**LAG written test of FEBRUARY 26, 2013**

**1.** Let $P = (5, 3, 1)$, $\mathbf{v} = (1, 1, -1)$ and $\mathbf{w} = (1, 2, 2)$. Let us consider the straight lines $R = \{P + t\mathbf{v} \,|\, t \in \mathbf{R}\}$, and $S = \{P + t\mathbf{w} \mid t \in \mathbf{R}\}$.
(a) Find the cartesian equation of the plain containing the straight lines $R$ and $S$
(b) Find all the right triangles with one vertex in $P$ and the other two vertices one in $R$ and one in $S$, such that their area is equal to $\sqrt{3}$.

**2.** We consider the following three planes in $\mathcal{V}_3$:
$M = \{(1, 1, -2) + t(1, 3, 1) + s(1, 1, 0) \mid t, s \in \mathbf{R}\}$, $\quad N : x - y + 2z = 5$, $\quad T : 3x + y + z = -1$.
Describe the following loci and find their equations:
(a) the locus of points $P \in \mathcal{V}_3$ such that $d(P, M) = 10$,
(b) the locus of points $P \in \mathcal{V}_3$ such that $d(P, M) = d(P, N)$,
(c) the locus of points $P \in \mathcal{V}_3$ such that $d(P, M) = d(P, T)$.

**3.** We consider the parametrized curve $\mathbf{r}(t) = (3t - t^3, 3t^2, 3t + t^3)$.
(a) For $t = 2$ express the acceleration vector as linear combination of the unit tangent vector and the unit normal vector.
(b) Compute the curvature of the underlying curve at the point $\mathbf{r}(2)$.

**4.** Let $V$ be the linear space of polynomials of degree at most 3. Given $p(x)$ and $q(x)$ in $V$, let us denote

$$< p(x), q(x) >= p(-1)q(-1) + p(0)q(0) + p(1)q(1) + p(2)q(2).$$

(a) Prove that $< p(x), q(x) >$ is an inner product on $V$.
(b) Let $W = L(x^2 + x, 2x^2 + 1)$ be the linear subspace of $V$ spanned by the polynomials $x^2 + x$ and $2x^2 + 1$ Compute the projection of $x^3 + 2x$ onto $W$. Compute the distance between $p(x) = x^3 + 2x$ and $W$.

**5.** Let us consider the quadratic form $q((x, y, z)) = (x, y, z)A \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, where $A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & 1 \\ -2 & 1 & 5 \end{pmatrix}$.
(a) Find an orthonormal basis of $\mathcal{V}_3$, say $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$, and scalars $c_1$, $c_2$ $c_3$ such that

$$q((x, y, z)) = c_1(x')^2 + c_2(y')^2 + c_3(z')^2,$$

where $(x, y, z) = x'\mathbf{u} + y'\mathbf{v} + z'\mathbf{w}$.

(b) Describe the set of $(x, y, z) \in \mathcal{V}_3$ such that $q((x, y, z)) = 0$.
(c) Find the maximum and the minimum of $q$ on the the unit sphere of $\mathcal{V}_3$. Describe the points of minimum and the points of maximum.