First Name: Last Name:

Linear Algebra and Geometry, written test, 09.08.2011

NOTE: In the solution of a given exercise you must (briefly) explain the line of your argument and show the main points of your calculations. Solutions without adequate explanations will not be evaluated.

1. Let L be the line $\{(1,0,1)+t(1,0,2)\}$. Moreover let M be the plane passing through the points $(0,1,0)$, $(1,2,-1)$, $(1,1,1)$. Find (if any) all points P of L such that $d(P,M)=1$.

Solution. $M=(0,1,0)+s(1,1,-1)+t(1,0,1)$. The cartesian equation of M is

$$((x,y,z)-(0,1,0))\cdot((1,1,-1)\times(1,0,1))=0,$$

hence $M: x-2(y-1)-z=0$, or $M: x-2y-z=-2$. A normal vector is $N=(1,-2,-1)$ and $Q\cdot N=-2$ for each $Q\in M$.
Let $P=(x,y,z)$. We have that $d(P,M)=\frac{|P-Q|}{\|N\|}$, where Q is any point of M. Therefore

$$d(P,M)=\frac{|P\cdot N+2|}{\sqrt{6}}=\frac{|x-2y-z-2|}{\sqrt{6}}$$

If $P\in L$, we have that $P=(x,y,z)=(1+t,0,1+2t)$. Substituting in the formula for the distance we get $d(P,M)=\frac{|-t-2|}{\sqrt{6}}$. hence we have to solve the equation: $\frac{|-t-2|}{\sqrt{6}}=1$. the solutions are $-2+\sqrt{6}$ and $\sqrt{6}-2$. Plugging these two values of t in the parametric equation of L we get the two points in L whose distance from M is equal to 1.

2. Let $\mathcal{C}$ be a conic section with eccentricity 2, focus at the origin and vertical directrix in the half-space $x>0$. Suppose that the point P of polar coordinates $(\rho,\theta)=(4,\pi/3)$ belongs to $\mathcal{C}$. Find: the directrix, the vertices, and the polar equations of (the branches of) $\mathcal{C}$.

Solution. Recall that since the focus is at the origin, $\mathcal{C}$ is the locus of points P such that $\| P \|=2d(P,L)$, where L is the directrix. Note that $P=(4(1/2,\sqrt{3}/3)=(2,2\sqrt{3})$. Therefore the directrix must be $x=4$ (because $d(O,L)=\rho\cos\theta+d(P,L)=2+d(P,L)$ and $d(P,L)=\rho/2=2$).The vertex on the left of the directrix must be $(8/3,0)$. the vertex on the right must be $(6,0)$ (the vertices V lie on the x-axis, and, again, we impose that $d(V,O)=2d(V,L)$). Since d, the distance between the focus on the directrix is equal to 4, the polar equation of the branch on the left (resp. on the right) is $\rho=8/(2\cos\theta+1)$ (resp. $\rho=8/(2\cos\theta-1)$.

3. For t varying in $\mathbf{R}$ let us consider the matrix $A_t=\begin{pmatrix}4 & t+6 & t-1 & t-1\\ t+1 & 2t+5 & 0 & 0\\ 3 & 5 & t-1 & t-1\\ 3 & 7 & t-2 & t-3\end{pmatrix}$.

(a) Find the values of t such that $\det(A_t)=0$.
(b) Find the values of t such that the system of linear equations whose *augmented* matrix is A_t has solutions. For such values of t is the solution unique?

Solution.

$$\det A_t \overset{A_1\to A_1-A_3}{=} \det\begin{pmatrix}1 & t+1 & 0 & 0\\ t+1 & 2t+5 & 0 & 0\\ 3 & 5 & t-1 & t-1\\ 3 & 7 & t-2 & t-3\end{pmatrix}$$

The matrix on the right is lower block-triangular. Therefore

$$\det(A_t)=((2t+5)-(t+1)^2)((t-1)(t-3)-(t-1)(t-2))=(t-1)(t^2-4)-(t-1)(t+2)(t-2).$$

Therefore $\det(A_t) = 0$ for $t = 1, 2, -2$.
(b) If $\det(A_t) \neq 0$ there is no solution, since the augmented matrix has rank equal to 4 and the rank of the matrix of coefficients is at most 3. If $\det(A_t) = 0$ one should check case-by-case the corresponding system. In all the three cases it follows that the rank of the matrix of coefficient is equal to 3. Therefore also $rk(A_t) = 3$ (it can't be 4 because the determinant vanishes). Therefore for $t = 0, 2, -2$ there a unique solution. Otherwise no solutions.

4 . Let $T : \mathcal{V}_3(\mathbf{R}) \to \mathcal{V}_3(\mathbf{R})$ be defined as follows: $T((x, y, z)) = (1, 2, -1) \times (x, y, z)$.
(a) Is T a linear transformation?
(b) Find dimension and a basis for the null-space of T and for the range of T.
(c) Find eigenvalues and eigenspaces of T. Is T diagonalizable?

Solution. (a) Let us denote $v = (1, 2, -1)$. The function T is a linear transformation since $T(u_1 + u_2) = v \times (u_1 + u_2) \overset{*}{=} v \times u_1 + v \times u_2 = T(u_1) + T(u_2)$, and $T(cu_1) = v \times (cu_1) \overset{*}{=} c\,v \times u_1 = cT(u_1)$, where the equalities with $*$ are properties of the cross-product.
(b) Let $u \in \mathcal{V}_3$ such that $T(u) = v \times u = O$. By the properties of the cross-product u must be parallel to v. This means that $N(T) = L(v) = L((1, 2, -1))$. From the nullity-plus-rank theorem, it follows that $rk(T) = 2$. Let us compute $T(e_1) = T((1, 0, 0)) = (0, -1, -2)$, $T(e_2) = (1, 0, 1)$. Since they are independent, they must be a basis of $T(\mathcal{V}_3)$. Therefore $T(\mathcal{V}_3 = L((0, -1, -2), (1, 0, 1))$.
(c) We already know that 0 is an eigenvalue, with eigenspace $E(0) = L((1, 2, -1))$. There are no other eigenvalues. If fact, given $u \in \mathcal{V}_3$, $(u) = v \times u$ is perpendicular to u. hence $T(u)$ cannot be of the form λu unless $\lambda = 0$. Since T has only one eigenvalue, and the corresponding eigenspace is 1-dimensional, T is not diagonalizable (as a *real* linear transformation, of course).

NOTE: alternatively, one can compute the matrix of T (withe respect to the canonical basis). It is the matrix whose columns are $T(e_1)$, $T(e_2)$, $T(e_3)$. The answers to (a), (b), (c) can be deduced from such matrix as well.

5. In the real linear space $\mathcal{C}([-1, 1])$ with inner product $< f, g >= \int_{-1}^{1} f(x)g(x)dx$, let $f(x) = e^x$. Find the polynomial of degree one g nearest to f and compute $\| f - g \|$.

Solution. Let us denote V_1 the linear space of polynomials of degree at most 1. In general, to anwer the question, one should find an orthogonal basis of V_1. In this case the natural basis $\{1, x\}$ is already orthogonal, since $\int_{-1}^{1} xdx = 0$. Therefore the required g is simply

$$\frac{< 1, e^x >}{\| 1 \|^2} + \frac{< x, e^x >}{\| x \|^2} x$$

Computing the integrals $< 1, e^x >$, $< 1, 1 >$, $< x, e^x >$ and $< x, x >$ it follows that

$$g(x) = \frac{e - \frac{1}{e}}{2} + \frac{3}{e} x$$

and that $\| e^x - g(x) \| = \sqrt{1 - \frac{7}{e^2}}$.