Linear Algebra and Geometry. Written test of september 12, 2013.

1. *We recall that, given three vectors* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{V}_3$, *their scalar triple product is defined as* $[\mathbf{u}\,\mathbf{v}\,\mathbf{w}] := \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.

Let $\mathbf{u} = (1, 2, 1)$, $\mathbf{v} = (1, 1, 1)$, $\mathbf{w} = (1, 1, -1)$. *Find* $\mathbf{a} = (a_1, a_2, a_3) \in \mathcal{V}_3$ *such that* $[\mathbf{a}\,\mathbf{v}\,\mathbf{w}] = 4$, $[\mathbf{u}\,\mathbf{a}\,\mathbf{w}] = -8$, $[\mathbf{u}\,\mathbf{v}\,\mathbf{a}] = 16$*).* (Hint: the shortest way for solving the problem uses Cramer's rule).

Solution. (Recall that the scalar triple product is nothing else than the determinant of the matrix whose rows are the three vectors (which equals the determinant of the matrix whose columns are the three vectors)).

Note that the determinant $[\mathbf{u}\,\mathbf{v}\,\mathbf{w}]$ is equal to 2 (easy calculation). Then Cramer's rule tells us that, for all $\mathbf{a} \in \mathcal{V}_3$, the system

$$x\mathbf{u} + y\mathbf{v} + z\mathbf{w} = \mathbf{a}$$

has the unique solution:
$x = [\mathbf{a}\,\mathbf{v}\,\mathbf{w}]/[\mathbf{u}\,\mathbf{v}\,\mathbf{w}]$, $y = [\mathbf{u}\,\mathbf{a}\,\mathbf{w}]/[\mathbf{u}\,\mathbf{v}\,\mathbf{w}]$, $z = [\mathbf{u}\,\mathbf{v}\,\mathbf{a}]/[\mathbf{u}\,\mathbf{v}\,\mathbf{w}]$.
In our case this numbers must be respectively 2, -4 and 8. Therefore the required $\mathbf{a}$ is $2\mathbf{u} - 4\mathbf{v} + 8\mathbf{w}$.

(Alternatively, noting that $[\mathbf{u}\,\mathbf{a}\,\mathbf{w}] = -[\mathbf{a}\,\mathbf{u}\,\mathbf{w}]$ one can compute explicitly $\mathbf{u}\times\mathbf{v}$, $\mathbf{u}\times\mathbf{w}$, $\mathbf{v} \times \mathbf{w}$ and find $\mathbf{a}$ by solving the 3×3 system

$$\begin{cases} \mathbf{a} \cdot \mathbf{v} \times \mathbf{w} & =4 \\ \mathbf{a} \cdot \mathbf{u} \times \mathbf{w} & =8 \\ \mathbf{a} \cdot \mathbf{u} \times \mathbf{v} & =16 \end{cases}$$

2. *Compute the cartesian equation of the hyperbola passing through the point* $(5, 7)$ *whose asymptotes are* $R_1 : 4x - 3y = 23$ *and* $R_2 : 4x + 3y = -7$.

Solution. The center of the hyperbola is the intersection of the asymptotes, namely $C = (2, -5)$. The union of the lines passing through the origin and parallel to the asymptotes is symmetric with respect to the $x-$ and $y-$ axes, therefore the cartesian equation of the hyperbola has the form

$$\frac{(x-2)^2}{a^2} - \frac{(y+5)^2}{b^2} = 1 \qquad \text{or} \qquad -\frac{(x-2)^2}{a^2} + \frac{(y+5)^2}{b^2} = 1$$

In both cases the asymptotes are parallel to the lines $y = (b/a)x$ and $y = -(b/a)x$. Therefore we know that

$$\frac{b}{a} = \frac{4}{3}$$

Hence, after easy calculations, the previous equations take the form

$$16(x-2)^2 - 9(y+5)^2 = 16a^2 \qquad \text{or} \qquad 16(x-2)^2 - 9(y+5)^2 = 16a^2$$

Imposing the passage through $(5,7)$ one sees that the first case is not possible. So we fall in the second case and we compute $a^2 = 72$ and $b^2 = 128$. Therefore the required equation is

$$-\frac{(x-2)^2}{72} + \frac{(y+5)^2}{128} = 1$$

3. *We recall that if the x and y-coordinates of a point $P = (x, y, z) \in \mathcal{V}_3$ are replaced by polar coordinates ρ and θ then (ρ, θ, z) are called cylindrical coordinates for the point P.*

Let us consider the motion of a point in $\mathcal{V}_3$ such that its position at time t has cylindrical coordinates

$$\rho = 2t, \qquad \theta = 2t, \qquad z = 2t$$

Find formulas for the velocity $\mathbf{v}$, the acceleration $\mathbf{a}$ and the curvature κ in function of t.

Solution. As done in class, the best way is to imitate the procedure for computing velocity, acceleration of a plane curve in terms of polar coordinates. We denote $\mathbf{u}_\rho = (\cos\theta, \sin\theta, 0)$, $\mathbf{u}_\theta = (-\sin\theta, \cos\theta, 0)$ and $\mathbf{k} = (0,0,1)$. They form a positively oriented orthonormal basis, so that

$$\mathbf{u}_\rho \times \mathbf{u}_\theta = \mathbf{k}, \qquad \mathbf{u}_\theta \times \mathbf{k} = \mathbf{u}_\rho, \qquad \mathbf{k} \times \mathbf{u}_\rho = \mathbf{u}_\theta \tag{$*$}$$

If θ is in function of t we have that

$$\mathbf{u}'_\rho(t) = \theta'(t)\mathbf{u}_\theta(t), \qquad \mathbf{u}'_\theta(t) = -\theta'(t)\mathbf{u}_\rho(t)$$

Therefore, since $\mathbf{r}(t) = \rho(t)\mathbf{u}_\rho(t) + z(t)\mathbf{k}$ we have (in the notation we omit the dependance on t)

$$\mathbf{v} = \rho'\mathbf{u}_\rho + \rho\theta'\mathbf{u}_\theta + z'\mathbf{k}$$

$$v = \big((\rho')^2 + \rho^2(\theta')^2 + (z')^2\big)^{1/2}$$

$$\mathbf{a} = (\rho'' - \rho(\theta')^2)\mathbf{u}_\rho + (2\rho'\theta' + \rho\theta'')\mathbf{u}_\theta + z''\mathbf{k}$$

In our case we have

$$\mathbf{v}(t) = 2\mathbf{u}_\rho + 4t\mathbf{u}_\theta + 2\mathbf{k}$$

$$v(t) = 2\sqrt{2}(1 + 8t^2)^{1/2}$$

$$\mathbf{a}(t) = -8t\mathbf{u}_\rho + 8\mathbf{u}_\theta$$

Using $(*)$ we have that $\mathbf{a} \times \mathbf{v} = (-8t\mathbf{u}_\rho + 8\mathbf{u}_\theta) \times (2\mathbf{u}_\rho + 4t\mathbf{u}_\theta + 2\mathbf{k}) = 16\mathbf{u}_\rho - 16t\mathbf{u}_\theta - 16(2t^2+1)\mathbf{k}$. Thetrefore

$$\| \mathbf{a} \times \mathbf{v} \| = 16\big(4t^4 + 3t^2 + 2\big)^{1/2}$$

In conclusion

$$\kappa(t) = \| \mathbf{a} \times \mathbf{v} \| / v^3 = (4t^4 + 3t^2 + 2\big)^{1/2} / (\sqrt{2}(1 + 8t^2)^{3/2})$$

4. *In $\mathcal{V}_3$, equipped with the usual dot product, let S be the unit sphere, that is the set of all elements of $\mathcal{V}_3$ having norm equal to 1. Let us consider the function $Q : S \to \mathbf{R}$ defined by*

$$Q((x,y,z)) = x^2 + 4xy + 4xz + y^2 + 4yz + z^2$$

Compute the minimum of Q and all points of $(x,y,z) \in S$ such that $Q((x,y,z))$ is minimum. Compute the maximum of Q and all points of $(x,y,z) \in S$ such that $Q((x,y,z))$ is maximum.

Solution. Q is a real quadratic form. We know from the theory that the maximum and minimum on S are the maximum and minimum eigenvalue. The matrix of Q is $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$. It is evident, without computing the characteristic polynomial, that adding 1 on the diagonal we get the matrix whose entries are all 2's, which has rank 1. This means that the matrix $1\,\mathrm{I}_3 - A$ has rank one. Therefore -1 is a double eiegenvalue (the matrix is diagonalizable, hence the multiplicity of -1 as eigenvalue equal the dimension of the eigenspace, which is $3 - rk(1\,\mathrm{I}_3 - A) = 2$. The other eigenvalue follows from $-1 - 1 + \lambda = Tr(A) = 3$. Hence $\lambda = 5$.

(Alternatively, one can compute the characteristic polynomial to find the eigenvalues).

Therefore the minimum is -1 and the maximum is 5.

We know from the theory that the points of minimum (resp. maximum) are the intersections of the eigenspaces with S, that is the eigenvectors whose norm equals 1. Therefore we need to compute the eigenspaces. By the above, $E(-1)$ has equation $x + y + z = 0$, that is $E(-1) = L((1,-1,0),(1,0,-1))$. The other eigenspace $E(5)$ is the orthogonal of $E(-1)$, namely $L(1,1,1))$. Therefore the points of maximum are only two, and they are easily found: $(1/\sqrt{3})(1,1,1)$ and $-(1/\sqrt{3})(1,1,1)$.

Instead, since the dimension of $E(-1)$ is two, the eigenvectors of -1 whose norm equals one are infinitely many (they are the unit circle in the plane provided by $E(-1)$). To describe them first we have to orthonormalize the basis of $E(-1)$, getting $\mathbf{u}, \mathbf{v}$. At this point the eigenvectors of norm one are those of the form $\cos\theta\,\mathbf{u} + \sin\theta\mathbf{v}$.

5. *For $X = (x_1, x_2, x_3) \in \mathcal{V}_3$ and $Y = (y_1, y_2, y_3) \in \mathcal{V}_3$, let us define*

$$< X, Y >= \ x_1y_1 + 2x_1y_2 + 2x_1y_3 + 2x_2y_1 + x_2y_2 + 2x_2y_3 + 2x_3y_1 + 2x_3y_2 + x_3y_3$$

Is this an inner product on $\mathcal{V}_3$? Prove your answer. (Hint: use the previous exercise)

Solution. The answer is NO. The property which does not hold is the positivity. In fact $< X, X >= x_1^2 + 4x_1x_2 + 4x_1x_3 + x_2^2 + 4x_2x_3 + x_3^2$, which is the quadratic form of the previous exercise. Since we know that it has also negative values (the minimum on the unit sphere is -1) we don't have that $< X, X > \geq 0$ for all $X \in \mathcal{V}_3$.