LAG, written test of JULY 20, 2012

1. In $\mathcal{V}_4$ let V be the space of solutions of the linear system $\begin{cases} 3x - y + 2z - t & =0 \\ x + y + z + t & =0 \end{cases}$, and let W be the space of solutions of the equation $3x - y + 2z - t = 0$.
Moreover, let $w = (1, 1, 2, -1)$.
(a) Write w as the sum of a vector in V and a vector in $V^\perp$.
(b)) Write w as the sum of a vector in W and a vector in $W^\perp$.

2. Let $A = (1, 0, -1)$, $B = (1, 2, 1)$. Moreover let L be the line $L = (1, 0, 0) + t(1, -1, 1)$, $t \in \mathbf{R}$.
Find (if any) all points $Q \in L$ such that the area of the triangle whose vertices are A, B and Q is equal to 4.

3. Let $u = (1, 1, -1)$ and $v = (1, 2, 0)$. Find all unit vectors in $L(u, v)$ forming an angle of $\pi/3$ with u.

4. Let $T : \mathcal{V}_2 \to \mathcal{V}_3$ be the linear transformation such that

$$T((1, 0, 0)) = (1, 1, 0) \qquad T((1, 2, 0)) = (1, 2, 0) \qquad T((1, 2, 1)) = (1, 1, -1)$$

(a) Is T invertible?
(b) If the answer is yes, find the matrix of T^{-1} with respect to the canonical basis of $\mathcal{V}_3$.

5. Exhibit two different examples of 3×3 matrices whose eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = -1$ and such that $u = (1, 2, -2)$ is an eigenvalue of $\lambda_1 = 1$. Arrange things in such a way that one of the examples is a symmetric matrix.

SOLUTIONS

1. (a) The required decomposition is $w = pr_V(w) + pr_{V^\perp}(w)$ where "pr" means projection. We have that $V^\perp = L((3, -1, 2, -1), (1, 1, 1, 1))$. An orthogonal basis of $V^\perp$ is $\{u, v\} = \{(1, 1, 1, 1), (9, -7, 5, -7)\}$. In conclusion $pr_{V^\perp}(w) = pr_u(w) + pr_v(w) = \frac{3}{4}(1, 1, 1, 1) + \frac{19}{204}(9, -7, 5, -7)$ and $pr_V(w) = w - pr_{V^\perp}(w)$.
(b) Here $W^\perp = L((3, -1, 2, -1))$. Hence $p_{W^\perp}(w) = \frac{7}{15}(3, -1, 2, -1)$ and $p_W(w) = w - pr_{W^\perp}(w)$.

2. The area of a triangle whose vertices are the points A,B, and Q is $\frac{1}{2} \parallel (B - A) \times (Q - A) \parallel$. Since a point of L is of the form $(1, 0, 0) + t(1, -1, 1)$ we get

$$\frac{1}{2} \parallel (0, 2, 2) \times (t, -t, t + 1) \parallel = \frac{1}{2} \parallel (4t + 2, 2t, -2t) \parallel = \sqrt{6t^2 + 4t + 1}$$

Hence we are looking for the t's such that $\sqrt{6t^2 + 4t = 1} = 4$, hence $t_1 = \frac{-2+\sqrt{94}}{6}$ and $t_2 = \frac{-2-\sqrt{94}}{6}$. The required points are $Q_1 = (1, 0, 0) + t_1(1, -1, 1)$ and $Q_2 = (1, 0, 0) + t_2(1, -1, 1)$

3. Let $v' = v - \frac{v \cdot u}{u \cdot u} u = (0,1,1)$, We have that $\{u, v'\}$ is an orthogonal basis of $L(u,v)$. Dividing by the norms, we get orthonormal basis $\{a, b\} = \{(\frac{1}{\sqrt{3}}(1,1,-1), \frac{1}{\sqrt{2}}(0,1,1)\}$. The required unit vectors are $\cos\frac{\pi}{3} a + \sin\frac{\pi}{3} b$ and $\cos\frac{\pi}{3} a - \sin\frac{\pi}{3} b$.

4. (a) As it is immediare to check, $\mathcal{B} = \{(1,0,0),(1,2,0),(1,2,1)\}$ is a basis of $\mathcal{V}_3$. The matrix representing T with respect to $\mathcal{B}$ (source) and $\mathcal{E}$(=canonical basis) (target) is

$$m_{\mathcal{E}}^{\mathcal{B}}(T) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

This matrix is clearly invertible (expanding from the last column one sees that the determinant is -1, in particular non-zero). Hence T is invertible, as we know that $m_{\mathcal{B}}^{\mathcal{E}}(T^{-1}) = (m_{\mathcal{E}}^{\mathcal{B}}(T))^{-1}$.
(b) We compute the matrix $(m_{\mathcal{E}}^{\mathcal{B}}(T))^{-1}$. This can be done using determinants or with gaussian elimination. For example, with gaussian elimination:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right)$$

Denoting X_1, X_2 and X_3 the rows of the inverse matrix, we find: $X_3 = (0,0,-1)$, $X_2 = (-1,1,0)$, and $X_1 = (1,0,0) - X_2 - X_3 = (2,-1,1)$. In conclusion

$$m_{\mathcal{B}}^{\mathcal{E}}(T^{-1}) = (m_{\mathcal{E}}^{\mathcal{B}}(T))^{-1} = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Hence

$$T^{-1}((1,0,0)) = 2(1,0,0) - (1,2,0) = (1,-2,0)$$

$$T^{-1}((0,1,0)) = -(1,0,0) + (1,2,0) = (0,2,0)$$

$$T^{-1}((0,0,1)) = (1,0,0) - (1,2,1) = (0,-2,-1)$$

In conclusion

$$m_{\mathcal{E}}^{\mathcal{E}}(T^{-1}) = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & -2 \\ 0 & 0 & -1 \end{pmatrix}$$

Here is an alternative way of finding the same results: one finds $m_{\mathcal{E}}^{\mathcal{E}}(T)$. This can be found using the change-of-basis formula. However, the case at hand is so simple that one can avoid that: since $(0,1,0) = \frac{1}{2}((1,2,0) - (1,0,0)$ one gets that

$$T((0,1,0)) = \frac{1}{2}(T(1,0,0) - T((1,2,0)) = (0, -\frac{1}{2}, 0)$$

Since $(0,0,1) = (1,2,1) - (1,2,0)$ one gets that

$$T((0,0,1)) = T((1,2,1) - T(1,2,0) = (0,-1,-1)$$

Hence

$$m_{\mathcal{E}}^{\mathcal{E}}(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -1 \\ 0 & 0 & -1 \end{pmatrix}$$

This matrix is invertible. Hence T is invertible, and the matrix $m_{\mathcal{E}}^{\mathcal{E}}(T^{-1})$ is the inverse of the matrix above.

5. Note that, since the eigenvalues are three different real numbers, the required matrices are diagonalizable.

Let $\mathcal{B}$ be any basis of $\mathcal{V}_3$, and let $C = m_{\mathcal{E}}^{\mathcal{B}}(\mathrm{id})$ be the matrix whose columns are the usual coordinates of the vectors of $\mathcal{B}$. We have that

$$diag(1,2,-1) = C^{-1}AC$$

where A is the matrix representing, with respect to the canonical basis, the linear tranformation T such that $m_{\mathcal{B}}^{\mathcal{B}} = diag(1,2,-1)$. Hence all matrices with (ordered) eigenvalues 1,2,-1 are of the form

$$A = C\, diag(1,2,-1)\, C^{-1}$$

where C is a non-singular(= invertible) 3×3 matrix. To satisfy the requirement that u is an eigenvalue of 1, we need that the first column of C is a vector parallel to u. In addition, if we want A to be symmetric, we can impose that C is an orthogonal matrix.

For example, we can take

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

Then compute C^{-1} and plug it in the above formula.

For a symmetric example, we can take

$$C = \begin{pmatrix} \frac{1}{3} & 0 & \frac{4}{\sqrt{18}} \\ \frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{18}} \\ -\frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} \end{pmatrix}$$