \begin{pmatrix} 3a & 3b \\ a & b \\ -2a & -2b \end{pmatrix}$$

for all $a, b \in \mathbf{R}$.
(a) (again). Of course, one can use the method of (b) to solve (a), too. The result is that a matrix X such that $AX = C$ is unique, and, with the gaussian elimination, one finds $X = \frac{1}{2}\begin{pmatrix} 5 & 1 \\ 3 & -1 \\ -9 & 3 \end{pmatrix}$.

4. Let $\mathbf{v} = (1, 1, -2)$. Let $T : V_3 \to V_3$ be the rotation of V_3 around the line $L(\mathbf{v})$of angle $\pi/6$ (the rotation is counterclockwise with respect to the orientation of $L(\mathbf{v})$ given by $\mathbf{v}$). Compute the matrix representing T with respect to the canonical basis. Compute all eigenvalues and eigenscpaces of T.

Solution. We follow the procedure indicated in the file "Supplementary notes and exercises on linear transformations and matrices, II", (see Week 14 of year 2012-'13), Examples

3.6 and 3.7. We first find a basis of the orthogonal plane $L(\mathbf{v})^\perp$. We get $L(\mathbf{v})^\perp = L((-1,1,0),(2,0,1))$. Orthogonalizing we get $L(\mathbf{v})^\perp = L((-1,1,0),(1,1,1))$. Normalizing we get the orthonormal basis of $L(\mathbf{v})^\perp$: $\{\mathbf{u},\mathbf{w}\} = \{(1/\sqrt{2})(-1,1,0),(1/\sqrt{3})(1,1,1)\}$. Note that $\mathcal{B} = \{\mathbf{u},\mathbf{w},\mathbf{v}\}$ is a positively-oriented basis (it satisfies the "right hand rule"), namely the determinant of the matrix $C = m^{\mathcal{B}}_{\mathcal{E}}(\mathrm{id}) = (\,\mathbf{u} \quad \mathbf{w} \quad \mathbf{v}\,)$ is positive. The notation means that the matrix C has as column-vectors the vectors $\mathbf{u},\mathbf{w},\mathbf{v}$ (in that order).

We have that the matrix representing T with respect to the basis $\mathcal{B}$ is

$$m^{\mathcal{B}}_{\mathcal{B}}(T) = U = \begin{pmatrix} \cos(\pi/6) & -\sin(\pi/6) & 0 \\ sin(\pi/6) & \cos(\pi/6) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore, by the change-of-basis-formula the required matrix is

$$m^{\mathcal{E}}_{\mathcal{E}}(T) = CUC^{-1}$$

Concerning the last question, the eigenvalues of T coincide with the eigenvalues of the representing matrix U. Clearly, 1 is the unique (and simple) eigenvalue. The eigenspace is $L(\mathbf{v})$.

5. Let $\mathcal{C}$ be the conic of equation $9x^2 + 16y^2 - 24xy - 212x - 134y - 150 = 0$. Find the type of conic and the canonical equation. Find the coordinates of the center/vertex, the equation(s) of the symmetry axis (axes) and draw a rough sketch of the curve.

Solution. The quadratic part is the quadratic form $Q(x,y) = 9x^2 + 16y^2 - 24xy$. The corresponding matrix is $A = \begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix}$. The eigenvalues are 25 and 0. $E(25) = L((1/5)(3,-4))$ and $E(0) = L((1/5)(4,3))$. Let $\mathcal{B} = \{(1/5)(3,-4)),(1/5)(4,3)\} = \{\mathbf{u},\mathbf{v}\}$. Let $C = m^{\mathcal{B}}_{\mathcal{E}}(\mathrm{id}) = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}$. We take components x', y' such that $(x,y) = x'\mathbf{u} + y'\mathbf{v}$. In practice

$$\begin{cases} x = (3/5)x' + (4/5)y' \\ y = (-4/5)x' + (3/5)y' \end{cases}$$

We have that $Q(x,y) = 25(x')^2$. Therefore the equation of $\mathcal{C}$ becomes

$$25(x')^2 - 212((3/5)x' + (4/5)y') - 134((-4/5)x' + (3/5)y') - 150 = 25(x')^2 - 100x' - 250y' - 150 =$$

$$= 25((x')^2 - 4x') - 250y' - 150 = 25((x') - 2)^2 - 4) - 270y' - 150 = 25(x' - 2) - 250y' - 250 =$$

$$= 25(x' - 2)^2 - 250(y' + 1)$$

The conic is a parabola whose equation in the (x',y')-coordinates is $(x'-2)^2 = 10(y'+1)$. The vertex has coordinates $(x',y') = (2,-1)$. Therefore, using the formulas relating (x,y) to (x',y'), we get $V = (2/5,-11/5)$. The symmetry axis is $C + L(\mathbf{u})$.