LAG, written test of FEBRUARY 13, 2013

1. Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be an orthogonal basis of $\mathcal{V}_3$ with $\| \mathbf{v}_1 \| = 4$, $\| \mathbf{v}_2 \| = \| \mathbf{v}_3 \| = 2$. Let $\mathbf{v} = (1/2)\mathbf{v}_1 + (1/2)\mathbf{v}_2 - \mathbf{v}_3$. For $i = 1, 2, 3$ let θ_i be the angle between $\mathbf{v}$ and $\mathbf{v}_i$.
(a) Find θ_i for $i = 1, 2, 3$.
(b) Let $\mathbf{w} = \mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3$. Write $\mathbf{w}$ as the sum of a vector parallel to $\mathbf{v}$ and a vector perpendicular to $\mathbf{v}$.

2. We consider parabolas with focus in $(0, 0)$, directrix parallel to $x - y = 0$, and such that they contain the point $(4, 3)$. For each such parabola find the directrix and the vertex.

3. We consider regular differentiable curves in $\mathcal{V}_3$ described by a point moving in $\mathcal{V}_3$ with position vector $\mathbf{r}(t)$. Let us denote $\mathbf{v}(t)$ and $\mathbf{a}(t)$ the velocity vector and the acceleration vector. For each of the following statements prove it (if true) or disprove it by showing a counterexample (if false).
(a) If $\mathbf{v}(t)$ is constant then the curve is contained in a plane.
(b) If $\mathbf{a}(t)$ is constant then the curve is contained in a plane.
(c) If both $\| \mathbf{v}(t) \|$ and $\| \mathbf{a}(t) \|$ are constant then the curve is contained in a plane.
(d) If $\mathbf{a}(t)$ is parallel to $\mathbf{r}(t)$, for each t, then the curve is contained in a plane.

4. Let $\mathbf{v}_1 = (1, 2, 1)$, $\mathbf{v}_2 = (1, 0, 0)$, $\mathbf{v}_3 = (1, -1, 0)$.
Let $L : \mathcal{V}_3 \to \mathcal{V}_3$ be the linear transformation such that

$$L(\mathbf{v}_1) = 2\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 \qquad L(\mathbf{v}_2) = \mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_3 \qquad L(\mathbf{v}_3) = \mathbf{v}_1 + \mathbf{v}_2 + 5\mathbf{v}_3$$

(a) Is L the unique linear transformation satisying the above requests?
(b) Compute $L((0, 1, 0))$.
(c) Compute the nullity and rank of L.

5. Find all symmetric matrices A such that $det(A) = -6$, $A\mathbf{v}_1 = 2\mathbf{v}_1$ and $A\mathbf{v}_2 = 2\mathbf{v}_2$, where $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

SOLUTIONS

1. (a) Since the basis is orthogonal, we have that

$$\| \mathbf{v} \|^2 = \| (1/2)\mathbf{v}_1 \|^2 + \| (1/2)\mathbf{v}_2 \|^2 + \| -\mathbf{v}_3 \|^2 = (1/4) \| \mathbf{v}_1 \|^2 + (1/4) \| \mathbf{v}_2 \|^2 + \| \mathbf{v}_3 \|^2 = 9$$

Hence $\| \mathbf{v} \| = 3$. Therefore $\cos\theta_1 = \frac{\mathbf{v}_1 \cdot \mathbf{v}}{\|\mathbf{v}\| \|\mathbf{v}_1\|} = \frac{(1/2)\mathbf{v}_1 \cdot \mathbf{v}_1}{\|\mathbf{v}\| \|\mathbf{v}_1\|} = 2/3$. (here we used that $\mathbf{v_1} \cdot \mathbf{v}_2 = \mathbf{v_1} \cdot \mathbf{v}_3 = 0$. Similarly, one computes $\cos\theta_2 = 1/3$ and $\cos\theta_3 = -2/3$.

(b) Again using the orthogonality of the basis we have that

$$\mathbf{w} \cdot \mathbf{v} = (1/2) \| \mathbf{v}_1 \|^2 - (1/2) \| \mathbf{v}_2 \|^2 - \| \mathbf{v}_3 \|^2 = 2.$$

Hence the projection of $\mathbf{w}$ along the line $L(\mathbf{v})$ is $\left(\frac{\mathbf{w} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = (2/9)\mathbf{v}$. By the property of the projection, the required decomposition is

$$\mathbf{w} = (2/9)\mathbf{v} + (\mathbf{w} - (2/9)\mathbf{v})$$

2. The distance between the point $(4, 3)$ and the focus is $\| (4, 3) - (0, 0) \| = 5$. Therefore, by the definition of parabola, the distance between the point $(4, 3)$ and the directrix must be equal to 5. Hence we should find the lines R_c of equation $x - y = c$, with $c \in \mathbf{R}$ whose distance from the point $(4,3)$ equals 5. We know (and it is easy to find) that $d((4,3), R_c) = \frac{|(4,3)\cdot N - c|}{\|N\|}$, where $N = (1,-1)$ is the normal vector to R_c provided by the cartesian equation. Hence $d((4,3), R_c) = \frac{|1-c|}{\sqrt{2}}$. Therefore we must have $|1-c| = 5\sqrt{2}$, which has the two solutions: $c_1 = 1 - 5\sqrt{2}$ and $c_2 = 5\sqrt{2} + 1$. Therefore there are two parabolas subject to the requests of the exercise.

The first parabola has directrix R_1 of equation $x - y = 1 - 5\sqrt{2}$. Let V be the vertex. We know that it must lie on the line passing trough the focus $F = (0,0)$ and perpendicular to the line R_1. Moreover $d(V,F) = d(V,R_1)$. Therefore V lies at the halfway between $F = (0,0)$ and R_1, on the line $\{tN \mid t \in \mathbf{R}\}$. The distance $d(F,R_1) = d((0,0),R_1)$ is easily computed as $d = \frac{|1-5\sqrt{2}|}{\sqrt{2}} = \frac{5\sqrt{2}-1}{\sqrt{2}}$. Since F is in the negative half-plane defined by N, V is $\frac{-(1/2)d}{\sqrt{2}}N = \frac{1-5\sqrt{2}}{2\sqrt{2}}(1,-1)$.

The second parabola has directrix R_2 of equation $x - y = 5\sqrt{2} + 1$. Let W be the vertex. We know that it must lie on the line passing trough the focus $F = (0,0)$ and perpendicular to the line R_2. Moreover $d(W,F) = d(V,R_2)$. Therefore W lies at the halfway between $F = (0,0)$ and R_2, on the line $\{tN \mid t \in \mathbf{R}\}$. The distance $d(F,R_2) = d((0,0),R_2)$ is easily computed as $d = \frac{|1+5\sqrt{2}|}{\sqrt{2}} = \frac{1+5\sqrt{2}}{\sqrt{2}}$. Since F is in the positive half-plane defined by N, V is $\frac{(1/2)d}{\sqrt{2}}N = \frac{1+5\sqrt{2}}{2\sqrt{2}}(1,-1)$.

3. (a) is true. Indeed if $\mathbf{r}'(t) \equiv \mathbf{u}$ then $\mathbf{r}(t) = t\mathbf{u} + \mathbf{w}$. therefore the curve is contained in the plane spanned by $\mathbf{u}$ and $\mathbf{w}$ (in fact it is a line).
(b) is true. Indeed, if $\mathbf{r}'' \equiv \mathbf{u}$ then $\mathbf{r}' = t\mathbf{u} + \mathbf{w}$ and $\mathbf{r}(t) = (t^2/2)\mathbf{u} + t\mathbf{w} + \mathbf{h}$. Hence the curve is contained in the plane $\{\mathbf{h} + \lambda\mathbf{u} + \mu\mathbf{w} \mid \lambda, \mu \in \mathbf{R}\}$.
(c) is false. Counterexample: the circular helix parametrized by $\mathbf{r}(t) = (a\cos t, a\sin t, bt)$ has constant speed and constant scalar acceleration.
(d) is true. The hypothesis means "radial" or "central" acceleration, and we know that in this case the curve is plane (for example, this is the very first step in order to show that the orbit of planets are conic sections, in fact ellipses). This is easy seen as follows: the condition can be written as $\mathbf{r}(t) \times \mathbf{a}(t) \equiv 0$. Since, as it is easily seen, $\mathbf{r}(t) \times \mathbf{a}(t) = (\mathbf{r}(t) \times \mathbf{v}(t))'$, this means that there is a vector $\mathbf{u}$ such that $\mathbf{r}(t) \times \mathbf{v}(t) \equiv \mathbf{u}$. If $\mathbf{u} \neq \mathbf{0}$ then the curve is contained in the plane $\mathbf{u} \cdot X = 0$. If $\mathbf{u} = \mathbf{0}$ then the velocity vector is always parallel to the position vector, and it is well known that in this case the curve is a line *, hence contained in a plane.

4. (a) The answer is yes. In fact, as it is easily checked, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent, hence they form a basis of $\mathcal{V}_3$. In this case we know a theorem ("the linear transformation with prescribed values on the vectors of a basis") ensuring that there is a unique linear transformation sending the vectors of a basis to prescribed vectors.
(b) here one could compute the matrix, say A, representing L with respect to the canonical basis. Then $L((0,1,0))$ would be $A\begin{pmatrix}0\\1\\0\end{pmatrix}$, that is the second column of A. However, since the text does not ask for the matrix representing L with respect to the canonical basis, but only for $L((0,1,0))$, it is shorter to argue directly. We need to find $c_1, c_2, c_3 \in \mathbf{R}$ such that $(0,1,0) = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$. After having done that, the answer is easy: $L((0,1,0)) = L(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) = c_1L(\mathbf{v}_1) + c_2L(\mathbf{v}_2) + c_3L(\mathbf{v}_3)$ and the answer can be given, since we know explicitly $L(\mathbf{v}_1), L(\mathbf{v}_2)$ and $L(\mathbf{v}_3)$.

To find the coordinates c_1, c_2 and c_3 amounts to solve the system of linear equations

$$\begin{cases} x + y + z = 0 \\ 2x - z = 1 \\ x = 0 \end{cases}$$

* This can be seen – for example – as follows: if $\mathbf{r}(t) = f(t)\mathbf{r}'(t)$ for some differentiable scalar function $f(t)$, differentiating one gets $(1 - f'(t))\mathbf{r}'(t) = f(t)\mathbf{r}''(t)$. Thus the acceleration is always parallel to the velocity, that is the accelaration has never a normal component, hence the path is a line.

whose solution is $c_1 = 0$, $c_2 = 1$, $c_3 = -1$ (that is, $(0,1,0) = (1,0,0) - (1,-1,0)$, which could have been seen immediately). Therefore $L((0,1,0)) = L(\mathbf{v}_2) - L(\mathbf{v}_3) = \mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_3 - (\mathbf{v}_1 + \mathbf{v}_2 + 5\mathbf{v}_3) = -2\mathbf{v}_2 - 6\mathbf{v}_3 = -2(1,0,0) - 6(1,-1,0) = (-8,6,0)$.
(c) The matrix representing L with respect to the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is

$$\begin{pmatrix} 2 & 1 & 1 \\ -1 & -1 & 1 \\ 1 & -1 & 5 \end{pmatrix}$$

The rank of L coincides with the rank of this matrix, which is 2, as it is easily seen with a gaussian elimination, or computing a determinant. Bt the nullity plus rank theorem, the nullity is 1.

5. Eigenvalues of A: the hypothesis tells us that 2 is at least a double eigenvalue. Since the determinant is the product of the eigenvalues, we have that the third eigenvalue is $-6/4 = -3/2$.
Eigenspaces of A: the eigenspace of 2 is $E(2) = L(\mathbf{v}_1, \mathbf{v}_2)$. The matrix A being symmetric, eigenspaces of distinct eigenvalues are orthogonal linear subspaces. Therefore $E(-3/2) = L(\mathbf{v}_1 \times \mathbf{v}_2) = L((3,0,-3))$.
Let $\mathcal{B} = \{\mathbf{v_1}, \mathbf{v}_2, \mathbf{v}_1 \times \mathbf{v}_2\}$ and let $C = m^{\mathcal{B}}_{\mathcal{E}}(id) = \begin{pmatrix} 1 & 1 & 3 \\ 2 & -1 & 0 \\ 1 & 1 & -3 \end{pmatrix}$. Moreover let L_A denote the linear transformations represented by the searched-for matrices A with respect to the canonical basis. We know that all L_A's are represented with respect to the basis $\mathcal{B}$ by the diagonal matrix $diag(2,2,-3/2)$. Hence there is only one L_A and, consequently, only one A. From the change-of-basis formula

$$diag(2,2,-3/2) = C^{-1}AC$$

we deduce that

$$A = C\, diag(2,2,-3/2)\, C^{-1}$$