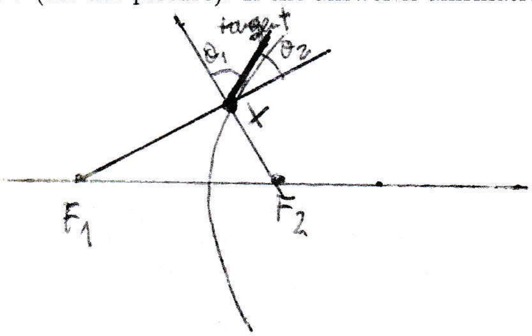
Linear Algebra and Geometry. Written test of february 10, 2014.
(Note: answers without adequate justification and proofs – especially in Exercises 2, 3 and 5 – will not be evaluated)

1. Let L be the line of $\mathcal{V}_3$: $L = \mathcal{L}((1,1,2))$ and let π be the plane pf $\mathcal{V}_3$: $\pi = \mathcal{L}((1,2,0),(0,1,1))$. Find the cartesian equations of all planes of $\mathcal{V}_3$ which are parallel to L, perpendicular to π, and such that their distance from the point $P = (1,3,1)$ is equal to $\sqrt{3}$. *(Recall that two planes are said to be perpendicular if and only if and only their normal vectors are perpendicular).*

2. Does the tangent line of a point X of a branch of hyperbola bisects the angle between the lines joining X to the two foci ? (see the picture). If the answer is affirmative, prove it. If the answer is negative, explain why.



3. In each of the following items, a transformation $T : \mathcal{V}_2 \longrightarrow \mathcal{V}_2$ is defined. In each case determine whether the transformation is linear. If it is linear, find its matrix with respect to the canonical bases, and its eigenvalues and eigenspaces.

(a) T maps a point of polar coordinates (ρ, θ) onto the point of polar coordinates $(2\rho,\ \theta + \pi/3)$. Also, T maps O onto itself.

(b) T maps a point of polar coordinates (ρ, θ) onto the point of polar coordinates $(\rho, 2\theta)$. Also, T maps O onto itself.

(c) T maps a point of polar coordinates (ρ, θ) onto the point of polar coordinates $(\rho,\ 2\pi/3 - \theta)$. Also, T maps O onto itself.

4. In the real linear space $\mathcal{C}([-1,1])$ with inner product $< f, g >= \int_{-1}^{1} f(x)g(x)dx$, let $f(x) = e^x$. Find the polynomial of degree one g nearest to f and compute $\| f - g \|$.

5. For each of the following real quadratic forms $Q : \mathcal{V}_4 \rightarrow \mathbf{R}$, determine whether they are positive, semipositive, negative, seminegative or indefinite.

(a) $Q(x,y,z,t) = x^2 + xy + xz + xt + y^2 + yz + yt + z^2 + zt + t^2$

(b) $Q(x,y,z,t) = x^2 + 2xy + 2xz + 2xt + y^2 + 2yz + 2yt + z^2 + 2zt + t^2$

(recall: positive, semipositive, negative, seminegative, indefinite mean respectively: $Q(X) > 0$ for each $X \neq O$, $Q(X) \geq 0$ for each $X \neq O$, $Q(X) < 0$ for each $X \neq O$, $Q(X) \leq 0$ for each $X \neq O$, none of the previous cases)

SOLUTIONS

1. This is exercise 1 of 4th session 2011-'12.

2. This is Example 4 of Section 14.5 of Apostol, Vol. 1.

3. Note: these exercises are very similar to Exercises 11, 12, 14, 15 of Section 16.4 of Apostol, Vol. 1.
(a) We have that $(x, y) = \rho(\cos\theta, \sin\theta) = (\rho\cos\theta, \rho\sin\theta)$. Thus T maps (x, y) to

$$(2\rho\cos(\theta + \pi/3), \rho\sin(\theta + \pi/3)) =$$

$$= (2\rho(\cos(\pi/3)\cos\theta - \sin(\pi/3)\sin\theta), 2\rho(\cos(\pi/3)\sin\theta + \sin(\pi/3)\cos\theta)) =$$

$$= (2\cos(\pi/3)x - 2\sin(\pi/3)y, 2\sin(\pi/3)x + 2\cos(\pi/3)y).$$

In matrix notation, this means that

$$T((x, y)) = \begin{pmatrix} 2\cos(\pi/3) & -2\sin(\pi/3) \\ 2\sin(\pi/3) & 2\cos(\pi/3) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Therefore T is the linear transformation associated to a matrix, so it is linear. Note: this linear transformation consists of the composition of two trasformations which are easy to visualize: the first one is a rotation of the plane, as it operates by rotating each point countercleckwise through the angle $\pi/3$. The second one is a homothety: it operates by stretching each vector (x, y) of a factor 2 (that is, maps (x, y) to $2(x, y)$).
(b) This transformation is NOT linear, as it is not additive. For example, let us consider $\mathbf{u} = (0, 1)$ and $\mathbf{v} = (0, -1)$. Thus $(T(\mathbf{u} + \mathbf{v}) = T(O) = O$. On the other hand the polar coordinates of $\mathbf{u}$ and $\mathbf{v}$ are respectively $(1, \phi/2)$ and $(1, 3\pi/2)$. Therefore $T(\mathbf{u})$ has polar coordinates $(1, \pi)$ hence it is $(1, 0)$. Similarly, $T(\mathbf{v})$ has polar coordinates $(1, 6\pi/2) = (1, \pi)$, hence it is also equal to $(1, 0)$. In conclusion $(\mathbf{u}) + T(\mathbf{v}) = (2, 0)$ is definitely different from $T(\mathbf{u} + \mathbf{v}) = (0, 0)$.
(c) This transformation is linear. In fact, arguing as in (a) one gets that

$$T((x, y)) = \begin{pmatrix} \cos(2\pi/3) & \sin(2\pi/3) \\ \sin(2\pi/3) & -\cos(2\pi/3) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

hence it is the linear transformation associated to matrix, in particular it is linear. Note: this transformation maps a point of polar coordinates (ρ, θ) onto the point of polar coordinates $(\rho, \pi/3 - (\theta - \pi/3))$. Thinking geometrically, this shows that it is the reflection with respect to the line $L(\cos(\pi/3), \sin(\pi/3)) = L((1/2, \sqrt{3}/2))$.

4. This is Exercise 5 of the third session of 2010-'11. The integrals are not difficult to compute.

5. (Compare with the file Exercises June 22, Ex.0.7).
(b) is really easy: one has that $Q(x, y, z, t) = (x + y + z + t)^2$. This shows that Q is semipositive, but not positive (if $x + y + z + t = 0$, but $(x, t, z, t) \neq (0, 0, 0, 0)$ then $Q((x, y, z, t) = 0$. Alternatively, one can argue as follows: the matrix of Q with respect to the canonical basis is $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ which has $\det = 0$, hence has 0 as eigenvalue. A closer inspection shows that 0 is in fact a triple eigenvalue, since the dimension of $E(0)$, namely the null-space of the matrix A, has dimension $4 - rk(A) = 3$. To find the fourth eigenvalue we can use the relation $\lambda_1 + \lambda_2 + \lambda + 3 + \lambda_4 = Tr(A) = 1 + 1 + 1 + 1 + 1 = 4$. Therefore $\lambda_4 = 4$. In conclusion, the eigenvalues of A are positive or zero, and the theory (reduction of a real quadratic form to diagonal form) shows that this means precisely that Q is semipositive.
(a) Arguing as above, we consider the matrix of Q with respect to the canonical basis:

$$B = \begin{pmatrix} 1 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1 \end{pmatrix}.$$

We see immediately that $(1/2)I_4 - B = -1/2A$ (where A is the matrix of point (a)). Therefore, as above, $1/2$ is a triple eigenvalue of B, and to compute the fourth one we can use that $1/2 + 1/2 + 1/2 + \lambda_4 = Tr(B) = 4$. Therefore $\lambda_4 = 5/2$. The eigenvalues are all positive, hence this Q is positive.

Note: let us recall why the last assertion holds. From the theory we know that we can find an (orthonormal) basis of $\mathcal{V}_4$ such that, denoting (x', y', z', t') the components of (x, y, z, t) with respect to that basis, we have

$$Q((x, y, z, t)) = (1/2)(x')^2 + (1/2)(y')^2 + (1/2)(z')^2 + (5/2)(t')^2$$

This shows that $Q((x, y, z, t) \geq 0$ for all (x, y, z, t). Moreover $Q(x, y, z, t) = 0$ if and only if $x' = y' = z' = t' = 0$. But this happens if and only if $x = y = z = t = 0$ (WHY?).