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Linear Algebra and Geometry, written test, 07.04.2011

NOTE: In the solution of a given exercise you must (briefly) explain the line of your argument. Solutions without adequate explanations will not be evaluated.

1. Let $V = L((1,1,-1,0),(1,1,0,1))$. Find a vector $v \in V$ and a vector w orthogonal to V such that $(1,0,0,0) = v + w$.

Solution. By the Orthogonal Decomposition Theorem, v (respectively w) must be the projection of $(1,0,0,0)$ on V (respectively on $V^{\perp}$). However we don't have to compute both of them, because, knowing v then $w = (1,0,0,0) - v$.

To compute $v = P_V((1,0,0,0)$ we first need to orthogonalize the given basis of V, We set $v_1 = (1,1,-1,0)$ and

$$v_2 = (1,1,0,1) - \frac{(1,1,0,1)\cdot(1,1,-1,0)}{\parallel (1,1,-1,0) \parallel^2} = (1,1,0,1) - \frac{2}{3}(1,1,-1,0) = (1/3,1/3,2/3,1)$$

To simplify the computation we can take as orthogonal basis of V $\{u_1,u_2\} = \{(1,1,-1,0),(1,1,2,3)\}$. Now

$$v = P_{L(u_1)}((1,0,0,0) + P_{L(u_2)}(1,0,0,0) = \frac{1}{3}(1,1,-1,0) + \frac{1}{15}(1,1,2,3) = \frac{1}{15}(6,6,-3,3)$$

Consequently

$$w = (1,0,0,0) - v = \frac{1}{15}(9,-6,3,-3)$$

2. Let $\mathcal{P}^2$ denote the linear space of all real polynomials of degree ≤ 2. Let moreover $p(t) = t^2 - t - 1$, $q(t) = 2t^2 - t$, $r(t) = t^2 - 2t$.

(a) Prove or disprove the following statement: $\{p(t), q(t), r(t)\}$ *is a basis of* $\mathcal{P}^2$.

(b) Let $f(t) = t + 1$. Is it possible to write $f(t)$ as a linear combination of $p(t)$, $q(t)$ and $r(t)$? If the answer is yes, find explicitly one such linear combination. Is it unique?

Solution. We know that $\dim \mathcal{P}^2 = 3$ ($\{1, t, t^2\}$ is a basis of $\mathcal{P}^2$). Therefore it is enough to see if $\{P(t), q(t), r(t)\}$ are linearly independent. Let $a, b, c \in \mathbf{R}$ such that $ap(t) + bq(t) + cr(t) = 0$ (right hand side is the zero polynomial $g(t) \equiv 0$). This means

$a(t^2 - t - 1) + b(2t^2 - t) + c(t^2 - 2t) = (a + 2b + c)t^2 + (-a - b - 2c)t - a = 0.$

Therefore we have the system of linear equations, in the unknowns a, b and c: $\begin{cases} a + 2b + c = 0 \\ -a - b - 2c = 0 \\ -a = 0 \end{cases}$ whose only solution is clearly $a = b = c = 0$. Therefore $\{p(t), q(t), r(t)$ are linearly independent, hence a basis of $\mathcal{P}^2$.

One could also think as follows: the function associating to a polynomial the triple (a, b, c) to the polynomial $at^2 + bt + c$ is clearly an isomorphism between $\mathcal{P}^2$ and V_3. Thinking in this way the question reduces to checking that $\{(1,-1,-1),(2,-1,0),(1,-2,0)\}$ is a basis of V_3.

(b) The answer is yes, and the linear combination is unique, since $\{p(t), q(t), r(t)\}$ is a basis of $\mathcal{P}^2$. Arguing as in (a) one finds that $t + 1 = -(t^2 - t - 1) + (2/3)(2t^2 - t) - (1/3)(t^2 - 2t)$.

3. Let us consider plane curves $\mathcal{C}$ with the following property: *for each point $P \in \mathcal{C}$, the normal line to $\mathcal{C}$ at P and a vertical line through P cut off a segment of length* 8 *on the x-axis.* Identify these curves and find their cartesian equations if they pass through the point $(1,4)$. Note: the normal line to $\mathcal{C}$ at P is, by definition, the line passing trough P perpendicular to the tangent line of $\mathcal{C}$ at P.

Solution. Let $(x(t),y(t))$ be a parametrization of our plane curve. The velocity vector is $(x'(t),y'(t))$. A vector perpendicular to the velocity vector is $A(t)=(y^prime(t),-x'(t)$. hence the normal line has parametric equation $L(t)=P(t)+sA(t)=(x(t),y(t))+s(y'(t),-x'(t))=(x(t)+sy'(t),y(t)-sx'(t))$, for $s\in\mathbf{R}$. the intersection of such line with the x-axis is obtained by solving $y(t)-sx'(t)=0$, hence $s=y(t)/x'(t)$. Therefore the intersection of $L(t)$ with the x-axis is $(x(t)+(y(t)/x'(t))y'(t),0)$. The intersection of a vertical line trough $(x(t),y(t))$ with the x-axis is obviously $(x(t),0)$. Hence the length of the segment cut off by the vertical line trough $(x(t),y(t)$ and the normal line trough $(x(t),y(t))$ is $|x(t)-(x(t)+(y(t)y'(t)/x'(t))|=|y(t)y'(t)/x'(t)|$. The condition on the curve is then

$$|\frac{y(t)y'(t)}{x'(t)}|\equiv\text{constant}=8$$

This splits in two cases:
(1) $x'(t)=y(t)y'(t)/8$. Integrating one gets $x^(t)=y^2(t)/16+C$, where C is a constant. Imposing the passage through $(1,4)$ one gets $C=0$. therefore we get the parabola of equation $x=y^2/16$.
(2) $x'(t)=-y(t)y'(t)/8$. As above, one gets $x^(t)=-y^2(t)/16+D$, where D is a constant. Imposing the passage trough $(1,4)$ one gets $D=2$. Therefore this curve is the parabola of equation $x^2=-(y^2/16)+2$.

In conclusion, there are exctly two curves satisfying the condition of the text. they are the two parabolas above.

4. Let $A=\begin{pmatrix}0&0&2&0\\0&0&0&-3\\2&0&0&0\\0&-3&0&0\end{pmatrix}$. (a) Find all eigenvalues of A.
(b) Are there a matrix U and a diagonal matrix D such that $U^TAU=D$? (note: if the answer is in the affirmative, it is not required to find explicitly the matrix U).

Solution. (a) The characteristic polynomial is

$$\det\begin{pmatrix}\lambda&0&-2&0\\0&\lambda&0&3\\-2&0&\lambda&0\\0&3&0&\lambda\end{pmatrix}=\lambda\det\begin{pmatrix}\lambda&0&3\\0&\lambda&0\\3&0&\lambda\end{pmatrix}-2\det\begin{pmatrix}0&-2&0\\\lambda&0&3\\3&0&\lambda\end{pmatrix}=\lambda^2(\lambda^2-9)-4(\lambda^2-9)=(\lambda^2-4)(\lambda^2-9)$$

Therefore the eigenvalues are $2,-2,3,-3$.
Such matrices U and D do exist because A is a symmetric matrix. In fact we can find an orthonormal basis of eigenvectors, corresponding respectively to the eigenvalues $2,-2,3,-3$.. The matrix U whose columns are the vectors of such basis is an orthogonal matrix, hence $U^T=U^{-1}$. Now, by base-change, we know that $U^{-1}AU=diag(2,-2,3,-3)$. Hence $U^TAU=diag(2,-2,3,-3)$.

5. Identify and make a sketch of the conic section $\mathcal{C}=\{(x,y)\in V_2\mid xy+y-2x-2=0\}$, specifying the coordinates of the center (if any), and the equations of the symmetry axes. All answers must be given using the usual (x,y)-coordinates.

Solution. The quadratic part is $Q(x,y)=xy$. the corresponding symmetric matrix is $A=\begin{pmatrix}0&1/2\\1/2&0\end{pmatrix}$. Its eigenvalues are $1/2,-1/2$. We have that $E(1/2)=L(\begin{pmatrix}1\\1\end{pmatrix})$ and $E(-1/2)=L(\begin{pmatrix}-1\\1\end{pmatrix})$. Therefore we can take as orthonormal basis $\mathcal{B} = \{v_1, v_2\} = \{\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}\}$. Let (x', y') be the components of (x, y) with respect to the basis $\mathcal{B}$. We have that $1/\sqrt{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$. Hence

$$\begin{cases} x = \frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}} \\ y = \frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}} \end{cases} \tag{1}$$

Therefore the equation of $\mathcal{C}$, which we donote $F(x, y) = 0$, becomes

$$F(x, y) = \frac{1}{2}x'^2 - \frac{1}{2}y'^2 + (\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}) - 2(\frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}}) - 2 = \frac{1}{2}x'^2 - \frac{1}{\sqrt{2}}x' - \frac{1}{2}y'^2 + \frac{3}{\sqrt{2}}y' - 2$$

Now we have to take care of the linear part, in order to write the equation as

$$\frac{1}{2}(x' - a)^2 - \frac{1}{2}(y' - b)^2 = C.$$

We complete the square as follows

$$F(x, y) = \frac{1}{2}(x'^2 - \sqrt{2}x') - \frac{1}{2}(y'^2 - 3\sqrt{2}y') - 2 = \frac{1}{2}\big((x' - \frac{1}{\sqrt{2}})^2 - \frac{1}{2}\big) - \frac{1}{2}\big((y' - \frac{3}{\sqrt{2}})^2 - \frac{9}{2}\big) - 2 =$$

$$= \frac{1}{2}(x' - \frac{1}{\sqrt{2}})^2 - \frac{1}{2}(y' - \frac{3}{\sqrt{2}})^2 = \frac{1}{2}\Big((x' - \frac{1}{\sqrt{2}}) - (y' - \frac{3}{\sqrt{2}})\Big)\Big((x' - \frac{1}{\sqrt{2}}) + (y' - \frac{3}{\sqrt{2}})\Big)$$

Therefore the equation of $\mathcal{C}$ can be written a

$$(x' - \frac{1}{\sqrt{2}})^2 - (y' - \frac{3}{\sqrt{2}})^2 = 0.$$

that is

$$\Big((x' - \frac{1}{\sqrt{2}}) - (y' - \frac{3}{\sqrt{2}})\Big)\Big((x' - \frac{1}{\sqrt{2}}) + (y' - \frac{3}{\sqrt{2}})\Big) = 0$$

It tunrs out that in fact it is not a hyperbola, but the union of two lines whose equations are

$$x' - \frac{1}{\sqrt{2}} = y' - \frac{3}{\sqrt{2}} \qquad \text{and} \qquad x' - \frac{1}{\sqrt{2}} = -(y' - \frac{3}{\sqrt{2}}).$$

They meet at the center $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} \end{pmatrix}$, that is, using (1),

$$C = (x, y) = (-1, 2).$$

In the (x, y)-coordinates the two lines are just $y = 2$ and $x = -1$.
Note that in fact $(x + 1)(y - 2) = xy + y - 2x - 2$.