LAG, written test of Sept 24, 2012

1. Let L be the line of $\mathcal{V}_3$: $L = L\big((1,1,2)\big)$ and let π be the plane pf $\mathcal{V}_3$: $\pi = L\big((1,2,0),(0,1,1)\big)$.
Find the cartesian equations of all planes of $\mathcal{V}_3$ which are parallel to L, perpendicular to π, and such that their distance from the point $P = (1,3,1)$ is equal to $\sqrt{3}$. *(Recall that two planes are said to be perpendicular if and only if and only their normal vectors are perpendicular).*

2. Let $\mathbf{v}_1 = (1,2,2)$ and $\mathbf{v}_2 = (1,0,1)$.
(a) Find an orthogonal basis of $\mathcal{V}_3$, say $\{\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\}$ such that $L(\mathbf{v}_1) = L(\mathbf{w}_1)$ and $L(\mathbf{v}_1,\mathbf{v}_2) = L(\mathbf{w}_1,\mathbf{w}_2)$.
(b) Find an orthogonal basis of $\mathcal{V}_3$, say $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$ such that $L(\mathbf{v}_1,\mathbf{v}_2) = L(\mathbf{u}_1,\mathbf{u}_2)$, and the angle between $\mathbf{V}_1$ and $\mathbf{u}_1$ is equal to $\pi/4$.

3. A point moves in the plane with position vector $\mathbf{r}(t) = \big(x(t),y(t)\big)$. The motion is such that $\mathbf{r}'' = -9\mathbf{r}(t)$. The initial position is $\mathbf{r}(0) = (0,3)$. The initial velocity is $\mathbf{r}'(0) = (-6,0)$.
(a) Find the function $\mathbf{r}(t)$.
(b) Find the cartesian equation of the trajectory. Make a sketch and indicate the direction of motion.

4. Let $V = L\big(1,1,0),(0,1,-1)\big)$. Let $P_V : \mathcal{V}_3 \to \mathcal{V}_3$ be the projection onto V. Write the matrix (with respect to the canonical basis) of P_V.

5. Reduce the quadratic form $Q(x,y,z,t) = 2\sqrt{3}xz + 4yt + 4z^2 - 3t^2$ to canonical form (without explicitly finding the basis). Is the form positive, semipositive, negative, seminegative or indefinite?

SOLUTIONS

1. A normal vector to π is $(1,2,0)\times(0,1,1) = (2,-1,1)$. A normal vector to a plane parallel to L has to be perpendicular to $(1,1,2)$. A normal vector to a plane perpendicluar to π has to be perpendicular to $(2,-1,1)$. Hence a normal vector to a plane which is both parallel to L and perpendicular to π has to be parallel to $(1,1,2)\times(2,-1,1) = (3,-3,3)$. Hence the cartesian equation of the planes we are looking for is of the form

$$x - y + z = d.$$

Denoting Q a point of our plane (any point), the distance of such a plane from the point $P = (1,3,1)$ is

$$\frac{|(P-Q)\cdot N)|}{\parallel N \parallel} = \frac{|P\cdot N - d|}{\parallel N \parallel} = \frac{|-1-d|}{\sqrt{3}}$$

Therefore we have to impose

$$\frac{|-1-d|}{\sqrt{3}} = \sqrt{3}$$

yielding $d = -4$ and $d = 2$. In conclusion the required planes are two, having as cartesian equations

$$x - y + z = -4 \qquad \text{and} \qquad x - y + z = 2$$

2. (a) We take $\mathbf{w}_1 = \mathbf{v}_1$. Then we find $\mathbf{w}_2$ using the formula

$$\mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \frac{1}{3}(2, -2, 1)$$

Therefore we can take $\mathbf{w}_2 = (2, -2, 1)$. Then we can take $\mathbf{w}_3 = \mathbf{w}_1 \times \mathbf{w}_2 = (6, 3, -6)$.
(b) Clearly we can take $\mathbf{u}_3 = \mathbf{w}_3 = (6, 3, -6)$. A convenient way to find $\mathbf{u}_1$ and $\mathbf{u}_2$ is as follows: we first normalize $\mathbf{v}_1 = \mathbf{w}_1$ and $\mathbf{w}_2$, thus obtaining two orthogonal unit vectors, say $\mathbf{w}_1'$ and $\mathbf{w}_2'$, parallel respectively to $\mathbf{v}_1$ and $\mathbf{w}_2$. At this point, two vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ as required are, for example, $\mathbf{u}_1 = (1/\sqrt{2})\mathbf{w}_1' + (1/\sqrt{2})\mathbf{w}_2'$ and $\mathbf{u}_2 = (1/\sqrt{2})\mathbf{w}_1' - (1/\sqrt{2})\mathbf{w}_2'$ (here $(1/\sqrt{2}) = \cos(\pi/4) = \sin(\pi/4)$). Hence

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}}(1, 0, 1) \qquad \text{and} \qquad \mathbf{u}_2 = \frac{1}{3\sqrt{2}}(-1, 4, 1)$$

3. The functions $x(t)$ and $y(t)$ satisfy the equations with initial conditions

$$\begin{cases} x''(t) = -9x(t), & x(0) = 0, \quad x'(0) = -6 \\ y''(0) = -9y(t), & y(0) = 3, \quad y'(0) = 0 \end{cases}$$

Since both functions are of the form $a\cos 3t + b\sin 3t$, one finds easily that

$$x(t) = -2\sin 3t \qquad \text{and} \qquad y(t) = 3\cos 3t.$$

The trajectory is the conic section of cartesian equation

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

The sense of motion is counterclockwise.

4. In general (that is, in $\mathcal{V} - n$, with arbitrary n) , the best way to answer a question like this is to first write down the matrix of P_V with respect to a convenient basis of $\mathcal{V}_n$ (for example a basis obtained by putting together the known basis of V and a basis of $V^\perp$). The matrix is $B = \operatorname{diag}(1, \ldots, 1, 0, \ldots, 0)$ (the number of 1's is the dimension of V). Then one can obtain the matrix with respect to the canonical basis from B via the change-of-basis procedure. You are invited to solve this exercise also with this method.

However, the case of this exercise is especially easy, because $V^\perp$ is one-dimensional. In this case the formula for the projection is so easy that it is shorter to proceed directly. We have that $V^\perp$ is generated by a normal vector, as for example $N = (1,1,0) \times (0,1,-1) = (-1,1,1)$. Hence the projection of a vector (x,y,z) onto V is

$$(x,y,z) - \frac{(x,y,z)\cdot N}{N \cdot N} N = (x,y,z) - \frac{-x+y+z}{3}(-1,1,1) =$$

$$= \frac{1}{3}(4x - y - z, x + 2y - z, x - y + 2z)$$

Therefore the matrix of the projection P_V with respect to the canonical basis of $\mathcal{V}_3$ is

$$A = \frac{1}{3}\begin{pmatrix} 4 & -1 & -1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

5. The characteristic polynomial is

$$P(\lambda) = \det\begin{pmatrix} \lambda & 0 & -\sqrt{3} & 0 \\ 0 & \lambda & 0 & -2 \\ -\sqrt{3} & 0 & \lambda - 4 & 0 \\ 0 & -2 & 0 & \lambda + 3 \end{pmatrix} = \lambda(\lambda^2 + 3\lambda - 4) - \sqrt{3}(\sqrt{3}(\lambda^2 + 3\lambda - 4) =$$

$$= (\lambda^2 + 3\lambda - 4)(\lambda^2 - 4\lambda - 3).$$

Its roots are $4, -1, 2+\sqrt{7}, 2+\sqrt{7}$. Therefore the canonical form is

$$Q(x,y,z,t) = 4(x')^2 + (2+\sqrt{7})(y')^2 + (2-\sqrt{7})(z')^2 - (t')^2$$

where (x', y', z', t') are the components of (x,y,z,t) with respect to an orthonormal basis of eigenvectors of the matrix above above. The form is indefinite.