LAG, written test of JULY 20, 2012

1. Let $\mathcal{C}$ be a parabola in $\mathbf{V}_2$, with focus in $(0,0)$, directrix of equation $4x+3y=c$, and such that $(1,0)\in\mathcal{C}$.
(a) Find c assuming that $c>0$. Find also the vertex and the equation of $\mathcal{C}$.
(b) Find c assuming that $c<0$.

2. A particle moves along the ellipse $4x^2+y^2=1$ with position vector $\mathbf{r}(t)=(f(t),g(t))$. The motion is such that $f'(t)=-g(t)$ for every t.
How much time is required for the particle to go once around the ellipse?

3. Let $T:\mathbf{V}_4\to\mathbf{V}_3$ defined by $T(\begin{pmatrix}x\\y\\z\\t\end{pmatrix})=\begin{pmatrix}x+y+z-t\\x+2y+z+t\\2x+3y+2z\end{pmatrix}$.
(a) Find dimensions and bases of $N(T)$ and $T(\mathbf{V}_4)$.
(b) Does $\begin{pmatrix}1\\-1\\0\end{pmatrix}$ belong to $T(\mathbf{V}_4)$? If the answer is yes, describe the set of all $\mathbf{v}\in\mathbf{V}_4$ such that $T(\mathbf{v})=\begin{pmatrix}1\\-1\\0\end{pmatrix}$.

4. Let $\mathbf{V}$ be the space of real polynomials, with inner product $(P.Q)=\int_{-1}^{1}P(x)Q(x)dx$. Moreover let $\mathbf{W}$ be the subspace of all polynomials of degree ≤ 2.
Write $P(x)=x^3$ as the sum of a polynomial in $\mathbf{W}$ and of a polynomial orthogonal to $\mathbf{W}$.

5. Find all 3×3 symmetric matrices A such that $\lambda=2$ is an eigenvalue of A, $L((1,1,1),(1,2,-1))$ is an eigenspace of A, and $\det(A)=-100$.

SOLUTIONS

1. (a) Let $F=(0,0)$ be the focus, let $P=(1,0)$ be the point on $\mathcal{C}$, and let $L:4x+3y=c$ be the directrix. We have that $d(P,F)=1$. Since – by definition of parabola – $d(P,F)=d(P,L)$, we have that $d(P,L)=1$. We compute

$$d(P,L)=\frac{|(1,0)\cdot(4,3)-c|}{\|\,(4,3)\,\|}=\frac{|4-c|}{5}$$

Hence $|4-c|=5$. Since $c>0$ then $c=9$. In this case the vertex is the point V lying on the line $L((4,3))$ such that $d(V,F)=d(V,L)$. Therefore $d(V,F)$ is the half of $d(F,L)=9/5$. Hence

$$V=\frac{9}{10}\frac{(4,3)}{5}=\frac{9}{50}(4,3)$$

Cartesian equation:

$$d((x,y),F)=\sqrt{x^2+y^2}=d((x,y),L)=\frac{|4x+3y-9|}{5}$$

Hence $25\,x^2+25\,y^2=16\,x^2+9\,y^2+81+24\,xy-72\,x-54\,y$, that is

$$9\,x^2+16\,y^2-24\,xy+72\,x+54\,y-81=0.$$

(b) From the equation $|4-c|=5$ we get that, if $c<0$, then $c=-1$.

2. We have that, for all t, $4(f(t))^2+(g(t))^2=1$. Hence $8f(t)f'(t)+2g(t)g'(t)=0$. Since $f'(t)=-g(t)$ we get $-g(t)(8f(t)+2g'(t))=0$. This means that $g'(t)=4f(t)$. Differentiating once again we get

$$\begin{cases} f''(t)=-g'(t)=-4f(t) \\ g''(t)=4f'(t)=-4g(t) \end{cases}$$

From what you know of the differential equation $x''=c\,x$ both $f(t)$ and $g(t)$ are of the form $a\cos 2t+b\sin 2t$. It follows that the time needed to go once around the ellipse is $T=\pi$.

3. (a) The matrix representing T with respect to the canonical bases of $\mathbf{V}_4$ and $\mathbf{V}_3$ is $A=\begin{pmatrix}1&1&1&-1\\1&2&1&1\\2&3&2&0\end{pmatrix}$. The rank of T is equal to the rank of A. From gaussian elimination

$$\begin{pmatrix}1&1&1&-1\\1&2&1&1\\2&3&2&0\end{pmatrix}\to\begin{pmatrix}1&1&1&-1\\0&1&0&2\\0&1&0&2\end{pmatrix}\to\begin{pmatrix}1&1&1&-1\\0&1&0&2\\0&0&0&0\end{pmatrix}$$

we have the rank is equal to 2. A basis of $T(\mathbf{V}_4$ is, for example, given by two independent columns of A, for example $\{\begin{pmatrix}1\\1\\2\end{pmatrix},\begin{pmatrix}1\\2\\3\end{pmatrix}\}$. By the nullity plus rank Theorem, $\dim N(T)=4-2=2$. To find a basis we solve the homogenous system associated to the matrix A. After the elimination we find $y=-2t$, $x=-z+t$. Hence $N(T)$ is the space of 4-tuples of the form $\begin{pmatrix}-z+3t\\-2t\\z\\t\end{pmatrix}$. Thus $N(T)=L(\begin{pmatrix}-1\\0\\1\\0\end{pmatrix},\begin{pmatrix}3\\-2\\0\\1\end{pmatrix})$.

(b) $\mathbf{w}=\begin{pmatrix}1\\-1\\0\end{pmatrix}$ belongs to $T(\mathbf{V}_4)$ if and only if the linear system $AX=\mathbf{w}$ has solutions. In this case the set of all $\mathbf{v}\in\mathbf{V}_4$ such that $T(\mathbf{v})=\mathbf{w}$ is precisely the set of solutions of the system $AX=\mathbf{w}$. With gaussian elimination we compute

$$A|\mathbf{w}=\begin{pmatrix}1&1&1&-1&1\\1&2&1&1&-1\\2&3&2&0&0\end{pmatrix}\to\begin{pmatrix}1&1&1&-1&1\\0&1&0&2&-2\\0&1&0&2&-2\end{pmatrix}\to\begin{pmatrix}1&1&1&-1&1\\0&1&0&2&-2\\0&0&0&0&0\end{pmatrix}$$

Hence $\mathbf{w} \in T(\mathbf{V}_4)$ and, solving the system, the set of all $\mathbf{v}$ such that $T(\mathbf{v}) = \mathbf{w}$ is the set of 4-tuples of the form $\begin{pmatrix} 3-z+3t \\ -2-2t \\ z \\ t \end{pmatrix}$, therefore

$$\begin{pmatrix} 3 \\ -2 \\ 0 \\ 0 \end{pmatrix} + L(\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 0 \\ 1 \end{pmatrix}).$$

4. Note that $\dim \mathbf{W} = 3$ and that $\{1, x, x^2\}$ is a basis of $\mathbf{W}$. By the Orthogonal Decomposition Theorem, the required decomposition is

$$x^3 = p_{\mathbf{W}}(x^3) + (x^3 - p_{\mathbf{W}}(x^3))$$

where $p_{\mathbf{W}}(x^3)$ is the projection of x^3 onto $\mathbf{W}$. To find that, we first need to find an orthogonal basis of $\mathbf{W}$, say $\{Q_1, Q_2, Q_3\}$. Then we have that

$$p_{\mathbf{W}}(x^3) = \frac{(x^3, Q_1)}{(Q_1, Q_1)} Q_1 + \frac{(x^3, Q_2)}{(Q_2, Q_2)} Q_2 + \frac{(x^3, Q_3)}{(Q_3, Q_3)} Q_3$$

The easiest orthogonal basis of $\mathbf{W}$ is found by applying the Gram-Schmidt orthogonalization to the basis $\{1, x, x^2\}$. Note that $(1, x) = 0$ so they are already orthogonal. Hence $Q_1 = 1$ and $Q_2 = x$. To find Q_3, we use that

$$Q_3 = x^2 - \frac{(x^2, Q_1)}{(Q_1, Q_1)} Q_1 - \frac{(x^2, Q_2)}{(Q_2, Q_2)} Q_2$$

It is easy to compute the integrals: $(x^2, Q_2) = 0$, $(x^2, Q_1) = (x^2, 1) = 2/3$, and $(Q_1, Q_1) = (1, 1) = 2$. Hence

$$Q_3 = x^2 - \frac{1}{3}$$

Next, we compute $p_{\mathbf{W}}(x^3)$ using the above formula. It is easy to compute that the integrals $(x^3, Q_1) = (x^3, Q_3) = 0$. Hence

$$p_{\mathbf{W}}(x^3) = \frac{(x^3, x)}{(x, x)} x = \frac{\frac{2}{5}}{\frac{2}{3}} x = \frac{3}{5} x$$

In conclusion, the (unique!) decomposition as requested by the exercise is

$$x^3 = \frac{3}{5} x + (x^3 - \frac{3}{5} x)$$

5. The condition on the eigenspace implies that A has a double eigenvalue. Moreover the product of the eigenvalues has to be equal to -100. Thus there are two possibilities: $\lambda = 1 = \lambda_2 = 2$, $\lambda_3 = -25$ and $\lambda_1 = 2$, $(\lambda_2)^2 = -50$. However, the latter possibility is ruled out by the fact that A is supposed to be symmetric, hence its eigenvalues need to be real.

Moreover, since A is symmetric, the eigenvectors of different eigenvalues are perpendicular. Therefore $E_A(2) = L((1,1,1),(1,2,-1))$ and $E_A(-25) = L((-3,2,1))$. Let $\mathcal{B}$ denote the basis $\{(1,1,1),(1,2,-1),(-3,2,1)\}$. Letting $T_A : \mathbf{V}_3 \to \mathbf{V}_3$ the linear transformation defined by $T_A(X) = AX$, we have that $m^{\mathcal{B}}_{\mathcal{B}}(T_A) = \mathrm{diag}(2,2,-25)$. Therefore, we consider the matrix $C = m^{\mathcal{B}}_{\mathcal{E}}(\mathrm{id}) = \begin{pmatrix} 1 & 1 & -3 \\ 1 & 2 & 2 \\ 1 & -1 & 1 \end{pmatrix}$. We have that $\mathrm{diag}(2,2,-25) = C^{-1}AC$. Hence

$$A = C\,\mathrm{diag}(2,2,-25)\,C^{-1}$$

Finding C^{-1} and then computing the above product one finds explicitly the matrix A. Note that it follows that there is a unique matrix A satisfying the requests of the exercise.