First Name: Last Name:

Linear Algebra and Geometry, Midterm exam, 02.23.2011

NOTE: you must give reasons for all solutions and/or assertions. In practice, for each exercise you must (briefly) explain the steps of the reasoning. Final solutions, without adequate explanations, will not be evaluated.

1. Let $A = (1, 1, -1)$, $B = (-1, 1, 1)$ and $C = (1, 2, 1)$. Find two vectors D and E such that: $C = D + E$, D is orthogonal to both A and B, and E is a linear combination of A and B. Is there a unique solution?

Solution. We know that C can be written in a unique way as the sum of two vectors D and E such that D is parallel to $A \times B$ and E is orthogonal to $A \times B$ (hence E belongs to $L(A, B)$). These are the requested vectors D and E, and we know that they are the unique vectors having such properties. We compute: $A \times B = (2, 0, 2)$. To simplify the computation we can take $(1, 0, 1)$ rather than $(2, 0, 2)$.

$$D = \frac{(1, 2, 1) \cdot (1, 0, 1)}{2}(1, 0, 1) = (1, 0, 1), \qquad E = (1, 2, 1) - D = (0, 2, 0)$$

Note $E = (0, 2, 0) = A + B$. Now $C = D + E$.

2. Let $A = (1, 1, 1)$, $B = (1, -1, -1)$ and, for t varying in $\mathbf{R}$, $C(t) = (1, t, 2)$. Find the values of t such that one of the three vectors A, B, $C(t)$ can be expressed as a linear combination of the remaining two, and, for each such value of t, write explicitly one such expression.

Solution. A vector of A, B and $C(t)$ can be expressed as a linear cobination of the remaining two if and only if $\{A, B, C(t)\}$ is a set of linearly dependent vectors. This happens if and only if the triple product $A \cdot (B \times C(t))$ is zero. The triple product is the determinant $\det \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & t & 2 \end{pmatrix} = 2t - 4$, which is zero only for $t = 2$. In this case, solving the system $xA + yB + zC(2) = O$ one easily finds the solutions $(-3y, y, 2y)$, $y \in \mathbf{R}$. For $y = 1$ one gets $(-3, 1, 2)$ which means that $-3A + B + 2C(2) = O$. Hence $B = 3A - 2C(2)$ (check!).

3. Let us consider the two lines in $\mathcal{V}_2$: $L = \{(1, -1) + t(1, 2)\}$ and $S : x + y = -3$. How many are the points $P \in \mathcal{V}_2$ such that $\begin{cases} d(P, L) = \sqrt{5} \\ d(P, S) = \sqrt{2} \end{cases}$? Find explicitly two of them.

Solution. The points are four. In fact the locus of points P whose distance from L (respectively M) is a fixed number, say d, is the union of two lines, parallel to L (resp. parallel to M))(one in each half-plane). Hence the intersection of the two loci is four points. In our specific case, let $P = (x, y)$. We have that

$$d((x, y), L) = \frac{|(x - 1, y + 1) \cdot (2, -1)|}{\sqrt{5}} = \frac{|2x - y - 3|}{\sqrt{5}}$$

Moreover

$$d((x, y), S) = \frac{|(x, y) \cdot (1, 1) + 3|}{\sqrt{2}} = \frac{|x + y + 3|}{\sqrt{2}}$$

Hence we are looking for points (x, y) such that

$$\begin{cases} \frac{|2x-y-3|}{\sqrt{5}} = \sqrt{5} \\ \frac{|x+y+3|}{\sqrt{2}} = \sqrt{2} \end{cases}$$

A first case is, for example,

$$\begin{cases} 2x - y - 3 = 5 \\ x + y + 3 = 2 \end{cases}$$

hence $P = (7/3, -10/3)$. A second case is, for example,

$$\begin{cases} -2x + y + 3 = 5 \\ x + y + 3 = 2 \end{cases}$$

hence $P = (-1, 0)$.

4. In $\mathcal{V}_3$ let us consider the motion $\mathbf{r}(t) = (\cos t, \sin t, 2\cos t)$. Show that the path is an ellipse. Find the cartesian equation of the plane containing the ellipse. Find the points of intersection with the two symmetry axes.

Solution. We can write $\mathbf{R}(t) = \cos t(1,0,2) + sint(0,1,0) = \sqrt{5}\cos t((1,0,2)/\sqrt{5} + \sin t(0,1,0)$. Since $(1,0,2)/\sqrt{5}$ and $(0,1,0)$ are orthogonal unit vectors, this an ellipse in the plane $M = \{s(1,0,2)/\sqrt{5} + t(0,1,0)\}$, whose cartesian equation is $(x,y,z)\cdot(-2,0,1) = 0$, that is $-2x + z - 0$. The symmetry axes are $\{t(1,0,2)/\sqrt{5}\}$ and $\{t(0,1,0)\}$. The intersection points with the first axis are $\sqrt{5}(1,0,2)/\sqrt{5} = (1,0,2)$ and $-(1,0,2)$. The intersection points with the other axis are $(0,1,0)$ and $-(0,1,0)$.

5. In $\mathcal{V}_2$ a parametrized curve $\mathbf{r}(t) = (x(t), y(t))$ is such that $\mathbf{a}(t) = -9\mathbf{r}(t)$ for all t. A the time $t = 0$ the initial position is $\mathbf{r}(0) = (6,0)$ and the initial velocity is $\mathbf{v}(0) = (0,9)$.
(a) Determine the components $x(t)$ and $y(t)$ in terms of t. (b) Find a cartesian equation for the curve, Sketch the curve and indicate the direction of motion. (c) How much time is needed to return to the initial position?

Solution. (a) $x(t)$ and $y(t)$ are both solution of the differential equation $f'' = -9f(x)$. Therefore they are functions of the form $k\cos 3t + h\sin 3t$, where the constants $k, h \in \mathbf{R}$ must be determined by means of the initial conditions. For $x(t)$ we have the initial conditions $x(0) = 6$ and $x'(0) = 0$. Hence it follows easily that $x(t) = 6\cos 3t$. For $y(t)$ the initial conditions are $y(0) = 0$ and $y'(0) = 9$. This yields that $y(t) = 3\sin 3t$.
(b) The curve has parametric equation $(6\cos 3t, 3\sin 3t)$. Hence the curve is an ellipse, whose cartesian equation is $x^2/36 + y^2/9 = 1$. The motion is anti-clockwise.
(c) $2\pi/3$.