LINEAR ALGEBRA AND GEOMETRY **written test of JULY 06, 2012**

1. In $\mathbf{V}_3$ (with dot product) let $\mathbf{u} = (1, 0, -1)$, $\mathbf{v} = (1, 1, -1)$, and $\mathbf{w} = (1, 0, 0)$.
(a) Find the reflection of $\mathbf{w}$ with respect to $L(\mathbf{u}, \mathbf{v})$.
(b) Compute the distance between $\mathbf{w}$ and $L(\mathbf{u}, \mathbf{v})$.

2. In $\mathbf{V}_3$, let $A = (1, 1, 1)$ and $B = (1, 0, 1)$. Let $\mathbf{r} : I \to \mathbf{V}_3$ be a regular parametrized curve such that $\mathbf{v}(t) = A \times \mathbf{r}(t)$ and $\mathbf{r}(t_0) = B$ (here I is an interval in $\mathbf{R}$ and $t_0 \in I$).
(a) Show that there is a plane Π such that $\mathbf{r}(t) \in \Pi$ for all $t \in I$ (hint: use that $\mathbf{v}(t) \cdot A \equiv 0$).
(b) Show that the speed is constant and compute it (Hint: use that $\mathbf{a}(t) \cdot \mathbf{v}(t) \equiv 0$).
(c) Compute the curvature $\kappa(t)$.
(d) Describe the underlying curve as precisely as you can.

3. Let $T : \mathbf{V}_3 \to \mathbf{V}_3$ be the linear transformation such that

$$T((1, 2, 1)) = (3, 6, 3) \qquad T((1, 1, 0)) = (-4, -4, 0) \qquad T((1, 0, 0)) = (1, 1, 0)$$

(a) Find the dimension and a basis of $T(\mathbf{V}_3)$.
(b) Find the dimension and a basis of $N(T)$.
(c) Find eigenvalues and eigenspaces of T.
(d) Compute $m_{\mathcal{E}}^{\mathcal{E}}(T)$ (the representative matrix of T with respect to the canonical basis of $\mathbf{V}_3$).

4. Let $\mathbf{V}$ be the space of real polynomials of degree ≤ 2. For $P, Q \in \mathbf{V}$, let

$$(P, Q) = P(0)Q(0) + P(1)Q(1) + P(-1)Q(-1)$$

(a) Show that the above is an inner product on $\mathbf{V}$.
(b) Find an orthogonal basis of $\mathbf{V}$ with respect to the above inner product.

5. Let $\mathcal{C}$ be the conic defined by the equation $2x^2 - 2\sqrt{3}\,xy + 2x + 2\sqrt{3}\,y - 5$.
Find the canonical form. Find the center and the symmetry axes (with respect to the canonical basis). make a rough sketch of the curve (in the reference system given by the canonical basis). If you have time, find the asymptotes.

SOLUTIONS

1. (a) $\mathbf{u} \times \mathbf{v} = (1, 0, 1)$. We have that $(L(\mathbf{u}, \mathbf{v}))^\perp = L(\mathbf{u} \times \mathbf{v})$. The projection of $\mathbf{w}$ onto $L(\mathbf{u} \times \mathbf{v})$, denoted $P_{L(\mathbf{u}\times\mathbf{v})}(\mathbf{w})$ is $(1/2)(1, 0, 1)$. Hence the projection of $\mathbf{w}$ onto $L(\mathbf{u}, \mathbf{v})$ is $\mathbf{w} - (1/2)(1, 01)$ and we have the orthogonal decomposition

$$\mathbf{w} = \big(\mathbf{w} - (1/2)(1, 0, 1)\,\big) + (1/2)(1, 0, 1)$$

The reflection of $\mathbf{w}$ with respect to $L(\mathbf{u}, \mathbf{v})$ is

$$\mathbf{w} = \big(\mathbf{w} - (1/2)(1, 0, 1)\,\big) - (1/2)(1, 0, 1) = (0, 0, -1)$$

(b) The required distance is $\| (1/2)(1,0,1) \| = \sqrt{2}/2$.

2. (a) We have that $\mathbf{r}'(t) \cdot A = (\mathbf{r}(t) \cdot A)' \equiv 0$. Hence $\mathbf{r}(t) \cdot A$ is constant. The constant is equal to $\mathbf{r}(t_0) \cdot A = B \cdot A = 2$. Therefore, for all t, $\mathbf{r}(t)$ belongs to the plane $(x,y,z) \cdot A = 2$, that is

$$x + y + z = 2$$

(b) Differentiating $\mathbf{v}(t) = A \times \mathbf{r}(t)$ one gets $\mathbf{a}(t) = A \times \mathbf{v}(t)$. This says that the acceleration vector is perpendicular to the velocity vector. Therefore there is no tangential component of the acceleration vector. As we know, this means that the speed is constant. The constant must be equal to $\| \mathbf{v}(t_0) \| = \| A \times B \| = \| (1,0,-1) \| = \sqrt{2}$.
(c) We know that

$$\kappa(t) = \frac{\| \mathbf{v}(t) \times \mathbf{a}(t) \|}{v(t)^3}$$

We know that the denominator is constant. Concerning the numerator, since $\mathbf{v}(t)$ is perpendicular tp $\mathbf{a}(t)$, and also to A, we have that

$$\| \mathbf{v}(t) \times \mathbf{a}(t) \| = \| \mathbf{v}(t) \| \| \mathbf{a}(t) \| = \| \mathbf{v}(t) \| \| A \times \mathbf{v}(t) \| = v(t)^2 \ \| A \| \equiv 2\sqrt{3}$$

Hence also the numerator is constant. In conclusion, the curvature is constant, equal to $\sqrt{3}/\sqrt{2}$.
(d) The underlying curve is a plane curve with constant curvature. Therefore it is (a part of) a circle of radius $1/\kappa = \sqrt{2}/\sqrt{3}$. We know that this circle is contained in the plane of item (a). The center is

$$\mathbf{r}(0) + (1/k)N(0) = B - (\sqrt{2}/\sqrt{3})N(0),$$

where N means unit normal vector. We have that $N(0) = \mathbf{a}(0)/ \| \mathbf{a}(0) \|$. Now we know that

$$\mathbf{a}(0) = A \times \mathbf{v}(0) = A \times (A \times \mathbf{r}(0)) = A \times (A \times B) = (-1, 2, -1).$$

Therefore $N(0) = (1/\sqrt{6})(-1, 2, -1)$ and, consequently, the center is $(2/3)(1,1,1)$.

3. (a) Let us denote

$$\mathbf{u} = (1,2,1) \qquad \mathbf{v} = (1,1,0) \qquad \mathbf{w} = (1,0,0)$$

It is easy to verify that $\mathcal{B} = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a basis of $\mathbf{V}_3$. Therefore $T(\mathbf{V}_3)$ is spanned by $\{T(\mathbf{u}), T(\mathbf{v}), T(\mathbf{w})\} = \{3\mathbf{u}, -4\mathbf{v}, \mathbf{v}\}$. From this it follows immediately that, for example, $\{\mathbf{u}, \mathbf{v}\}$ is a basis of $T(\mathbf{V}_3)$. The dimension is two.
(b) $\dim(N(T)) = 3 - 2 = 1$. Since $T(\mathbf{v}) = -4T(\mathbf{w}) = T(-4\mathbf{w})$, we have that $T(\mathbf{v} + 4\mathbf{w}) = \mathbf{0}$. Hence $N(T)$ is spanned by $\mathbf{v} + 4\mathbf{w} = (1,1,4)$.
(c) Clearly 3 and -4 are eigenvalues. From point (b) also 0 is an eigenvalue. Thus the eigenvalues are $3, -4$ and 0. Eigenspaces:

$$E(3) = L(\mathbf{u}) \qquad E(-4) = L(\mathbf{v}) \qquad E(0) = N(T) = L(\mathbf{v} + 4\mathbf{w}).$$

(d) For example, we have that

$$m_{\mathcal{B}}^{\mathcal{B}}(T) = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -4 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

We have that

$$m_{\mathcal{E}}^{\mathcal{E}}(T) = C \begin{pmatrix} 3 & 0 & 0 \\ 0 & -4 & 1 \\ 0 & 0 & 0 \end{pmatrix} C^{-1}$$

where

$$C = m_{\mathcal{E}}^{\mathcal{B}}(\mathrm{id}) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Note: it is easy to compute that

$$C^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -1 & 1 \end{pmatrix}$$

4. (a) The three properties

$$(P, Q) = (Q, P)$$
$$(P + R, Q) = (P, Q) + (R, Q)$$
$$(cP, Q) = c(P, Q)$$

(for all $P, Q, R \in \mathbf{V}$ and $c \in \mathbf{R}$) are immediate. It remains to check the positivity. We have that

$$(P, P) = P(0)^2 + P(1)^2 + P(-1)^2$$

which is certainly non-negative. It is zero if and only if $P(0) = P(1) = P(-1) = 0$ so that P has three zeroes. Since the degree of P is ≤ 2, this happens if and only if P is the zero polynomial (alternatively, one can show, solving an easy linear system, that if $(P, P) = 0$ then all coefficients of P must vanish).
(b) The easiest way seems to apply the Gram-Schmidt orthogonalization to the canonical basis $\{1, x, x^2\}$. We have that $(1, x) = 0 + 1 - 1 = 0$, hence 1 and x are already orthogonal. The third polynomial is

$$x^2 - (\frac{(x^2, 1)}{(1, 1)})\, 1 - (\frac{(x^2, x)}{(x, x)})\, x = x^2 - \frac{2}{3}$$

Hence the basis $\{1, x, x^2 - (2/3)\}$ is orthogonal.

5. The quadratic part has matrix $A = \begin{pmatrix} 2 & -\sqrt{3} \\ -\sqrt{3} & 0 \end{pmatrix}$. The eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -1$. The eigenspaces are $E(3) = L((-\sqrt{3}, 1)$ and $E(-1) = L((1, \sqrt{3})$. The matrix

$C = \frac{1}{2}\begin{pmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{pmatrix}$ is an orthogonal basis (with $\det = 1$) whose columns are eigenvectors of A. Therefore, letting $\mathcal{B} = \{\frac{1}{2}(\sqrt{3}, -1), \frac{1}{2}(1, \sqrt{3})\}$, the components of (x, y) with respect to $\mathcal{B}$, denoted (x', y'), satisfy the relations

$$\begin{cases} x = \frac{\sqrt{3}}{2}x' + \frac{1}{2}y' \\ y = -\frac{1}{2}x' + \frac{\sqrt{3}}{2}y' \end{cases}$$

Therefore we have that

$$2x^2 - 2\sqrt{3}\,xy + 2x + 2\sqrt{3}\,y - 5 = 3{x'}^2 - {y'}^2 + 4y' - 5 = 3{x'}^2 - (y' - 2)^2 - 1$$

The canonical form is

$$3{x'}^2 - (y' - 2)^2 = 1$$

The center is $(x', y') = (0, 2)$. Using the above relation, it follows that the coordinates (with respect to the canonical basis) of the center are $(x, y) = (1, \sqrt{3})$. The symmetry axes are $\{(1, \sqrt{3}) + t((\sqrt{3}, -1)\}$ and $\{(1, \sqrt{3} + t(1, \sqrt{3})\}$. Asymptotes: in the (x', y') coordinates, they are parallel to the lines $y' = \pm\sqrt{3}x'$, hence $L((1, \sqrt{3}))$ and $L((1, -\sqrt{3}))$. In the (x, y)-coordinates, they are the lines passing trough the center and parallel to the vectors $(\sqrt{3}, -1) + \sqrt{3}(1, \sqrt{3})$ and $(\sqrt{3}, -1) - \sqrt{3}(1, \sqrt{3})$.