First Name: Last Name:

Linear Algebra and Geometry, Midterm exam, 02.16.2011

NOTE: In the solution of a given exercise you must (briefly) explain the line of your argument. Solutions without adequate explanations will not be evaluated.

1. Let $W = L((1,0,-1,2),(0,1,1,0))$. Find an orthogonal basis $\{v_1, v_2, v_3, v_4\}$ of V_4 such that $L(v_1, v_2) = W$.

Solution. The shortest way is to find two orthogonal vectors $\{v_1, v_2\}$ such that $L(v_1, v_2) = W$ and two orthogonal vectors v_3, v_4 such that $L(v_3, v_4) = W^\perp$. At this point $\{v_1, v_2, v_3, v_4\}$ will be a basis a required, since v_3 and v_4 belong to $W^\perp$, and therefore they are orthogonal to both v_1 and v_2.

We set $v_1 = (1,0,-1,2)$. Then

$$v_2 = (0,1,1,0) - \frac{(1,0,-1,2)\cdot(0,1,1,0)}{\|(1,0,-1,2)\|^2}(1,0,-1,2) = (1/6, 1, 5/6, 1/3)$$

Next, let us compute a basis of $W^\perp : \begin{cases} (1,0,-1,2)\cdot(x,y,z,t) = 0 \\ (0,1,1,0)\cdot(x,y,z,t) = 0 \end{cases}$. Solving the system we get that $W^\perp = L((1,-1,1,0),(-2,0,0,1))$. Proceeding as above, we set $v_3 = (1,-1,1,0)$. Now

$$v_4 = ((-2,0,0,1) - \frac{(-2,0,0,1)\cdot(1,-1,1,0)}{3}(1,-1,1,0) = (-4/3, -2/3, 2/3, 1).$$

2. Let $v_1 = (1,1,0)$, $v_2 = (1,0,1)$, $v_3 = (-1,0,1)$, $u = (1,2,3)$. For a varying in $\mathbf{R}$, let $T_a : V_3 \to V_3$ be the linear transformation defined by: $T_a(v_1) = av_1 + 2v_2$, $T_a(v_2) = v_1 + av_2 + v_3$, $T_a(v_3) = 2v_2 + av_3$. Find the values of a such that $u \in T_a(V_3)$ and the values of a such that there is a unique $v \in V_3$ such that $T_a(v) = u$.

Solution. Let $\mathcal{B} = \{v_1, v_2, v_3\}$. We have that $m_{\mathcal{B}}^{\mathcal{B}}(T) = \begin{pmatrix} a & 1 & 0 \\ 2 & a & 2 \\ 0 & 1 & a \end{pmatrix}$. We have that the linear transformation T_a is bijective (that is, invertible) if and only if $\det m_{\mathcal{B}}^{\mathcal{B}}(T_a) \neq 0$. One computes easily that $\det m_{\mathcal{B}}^{\mathcal{B}}(T) = a(a^2-4) = a(a-2)(a+2)$. Therefore, if $a \neq 0, 2, -2$, $T_a(V_3) = V_3$ hence $u \in T_a(V_3)$. Moreover, since T_a is bijective, there is a unique v such that $T_a(v) = u$. The cases $a = 0, 2, -2$ need to be examined separately.

$a = 0$. In this case $T_0(v_1) = 2v_2$, $T_0(v_2) = v_1 + v_3$, $T_0(v_3) = 2v_2$. Hence, clearly, $T(V_3) = L(v_2, v_1 + v_3) = L((1,0,1),(0,1,1))$. We need to check if $u = (1,2,3)$ belongs to $L((1,0,1),(0,1,1))$, that is if there are scalars x and y such that $x(1,0,1) + y(0,1,1) = (1,2,3)$. Solving the corresponding system one discovers that the answer is affirmative. Hence $u \in T_0(V_3)$. However in this case the set of v such that $T(v) = u$ is infinite (why?).

$a = 2$. In this case $T(V_3) = L(2v_1 + 2v_2, v_1 + 2v_2 + v_3, 2v_2 + 2v_3)$. It is enough to take two generators (we know that the dimension is smaller than three, and, since the geneators are non-parallel, it has to be equal to two). For example, we can take $v_1 + v_2 = (2,1,1)$ and $v_2 + v_3 = (0,0,2)$. Again, we have to see if there are x and y such that $x(2,1,1) + y(0,0,2) = (1,2,3)$, which is impossible. hence $u \notin T_2(V_3)$.

$a = -2$. In this case $T(V_3) = L(-2v_1 + 2v_2, v_1 - 2v_2 + v_3, 2v_2 - 2v_3)$. Again, we can take two generators, for example $v_1 - v_2 = (0,1,-1)$ and $v_2 - v_3 = (2,0,0)$. Arguing as above, one sees that $u \notin T_{-2}(V_3)$.

Summarizing, there is $v \in V_3$ such that $T_a(v) = u$ if and only if $a \neq 2, -2$. Such v is not unique if and only if $a = 0$.

3. (a) Exhibit an example of a non-diagonal symmetric matrix in $M_{2,2}(\mathbf{R})$ whose eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -3$. (b) Exhibit an example of a non-diagonal non-symmetric matrix in $M_{2,2}(\mathbf{R})$ whose eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -3$. (c) Exhibit an example of non-block-diagonal symmetric matrix in $M_{3,3}(\mathbf{R})$ whose eigenvalues are $\lambda_1 = 4$, $\lambda_2 = -1$, $\lambda_3 = -3$.

Solution. (a) Perhaps the easy way is a s follows: let $A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$. We want $Tr(A) = a + d = 2 - 3 = 1$ and $\det(A) = ad - b^2 = 2(-3) = -6$. We can set, for example, $a = 1$. Then, from the first equation,

$b=-2$. Then, from the second equation, $b^2=-2+6=4$. Therefore we can take $b=2$. In conclusion $A=\begin{pmatrix}1 & 2\\ 2 & -2\end{pmatrix}$ is a symmetric matrix whose eingenvalues are 2 and -3.

(b) One can argue as above, with the difference that now we are looking for a matrix of the form $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ with $b\neq c$. In the same way, we find, for example, $\begin{pmatrix}1 & 1\\ 4 & -2\end{pmatrix}$.

(c) Here one could proceed as above, but the computations are a bit more complicated. Another way (we can be applied also to the previous question (a)) is as follows. We take an orthogonal basis $\mathcal{B}=\{v_1,v_2,v_3\}$, and the matrix whose columns are v_1,v_2,v_3, for example $C=\begin{pmatrix}1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 0 & -2\end{pmatrix}$. We look for a matrix A having v_1 as eigenvector of 4, v_2 as eigenvector of -1 and v_3 as eigenvector of -3. Such a matrix must be symmetric because the corresponding linear transformation T_A is symmetric (this is because $m_{\mathcal{B}'_{\mathcal{B}'}}(T_A)=diag(4,-1,-3)$ is a symmetric matrix, where $\mathcal{B}'$ is the orthogonal basis contained by $\mathcal{B}$ by dividing each vector by its norm). Since

$$diag(4,-1,-3)=C^{-1}AC$$

we will have

$$A=C\,diag(4,-1,-3)\,C^{-1}$$

(to find A explicitly one has to compute C^{-1} and the product).

4. Let $Q(x,y,z,t)=2(xz+yt)$. Reduce Q to diagonal form. Exhibit a point of maximum and a point of minimimum of Q on the unit sphere.

Solution. The associated matrix is $A=\begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}$. The characteristic polynomial is $P_A(\lambda)=$

$$\det\begin{pmatrix}\lambda & 0 & -1 & 0\\ 0 & \lambda & 0 & -1\\ -1 & 0 & \lambda & 0\\ 0 & -1 & 0 & \lambda\end{pmatrix}=$$

$$=\lambda\det\begin{pmatrix}\lambda & 0 & -1\\ 0 & \lambda & 0\\ -1 & 0 & \lambda\end{pmatrix}-\det\begin{pmatrix}0 & -1 & 0\\ \lambda & 0 & -1\\ -1 & 0 & \lambda\end{pmatrix}=\lambda^2(\lambda^2-1)+\lambda^2-1=(\lambda-1)^2(\lambda+1)^2.$$

Therefore there are the two double eigenvalues $\lambda_1=1$ and $\lambda_2=-1$. We have that $E(1)$ is the space of solutions of the homogeneous system $\begin{cases}x-z=0\\ y+t=0\end{cases}$, that is $L((1,0,1,0),(0,1,0,-1)=L(v_1,v_2)$. Similarly $E(-1)=L((1,0,-1,0),(0,1,0,1))=L(v_3,v_4)$. Note that v_1 and v_2 (resp. v_3 and v_4 are already orthogonal). Therefore one can take as orthonormal basis $\mathcal{B}=\{(1/\sqrt{2})v_1,(1/\sqrt{2})v_2,(1/\sqrt{2})v_3,(1/\sqrt{2})v_4\}$. Letting (x',y',z',t') the coodinates of (x,y,z,t) with respect to the basis $\mathcal{B}$, we have that

$$Q(x,y,z,t)=x'^2+y'^2-z'^2-t'^2.$$

A point of maximum of the form on the unit sphere is, for example, $(1/\sqrt{2})v_1$. A point of mininimum is, for example, $(1/\sqrt{2})v_3$.

5. In the real linear space $V=\mathcal{C}([-1,1])$ with inner product $<f,g>=\int_{-1}^{1}f(t)g(t)dt$, let $f(x)=e^x$. Let $W=\mathcal{P}^2$ be the linear subspace of V whose elements are polynomials of degree at most two. Find $p\in W$ and $g\in W^\perp$ such that $f=p+g$.

Solution. p must be the projection of f on W and $g=f-p$. To find p we have to find an orthogonal basis of W, say $\{p_0,p_1,p_2\}$. Then p will be the sum of the projections of f on the p_i's:

$$p=\frac{<f,p_0>}{<p_0,p_0>}p_0+\frac{<f,p_1>}{<p_1,p_1>}p_1+\frac{<f,p_2>}{<p_2,p_2>}p_2 \qquad (1)$$

A natural basis for W is $\{1, x, x^2\} = \{w_0, w_1.w_2\}$ but this is not orthogonal: it is easily checked that $< w_0, w_1 >= 0$ and $< w_1, w_2 >= 0$ but $< w_0, w_2 >= 2/3$ since $\int_{-1}^{1} t^2 dt = 2/3$. Therefore we have orthogonalize this basis according to Gram-Schmidt: we set

$$p_0 = w_0 = 1, \quad p_1 = w_1 = x, \quad \text{and} \quad p_2 = w_2 - \frac{< w_2, p_0 >}{< p_0, p_0 >} p_0 - \frac{< w_2, p_1 >}{< p_1, p_1 >} p_1 = w_2 - 1/3 = x^2 - 1/3.$$

Now let us compute: $< p_0, p_0 >= 2$
$< p_1, p_1 >= \int_{-1}^{1} t^2 dt = 2/3$.
$< p_2, p_2 >= \int_{-1}^{1} (t^2 - 1/3)^2 dt = 2/5 + 4/9 + 2/9 = 2/5 + 2/3 = \frac{16}{15}$.
$< f, p_0 >= \int_{-1}^{1} e^t dt = e - e^{-1}$.
$< f, p_1 >= \int_{-1}^{1} e^t t dt = ..byparts.. = 2e^{-1}$.
$< f, p_2 >= \int_{-1}^{1} e^t (t^2 - 1/3) dt = int_{-1}^{1} e^t t^2 dt - 1/3(e - e^{-1}) = ..byparts.. = e - e^{-1} - 4e^{-1}$.
Plugging everything in (1) you will get the answer.