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Linear Algebra and Geometry, written test, 07.18.2011

NOTE: In the solution of a given exercise you must (briefly) explain the line of your argument and show the main points of your calculations. Solutions without adequate explanations will not be evaluated.

1. Consider the lines in V_3: $L = \{(2,4,-2)+t(1,1,0)\}$ and $M = \{(-3,7,-4)+t(2,-2,1)\}$.
(a) Show that the intersection of L and M is a point and find it. Let us denote it P.
(b) Find a point $Q \in L$ and a point $R \in M$ such that $\| Q-P \| = \| R-P \|$ and the area of the triangle with vertices P, Q, R is 2.

Solution. (a) The intersection point (if any) is a point P such that there are a $t \in \mathbf{R}$ a $s \in \mathbf{R}$ such that $P = (2,4,-2)+t(1,1,0) = (-3,7,-4)+s(2,-2,1)$. Therefore $t(1,1,0)-s(2,-2,1) = (-5,3,-2)$. One finds easily, for example, that $-s = 2$. Hence $s = -2$ so the intersection point is $P = (-3,7,-4)-2(2,-2,1) = (1,3,-2)$.
(b) We can write $L = \{(1,3,-2)+s\frac{1}{\sqrt{2}}(1,1,0)\}$ and $M = \{(1,3,-2)+s\frac{1}{3}(2,-2,1)\}$. Therefore $Q-P = s(1/\sqrt{2}(1,1,0))$ and $\| Q-P \| = |s|$. Moreover $R-P = \lambda(1/3(2,-2,1)$ and $\| R-P \| = |\lambda|$. Since we want $\| Q-P \| = \| R-P \|$, we can take $s = \lambda$. Moreover the area of the triangle with vertices P, Q, R is $1/2 \| (Q-P)\times(R-P) \| = (1/2)(\sqrt{18}/3\sqrt{2})s^2 = s^2/2$. Since the area has to be equal to 4, we can take $s = 2$ (or $s = -2$). Therefore $Q = (1,3,-2)+2(1/\sqrt{2})(1,1,0)$ and $R = (1,3,-2)+2/3(2,-2,1)$ are points satisfying the requests of (b).

2. Find the cartesian equation of the hyperbola whose asymptotes are $L_1 : y = 3x+1$, $L_2 : y = -3x-5$ and passing trough the point $(-3,-5)$.

Solution. The intersection of the two asymptotes is the point $(-1,-2)$. Therefore the center of symmetry of the hyperbola is $(-1,2)$. Note the union of the two asymptotes is symmetric to the lines parallel to the x-axis and to the y-axis, and passing trough the center. Therefore the form of the cartesian equation must be either $(x+1)^2/a^2-(y+2)^2/b^2 = 1$ or $-(x+1)^2/a^2+(y+2)^2/b^2 = 1$. In both cases the asymptotes are of the form $y = (b/a)x+\text{constant}$ or $y = -(b/a)x+\text{constant}$. Therefore $b/a = 3$, that is $b = 3a$. Therefore the form of the equation is either $(x+1)^2/a^2-(y+2)^2/9a^2 = 1$ or $-(x+1)^2/a^2+(y+2)^2/9a^2 = 1$. Imposing the passage through $(-3,-5)$ one sees easily that the former case holds, and $a^2 = 3$. Therefore $b^2 = 27$ and the equation is

$$\frac{(x+1)^2}{3} - \frac{(y+2)^2}{27} = 1$$

3. Consider the motion whose position vector is $\mathbf{r}(t) = (3t-t^3, 3t^2, 3t+t^3)$.
(a) For $t = 0$ write the acceleration $\mathbf{a}(t)$ as a linear combination of the unit tangent vector $T(t)$ and of the unit normal vector $N(t)$.
(b) Determine the curvature $k(t)$ for $t = 0$.

Solution. (a) $\mathbf{v}(t) = (3-3t^2, 6t, 3+3t^2)$ and $\mathbf{a}(t) = (-6t, 6, 6t)$. Therefore $\mathbf{v}(1) = (0,6,6)$, $v(1) = \| (0,6,6) \| = 6\sqrt{2}$, so $T(1) = \frac{1}{\sqrt{2}}(0,1,1)$ and $\mathbf{a}(1) = (-6,6,6)$. In general we know that

$$\mathbf{a}(t) = v'(t)T(t) + v(t) \| T'(t) \| N(t) \tag{1}$$

This is the required linear combination. To find it explicitly, the shortest way is to note that the two summands in the linear combination above are perpendicular. Therefore $v'(t)T(t)$ must be $pr_{T(t)}(\mathbf{a}(t))$, the projection of $\mathbf{a}(t)$ on $T(t)$, and $v(t) \| T'(t) \| N(t)$ must be $\mathbf{a}(t) - pr_{T(t)}(\mathbf{a}(t))$ and the formula (1) above is the orthogonal decomposition. In the case of $t = 1$ everything is very easy: the orthogonal decomposition of $(-6,6,6)$ with respect to $L((0,1,1))$ is clearly $(-6,6,6) = (0,6,6)+(-6,0,0) = 6\sqrt{2}(0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}) - 6(1,0,0)$.

Since in (1) the coefficient of $N(t)$ is positive, we have that $N(1) = (-1,0,0)$ and the required linear combination is

$$(-6,6,6) = 6\sqrt{2}(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) + 6(-1,0,0)$$

Note that, alternatively, one can compute $N(1)$ using its definition, that is $N(t) = T'(t)/ \parallel T'(t) \parallel$ and then computing the functions appearing in (1).
(b) We apply the formula $\frac{k(t)=\|\mathbf{v}(t)\times\mathbf{a}(t)\|}{v^3(t)}$. It comes out that $k(1) = \frac{1}{12}$.

4. Let $A = \begin{pmatrix} -1 & 4 \\ 1 & 2 \end{pmatrix}$. (a) Compute the eigenvalues and eigenspaces of A. (b) Compute A^{100} (hint: use (a)). Note: the solution to (b) can be written as a product of explicitly written matrices (without computing explicitly the product). Moreover, powers like a^b does not have to be explicitly computed.

Solution. (a) Eigenvalues: $\lambda_1 = 3$, $\lambda_2 = -2$. Eigenspaces $E(3) = L((1,1))$, $E(-2) = L((-4,1))$.
(b) We know that 3^{100} and $(-2)^{100}$ are eigenvalues of A^{100}, with the same eigenvectors. Therefore, letting $B = \begin{pmatrix} 1 & -4 \\ 1 & 1 \end{pmatrix}$, we have that $B^{-1}A^{100}B = diag(3^{100}, (-2)^{100})$. Hence

$$A^{100} = B\, diag(3^{100}, (-2)^{100})\, B^{-1}$$

The only thing that remains to compute is B^{-1}. We compute easily that $B^{-1} = \frac{1}{5}\begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$.

5. Let $A = \begin{pmatrix} -2 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -2 \end{pmatrix}$.
(a) Find an orthogonal matrix $U \in \mathcal{M}_{3,3}(\mathbf{R})$ and a diagonal matrix $D \in \mathcal{M}_{3,3}(\mathbf{R})$ such that $U^T A U = D$.
(b) Consider the real quadratic form $Q(x,y,z) = (\, x\; y\; z\,)\, A \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Find the maximum and minimum value of Q on the unit sphere $S^2 = \{(x,y,z) \in V_3 \mid \; \parallel (x,y,z) \parallel = 1\,\}$. Describe the points of maximum and the points of minimum.

Solution. We compute the eigenvalues of A. One finds $\lambda_1 = 3$, $\lambda_2 = \lambda_3 = -3$. The eigenspaces are $E(3) = L((1,2,1)$ and $E(-3) = L((-2,1,0),(-1,0,1))$. To construct the matrix U we need to find an orthonormal basis of V_3 composed of eigenvectors of A (note that A is symmetric). First we have to orthogonalize the basis of $E(-3)$. We let $u_2 = (-2,1,0)$ and $u_3 = (-1,0,1) - \frac{(-1,0,1)\cdot(-2,1,0)}{5}(-2,1,0) = (\frac{1}{5}, -\frac{2}{5}, 1)$. Then we get the perpendicular unit vectors $v_2 = \frac{1}{\sqrt{5}}(-2,1,0)$ and $v_3 = \frac{1}{\sqrt{30}}(1,-2,5)$. Finally, we take a unit vector generating the other eigenspace $v_1 = \frac{1}{\sqrt{6}}(1,2,1)$. We have that $\mathcal{B} = \{v_1, v_2, v_3\}$ is an orthonormal basis of V_3 composed of eigenvectors of A. Therefore we can take as U the matrix whose columns are v_1, v_2 and v_3, that is

$$U = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{5}{\sqrt{30}} \end{pmatrix}$$

We have that $U^{-1}AU = diag(3,-3,-3)$. Since U is an orthogonal matrix, $U^{-1} = U^T$. Hence $U^T A U = diag(3,-3,-3)$.
(b) I briefly review the theory: the diagonal form of Q is $Q(x',y',z') = 3{x'}^2 - 3{y'}^2 - 3{z'}^2$. where (x',y',z') is the vector of components of (x,y,z) with respect to the basis $\mathcal{B}$. Since te basis $\mathcal{B}$ is orthonormal, we have that $S^2 = \{(x,y,z) \in V_3 \mid \; \parallel (x',y',z') \parallel = 1\,\}$. Therefore it follows easily from the diagnal form that the maximum of Q on S^2 is the maximal eigenvalue, that is 3, and the minimum is the minimal eigenvalue, that is -3. The points of maximum are then the unit vectors such that $y' = z' = 0$, that is v_1 and $-v_1$. The

points of minimum are the unit vectors such that $x' = 0$, that is the vectors of the form $y'v_2 + z'v_3$ with ${y'}^2 + {z'}^2 = 1$.