LINEAR ALGEBRA AND GEOMETRY **written test of JULY 10, 2013**

1. In V_3, let us consider the two straight lines $L = \{(-2,-5,4) + t(1,1,-1)\}$ and $R = \{(0,1,3) + t(-1,3,2)\}$.
(a) Compute the intersection $L \cap R$.
(b) Is there a plane containing both L and R? if the answer is yes, compute its cartesian equation.

Solution. (a) A point P lies in the intersection if and only if there are scalars t_0 and s_0 such that $(-2,-5,4) + t_0(1,1,-1) = (0,1,3) + s_0(-1,3,2) = P$. Therefore $t_0(1,1,-1) - s_0(-1,3,2) = (2,6,-1)$. Solving the system we find $-s_0 = 1$, hence $s_0 = -1$. Therefore $P = (0,1,3) - (-1,3,2) = (1,-2,1)$.
(b) The plane is $\{(1,-2,1)+t(1,1,-1)+s(-1,3,2)\}$. Cartesian equation: $5x-y+4z = 11$.

2. Let $V = \{(1,0,1,0),(0,1,0,-1)\}$ and let $R_V : V_4 \to V_4$ be the reflection with respect to V. Compute the matrix representing R_V with respect to the canonical basis of V_4 (equivalently: for (x,y,z,t) compute $R_V((x,y,z,t))$).

Solution. Let us call $\{v_1, v_2\}$ the given basis of V. It is already orthogonal. One sees easily that $\{w_1, w_2\} = \{(1,0,-1,0)(0,1,01)\}$ is an (orthogonal) basis of $V^{\perp}$. Given $(x,y,z,t) \in V_4$, its decomposition with respect to V is

$$(x,y,z,t) = \frac{x+z}{2}v_1 + \frac{y-t}{2}v_2 + \frac{x-z}{2}w_1 + \frac{y+t}{2}w_2$$

Therefore, by definition,

$$R_V((x,y,z,t)) = \frac{x+z}{2}v_1 + \frac{y-t}{2}v_2 - \frac{x-z}{2}w_1 - \frac{y+t}{2}w_2 = ... = (z,-t,x,-y)$$

Therefore the required matrix is

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Alternatively, one takes as

$$C = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

and

$$A = C \cdot diag(1,1,-1,-1) \cdot C^{-1}$$

If the calculations are correct (CHECK!!) one arrives to the same matrix.

3. Let $\mathbf{r} : (a,b) \to V_3$ be a motion such that $\mathbf{r}(t) \neq \mathbf{0}$ for all $t \in (a,b)$ and $\mathbf{r}'(t)$ is always parallel to $\mathbf{r}(t)$. Describe the underlying curve.

Solution. The underlying curve is a piece of a half-line of a line passing trough $\mathbf{0}$. Indeed let $T(t)$ be the unit tangent vector. By hypothesis there is a (never vanishing) function $\lambda(t)$ such that $T(t) = \lambda(t)\mathbf{r}(t)$. Therefore $T'(t) = \lambda'(t)\mathbf{r}(t) + \lambda(t)\mathbf{r}'(t)$ is parallel to $T(t)$. But we know that $T'(t)$ is always perpendicular to $T(t)$. Therefore $T'(t) \equiv \mathbf{0}$. Hence $T(t) \equiv \mathbf{a}$ for some constant unit vector $\mathbf{a}$. Hence $\mathbf{r}(t)$, which is, by hypothesis, parallel to $T(t)$ is of the form $\mu(t)\mathbf{a}$ for a scalar function $\mu(t)$ of constant sign. (see also Apostol, Vol. 1, Ex. 24 of section 14.4).

4. Let V be the linear space whose elements are continuous functions $F : [0, \pi] \to \mathbf{R}$. Let W be the linear subspace of V of functions $f \in V$ having continuous second derivative and such that $f(0) = f(\pi) = 0$. Let $T : W \to V$ defined by $T(f) = f''$. Compute all eigenvalues and eigenspaces of T.

Solution. The answer is: the eigenvalues are all numbers of the form $-n^2$, with n a (non-zero) natural number. The corresponding eigenspace is 1-dimensional, precisely $E(-n^2) = L(\sin nt)$. One arrives to this result by analyzing what is known for the solutions of the differential equation $f'' = \lambda f$. I don't repeat this since I did it in class. See also Apostol, Vol. II, Ex. 9 of Section 4.4.

5. Let us consider the quadratic form

$$Q(x, y, z) = 3x^2 + 6xy - 6xz - \frac{3}{2}y^2 - 3yz - \frac{3}{2}z^2$$

(a) Reduce Q to canonical form. (b) Find the maximum and minimum of Q on the unit sphere of V_3 and describe the points of minimum and maximum.

Solution. The matrix of Q is

$$A = \begin{pmatrix} 3 & 3 & -3 \\ 3 & -\frac{3}{2} & -\frac{3}{2} \\ -3 & -\frac{3}{2} & -\frac{3}{2} \end{pmatrix}$$

The characteristic polynomial:

$$\det \begin{pmatrix} \lambda - 3 & -3 & 3 \\ -3 & \lambda + \frac{3}{2} & \frac{3}{2} \\ 3 & \frac{3}{2} & \lambda + \frac{3}{2} \end{pmatrix} = \det \begin{pmatrix} \lambda - 3 & -3 & 3 \\ -3 & \lambda + \frac{3}{2} & \frac{3}{2} \\ 0 & \lambda + 3 & \lambda + 3 \end{pmatrix} = \cdots = (\lambda - 6)(\lambda + 3)^2$$

Therefore the eigenvalues are $\lambda_1 = 6$, $\lambda_2 = \lambda_3 = -3$. Eigenspaces:
$E(-3) = L((1, -2, 0), (0, 1, 1))$. Orthonormalizing: $E(-3) = L((1, -2, 0), \frac{1}{\sqrt{29}}(2, 1, 5))$.
$E(6) = L((1, 2, 0) \times (0, 1, 1)) = L((-2, -1, 1)) = L(\frac{1}{\sqrt{6}}(-2, -1, 1))$.
Let $\mathcal{B} = \{v_1, v_2, v_3\}$ be the orthonormal basis obtained putting together the orthonormal bases of $E(6)$ and $E(-3)$. Writing $(x, y, z) = x'v_1 + y'v_2 + z'v_3$ we have that

$$Q(x, y, z) = 6(x')^2 - 3(y')^3 - 3(z')^2$$

(b) The maximum on the unit sphere is $\lambda_1 = 6$. The points of maximum are two: $+v_1$ and $-v_1$.
The minimum on the unit sphere is -3. The points of minimum are infinitely many: they are all vectors of the form $\lambda v_2 + \mu v_3$ with $\lambda^2 + \mu^2 = 1$.