SOME BASIC RESULTS ON IRREGULAR VARIETIES REVISITED WITH THE FOURIER-MUKAI TRANSFORM

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Abstract. Recently Fourier-Mukai methods have proved to be a valuable tool in the study of the geometry of irregular varieties. An especially interesting point is the interplay between the generic vanishing theorems of Green and Lazarsfeld and the notion of *generic vanishing sheaves*, naturally arising in the Fourier-Mukai context. The purpose of this mainly expository paper is to exemplify this circle of ideas by revisiting some basic results. In particular we focus on the birational characterization of abelian varieties due to Chen and Hacon. We also provide a treatment, along the same lines, of work of Ein and Lazarsfeld, where both the original and the present proof of the theorem of Chen-Hacon find their roots. We complete the exposition by revisiting results on theta-divisors. Two preliminary sections of background material are included.

In recent years the systematic use of the classical Fourier-Mukai transform between dual abelian varieties, and of related integral transforms, has proved to be a valuable tool for investigating the geometry of irregular varieties. An especially interesting point is the interplay between the generic vanishing theorems of Green and Lazarsfeld and the notion of *generic vanishing sheaves* (GV-sheaves for short) naturally arising in the Fourier-Mukai context. The purpose of this mainly expository paper is to exemplify this circle of ideas by revisiting some basic results.

To be precise, we focus on the theorem of Chen and Hacon characterizing (birationally) abelian varieties by means of the conditions $q(X) = \dim X$ and $h^0(K_X) = h^0(2K_X) = 1$ ([CH1], Theorem 4.2 below). We show that the Fourier-Mukai/Generic Vanishing package, in combination with Kollár's theorems on higher direct images of canonical bundles, produces a surprisingly quick and transparent proof of such a result. Along the way, we provide a unified Fourier-Mukai treatment of most of the results of the paper [EL] of Ein and Lazarsfeld, where both the original and the present proof of the theorem of Chen-Hacon find their roots [1]. We complete the exposition with Hacon's cohomological characterization of desingularizations of theta-divisors, as it fits well in the same framework.

Although many of the results treated here served as base for further developments (e.g. [CH3], [HP1], [J], [DH]), we have not attempted to recover those as well with the present approach. However, we hope that the point of view illustrated in this paper will be useful in the further study of irregular varieties with low invariants. In particular the main lemma used to prove Chen-Hacon's theorem, (Lemma 4.2 and Cor. 4.3) is new and can be useful in other situations. Moreover the present version of Hacon's characterization of desingularized theta-divisors (Theorem 5.1) improves slightly the ones appearing in the literature.

[1] however, our treatment of the results of Ein and Lazarsfeld differs from the original arguments only in some aspects

The paper is organized as follows: there are two preliminary sections where the background material is recalled and informally discussed at some length. The first one is about the on the Fourier-Mukai transform, related integral transforms, and GV-sheaves. The second one is on generic vanishing theorems, including Hacon's generic vanishing theorem for higher direct dimages of canonical sheaves, which is already one of the most relevant applications of the Fourier-Mukai methods in the present context. The last three sections are devoted respectively to the work of Ein-Lazarsfeld, to Chen-Hacon characterization of abelian varieties, and to Hacon's characterization of desingularizations of theta-divisors.

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1. Background material I: Fourier-Mukai transform, cohomological support loci, and GV-sheaves.

Unless otherwise stated, in this paper we will deal with smooth complex projective varieties (but, as it will be pointed out in the sequel, some results work more generally for complex Kähler manifolds and some others for smooth projective varieties on any algebraically closed field). By *sheaf* we will mean always coherent sheaf.

Given a smooth complex projective variety X, its *irregularity* is $q(X) = h^0(\Omega^1_X) = h^1(\mathcal{O}_X) = \frac{1}{2}b_1(X)$, and X is said to be *irregular* if $q(X) > 0$. Its *Albanese variety* $\mathrm{Alb}\, X := H^0(\Omega^1_X)^*/H_1(X,\mathbb{Z})$ is a $q(X)$-dimensional complex torus which, since X is assumed to be projective, is an abelian variety. The *Albanese morphism* $alb : X \to \mathrm{Alb}\, X$ is defined by making sense of $x \to (\omega \mapsto \int_{x_0}^{x} \omega)$, where x_0 is a fixed point of X. Note that that it is defined up to a translation in $\mathrm{Alb}\, X$. The Albanese morphism is a universal morphism of X to abelian varieties (or, more generally, complex tori). The *Albanese dimension* of X is the dimension of the image of its Albanese map. It is easily seen that the Albanese dimension of X is positive as soon as X is irregular (we refer to [U], §9 for a thorough treatment of the Albanese morphism). X is said *of maximal Albanese dimension* if $\dim alb(X) = \dim X$.

The dual abelian variety of the Albanese variety is $\mathrm{Pic}^0 X = H^1(\mathcal{O}_X)/H^1(X,\mathbb{Z})$. The exponential sequence shows that $\mathrm{Pic}^0 X$ parametrizes those line bundles on X whose first Chern class vanishes ([GrHa] p.313). Hence $\mathrm{Pic}^0 X$ is a (smooth and compact) space of deformations of the structure sheaf of X. So all sheaves $\mathcal{F}$ on X have a "common family of deformations": $\{\mathcal{F}\otimes\alpha\}_{\alpha\in\mathrm{Pic}^0 X}$. Since Riemann, a quite natural thing to do, rather than considering $H^*(X,\mathcal{F})$, the cohomology of $\mathcal{F}$, is to consider the full family $\{H^*(X,\mathcal{F}\otimes\alpha)\}_{\alpha\in\mathrm{Pic}^0 X}$. For example, a good part of the geometry of curves is captured by the Brill-Noether varieties $W^r_d(C) = \{\alpha \in \mathrm{Pic}^0 C \mid h^0(L\otimes\alpha) \geq r+1\}$, where L is a line bundle on C of degree d ([ACGH]). In fact, it is often convenient to do a related thing. Since $\mathrm{Pic}^0 X$ is a fine moduli space, i.e. $X \times \mathrm{Pic}^0 X$ carries a universal line bundle P, the Poincaré line bundle, one can consider the *integral transform* $\mathbf{R}q_*(p^*(\cdot)\otimes P) : \mathbf{D}(X) \to \mathbf{D}(\mathrm{Pic}^0 X)$, where p and q are respectively the projections on X and $\mathrm{Pic}^0 X$. Given a sheaf $\mathcal{F}$, the cohomology sheaves of $\mathbf{R}q_*(p^*(\mathcal{F})\otimes P)$ are isomorphic to $R^iq_*(p^*(\mathcal{F})\otimes P)$. They are naturally related to the family of cohomology groups $H^i(X,\mathcal{F}\otimes\alpha)_{\alpha\in\mathrm{Pic}^0 X}$ via base change (see e.g. 1.3 below).

The pullback map $alb^* : \mathrm{Pic}^0(\mathrm{Alb}\, X) \to \mathrm{Pic}^0 X$ is an isomorphism ([GrHa] p.332), and, via such identification, the Poincarè line bundle on $X \times \mathrm{Pic}^0 X$ is the pullback of the Poincarè line bundle on $\mathrm{Alb}\, X \times \mathrm{Pic}^0(\mathrm{Alb}\, X)$. Therefore the above transform should be thought as a tool for studying the part of the geometry of X coming from its Albanese morphism.

1.1. **Integral transform associated to the Poincarè line bundle, cohomological support loci, $\mathbf{GV_{-k}}$-sheaves.** In practice it is convenient to consider an integral transform as above for an arbitrary morphism from X to an abelian variety.

Definition 1.1 (Integral transforms associated to Poincarè line bundles)**.** Let X be a projective variety of dimension d, equipped with a morphism to a q-dimensional abelian variety

$$a : X \to A.$$

Let $\mathcal{P}$ (script) be a Poincaré line bundle on $A \times \mathrm{Pic}^0 A$. We will denote

$$P_a = (a \times \mathrm{id}_{\mathrm{Pic}^0 A})^*(\mathcal{P})$$

and p, q the two projections of $X \times \mathrm{Pic}^0 A$. Given a sheaf $\mathcal{F}$ on X, we define

$$\Phi_{P_a}(\mathcal{F}) = q_*(p^*(\mathcal{F}) \otimes P_a).$$

We consider the derived functor of the functor Φ_{P_a}:

$$\mathbf{R}\Phi_{P_a} : \mathbf{D}(X) \to \mathbf{D}(\mathrm{Pic}^0 A).$$

Sometimes we will have to consider the analogous derived functor $\mathbf{R}\Phi_{P_a^{-1}} : \mathbf{D}(X) \to \mathbf{D}(\mathrm{Pic}^0 A)$ as well. Since $\mathcal{P}^{-1} \cong (1_A \times (-1)_{\mathrm{Pic}^0 A})^* \mathcal{P}$, there is not much difference between the two:

$$\mathbf{R}\Phi_{P_a^{-1}} = (-1_{\mathrm{Pic}^0 A})^* \mathbf{R}\Phi_{P_a} \tag{1}$$

Finally, when the map a is the Albanese map of X, (denoted $alb : X \to \mathrm{Alb}\, X$), the map alb^* identifies $\mathrm{Pic}^0(\mathrm{Alb}X)$ to $\mathrm{Pic}^0 X$ and the line bundle P_{alb} is identified to the Poincaré line bundle on $X \times \mathrm{Pic}^0 X$. We will simply denote $P_{alb} := P$. When X is an abelian variety then its Albanese morphism is (up to translation) the identity. In this case the transform Φ_P is called the *Fourier-Mukai transform* (see §1.5 below).

In the sequel, we will adopt the following notation: given a line bundle $\alpha \in \mathrm{Pic}^0 A$, we will denote $[\alpha]$ the point of $\mathrm{Pic}^0 A$ parametrizing α (via the Poincaré line bundle $\mathcal{P}$). In other words $\alpha = \mathcal{P}_{|A \times [\alpha]}$.

Definition 1.2 (Cohomological support loci)**.** Given a coherent sheaf $\mathcal{F}$ on X, its *i-th cohomological support locus with respect to a* is

$$V^i_a(X, \mathcal{F}) = \{[\alpha] \in \mathrm{Pic}^0 A \mid h^i(X, \mathcal{F} \otimes a^*\alpha) > 0\}$$

In the special case when a is the Albanese map of X, we omit the reference to the map, writing

$$V^i(X, \mathcal{F}) = \{[\alpha] \in \mathrm{Pic}^0 X \mid h^i(X, \mathcal{F} \otimes \alpha) > 0\}.$$

As is customary for cohomology groups, when possible we will omit the variety X in the notation, writing simply $V^i_a(\mathcal{F})$ or $V^i(\mathcal{F})$ instead of $V^i_a(X, \mathcal{F})$ and $V^i(X, \mathcal{F})$.

Finally, we will adopt the following notation

$$R\Delta(\mathcal{F}) = \mathbf{R}\mathcal{H}om(\mathcal{F}, \omega_X)$$

1.3 ((Hyper)cohomology and base change)**.** We will use the following terminology: given a sheaf, or more generally, a complex of sheaves $\mathcal{G}$ on X, the sheaf $R^i\Phi_{P_a}(\mathcal{G})$ is said to *have the base change property at a given point* $[\alpha]$ *of* $\mathrm{Pic}^0 X$ if the natural map $R^i\Phi_{P_a}(\mathcal{G})\otimes\mathbb{C}([\alpha])\to H^i(X,\mathcal{G}\otimes a^*\alpha)$ is an isomorphism (here $\mathbb{C}([\alpha])$ denotes the residue field at the point $[\alpha]\in\mathrm{Pic}^0 X$). We will frequently use the following well-known base change result (applied to our setting): given a sheaf (or, more generally, a bounded complex) $\mathcal{G}$ on X, *if* $h^{i+1}(X,\mathcal{G}\otimes a^*\alpha)$ *is constant in a neighborhood of* $[\alpha]$ [2], *then both* $R^{i+1}\Phi_{P_a}(\mathcal{G})$ *and* $R^i\Phi_{P_a}(\mathcal{G})$ *have the base-change property in a neighborhood of* $[\alpha]$. When $\mathcal{G}$ is a sheaf this is well known, see e.g. in [M], Cor.2 p.52. For more general case of complexes see [EGA III] 7.7. It follows that, if $\mathcal{F}$ is a *sheaf*, then $R^i\Phi_{P_a}(\mathcal{F})$ and $R^i\Phi_{P_a}(R\Delta\mathcal{F})$ vanish for $i>\dim X$, and both $R^d\Phi_{P_a}(\mathcal{F})$ and $R^d\Phi_{P_a}(R\Delta\mathcal{F})$ have the base change property at all $[\alpha]\in\mathrm{Pic}^0 A$.

The following basic result compares two types of vanishing notions. The first one is *generic vanishing of higher cohomology groups* i.e., roughly speaking, that certain cohomological support loci $V^i(\mathcal{F})$ are *proper* closed subsets of $\mathrm{Pic}^0 A$. The second one is the vanishing of cohomology *sheaves* when applying the transform above to the derived dual of $\mathcal{F}$.

Theorem 1.4 ([PaPo3] Thm A, [PaPo4] Thm 2.2))**.** *Let $\mathcal{F}$ be a sheaf on X and let k be a non-negative integer. The following are equivalent*

(a) $\mathrm{codim}_{\mathrm{Pic}^0 A} V^i_a(\mathcal{F})\geq i-k$ *for all* $i\geq 0$ [3]*;*

(b) $R^i\Phi_{P_a}(R\Delta\mathcal{F})=0$ *for all* $i\notin[d-k,d]$.

An indication of proof will appear in the next subsection.

Definition 1.5 (GV_{-k}-sheaves)**.** When one of the two equivalent conditions of the above Theorem holds, the sheaf $\mathcal{F}$ is said to be a *GV_{-k}-sheaf with respect to the morphism a.* When possible, we will omit the reference to the morphism a and say simply *GV_{-k}-sheaf.*

1.2. **GV-sheaves.** We focus on the special case $k=0$ in Theorem 1.4. For sake of brevity, a GV_0-sheaf will be simply referred to as a GV-sheaf. Note that in this case it follows from condition (a) of Theorem 1.4 that, for *generic* $\alpha\in\mathrm{Pic}^0 A$, the cohomology groups $H^i(\mathcal{F}\otimes a^*\alpha)$ vanish for all $i>0$. The second equivalent condition of Theorem 1.4 says that, for a GV-sheaf $\mathcal{F}$, the full transform $\mathbf{R}\Phi_{P_a}(R\Delta(\mathcal{F}))$ is a sheaf concentrated in degree $d=\dim X$:

$$\mathbf{R}\Phi_{P_a}(R\Delta\mathcal{F})=R^d\Phi_{P_a}(R\Delta\mathcal{F})[-d]$$ [4]

Then one usually denotes

$$R^d\Phi_{P_a}(R\Delta\mathcal{F})=\widehat{R\Delta\mathcal{F}}.$$

The following proposition provides two basic properties of the sheaf $\widehat{R\Delta\mathcal{F}}$.

[2] by semicontinuity, this holds if $h^{i+1}(X,\mathcal{G}\otimes a^*\alpha)=0$

[3] if $V^1_a(\mathcal{F})$ is empty we declare that its codimension is ∞

[4] In a terminology frequently used in Fourier-Mukai theory this condition is expressed by saying that $R\Delta\mathcal{F}$ *satisfies the Weak Index Theorem (WIT) with index d.*

Proposition 1.6. *Let $\mathcal{F}$ be a GV-sheaf on X, with respect to a. Then*

(a) $rk\,\widehat{R\Delta\mathcal{F}} = \chi(\mathcal{F})$;

(b) $\mathcal{E}xt^i_{\mathcal{O}_{\mathrm{Pic}^0 A}}(\widehat{R\Delta\mathcal{F}}, \mathcal{O}_{\mathrm{Pic}^0 A}) \cong (-1_{\mathrm{Pic}^0 A})^* R^i\Phi_{P_a}(\mathcal{F})$.

Proof. (a) By Serre duality and base change, the rank of $\widehat{R\Delta\mathcal{F}}$ at a general point is the generic value of $h^0(\mathcal{F}\otimes a^*\alpha)$, which coincides with $\chi(\mathcal{F}\otimes a^*\alpha)$ (the higher cohomology vahishes for generic $\alpha \in \mathrm{Pic}^0 X$). Then (a) follows from the deformation invariance of the Euler characteristic.

(b) In the context of Definition 1.1, Grothedieck duality says that

$$\mathbf{R}\mathcal{H}om(R\Phi_{P_a}(\mathcal{F}), \mathcal{O}_{\mathrm{Pic}^0 A}) \cong \mathbf{R}\Phi_{P_a^{-1}}(R\Delta\mathcal{F})[d] \tag{2}$$

(see [PaPo3], Lemma 2.2). Therefore (b) of Theorem 1.4, combined with (1), yields

$$\mathbf{R}\mathcal{H}om(R\Phi_{P_a}(\mathcal{F}), \mathcal{O}_{\mathrm{Pic}^0 A}) \cong (-1_{\mathrm{Pic}^0 X})^* \widehat{R\Delta\mathcal{F}}$$

Since $\mathbf{R}\mathcal{H}om(\cdot, \mathcal{O}_{\mathrm{Pic}^0 X})$ is an involution on $\mathbf{D}(\mathrm{Pic}^0 X)$, we have also

$$\mathbf{R}\Phi_{P_a}(\mathcal{F}) \cong (-1_{\mathrm{Pic}^0 X})^* \mathbf{R}\mathcal{H}om(\widehat{R\Delta\mathcal{F}}, \mathcal{O}_{\mathrm{Pic}^0 X}) \tag{3}$$

which proves (b). □

Indication of proof of Theorem 1.4. The implication $(b) \Rightarrow (a)$ of Theorem 1.4 in the case $k = 0$ is proved as follows. To begin with, we recall that, by the Auslander-Buchsbaum-Serre formula, if $\mathcal{G}$ is a sheaf on $\mathrm{Pic}^0 A$, then the support of $\mathcal{E}xt^i(\mathcal{G}, \mathcal{O}_{\mathrm{Pic}^0 A})$ has codimension $\geq i$ in $\mathrm{Pic}^0 A$ (see e.g. [OScSp] Lemma II.1.1.2). Applying this to the sheaf $\widehat{R\Delta\mathcal{F}}$, from Proposition 1.6(b) (which is a consequence of hypothesis (b) of Theorem 1.4) we get that

$$\mathrm{codim}_{\mathrm{Pic}^0 A}\, Supp\, R^i\Phi_{P_a}(\mathcal{F}) \geq i \qquad \text{for all } i \geq 0. \tag{4}$$

To show that (4) is equivalent to (a) of Theorem 1.4, we argue by descending induction on i. For $i = d$ this is immediate since $R^d\Phi_{P_a}(\mathcal{F})$ has the base change property. Now suppose that $\mathrm{codim}\, V^{\bar{i}}_a(\mathcal{F}) < \bar{i}$ for a given $\bar{i} < d$, and let $[\alpha]$ be a general point of a component of $V^{\bar{i}}_a(\mathcal{F})$ achieving the dimension. Because of (4) it must be that $R^{\bar{i}}\Phi_{P_a}(\mathcal{F})$ does not have the base change property in the neighborhood of $[\alpha]$. Hence, by 1.3, such component has to be contained in $V^{\bar{i}+1}_a(\mathcal{F})$. Therefore $\mathrm{codim}\, V^{\bar{i}+1}_a(\mathcal{F}) < \bar{i}$, violating the inductive hypothesis. Hence $\mathrm{codim}\, V^i_a(\mathcal{F}) \geq i$ for all $i \geq 0$. This proves the implication $(b) \Rightarrow (a)$ for $k = 0$. For arbitrary k one uses the same argument, replacing Prop. 1.6(b) with the spectral sequence arising from (3).

The implication $(a) \Rightarrow (b)$ can be proved, with more effort, using the same ingredients (Grothendieck duality, Auslander-Buchsbaum-Serre formula and base change, see [PaPo4] Thm 2.2). □

A peculiar property of GV-sheaves is in the following

Lemma 1.7 ([H2] Cor.3.2). *Let $\mathcal{F}$ be a GV-sheaf on X, with respect to a. Then*

$$V^d_a(\mathcal{F}) \subseteq \cdots \subseteq V^1_a(\mathcal{F}) \subseteq V^0_a(\mathcal{F})$$

Proof. Let $i > 0$ and assume that $[\alpha] \in V^i_a(\mathcal{F}) = -V^{d-i}_a(R\Delta(\mathcal{F}))$ (the equality follows from Serre duality). Since $R^{d-i}\Phi_{P_a}(R\Delta(\mathcal{F})) = 0$, it follows by base change that $[\alpha] \in -V^{d-i+1}_a(R\Delta\mathcal{F}) = V^{i-1}_a(\mathcal{F})$. □

The usefulness of the concept of GV-sheaf stems from the fact that some features of the cohomology groups $H^i(\mathcal{F}\otimes a^*\alpha)$ and of the cohomological support loci $V^i_a(\mathcal{F})$ can be detected by local and sheaf-theoretic properties of the transform $\widehat{R\Delta\mathcal{F}}$. The following simple example will be repeatedly used in §3 and §4.

Lemma 1.8 ([PaPo3] Prop.3.15[5])**.** *Let $\mathcal{F}$ be a GV-sheaf on X, let W be an irreducible component of $V^0_a(\mathcal{F})$, and let $k = \mathrm{codim}_{\mathrm{Pic}^0 A} W$. Then W is also a component of $V^k_a(\mathcal{F})$. Therefore* $\dim X \geq k$. *In particular, if $[\alpha]$ is an isolated point of $V^0_a(\mathcal{F})$ then $[\alpha]$ is also an isolated point of $V^q_a(\mathcal{F})$* (here $q = \dim A$)*. Therefore* $\dim X \geq \dim A$.

Proof. Since $\widehat{R\Delta\mathcal{F}}$ has the base-change property, it is supported at $V^d_a(R\Delta\mathcal{F}) = -V^0_a(\mathcal{F})$. Hence $-W$ is a component of the support of $\widehat{R\Delta\mathcal{F}}$. Let $[\alpha]$ be a general point of W. Since $R^i\Phi_{P_a}(\mathcal{F}) = (-1_{\mathrm{Pic}^0 A})^*\mathcal{E}xt^i(\widehat{R\Delta\mathcal{F}}, \mathcal{O}_{\mathrm{Pic}^0 A})$, from well known properties of $\mathcal{E}xt$'s it follows that, in a suitable neighborhood in $\mathrm{Pic}^0 A$ of $[\alpha]$, $R^i\Phi_{P_a}(\mathcal{F})$ vanishes for $i < k$ and is supported at W for $i = k$. Therefore, by base change, W is contained in $V^k_a(\mathcal{F})$ (and in fact it is a component since, again by Theorem 1.4, $\mathrm{codim}\, V^k_a(\mathcal{F}) \geq k$). □

From the previous Lemma it follows that, if $\mathcal{F}$ is a GV-sheaf, then either $V^0_a(\mathcal{F}) = \mathrm{Pic}^0 A$ or there is a positive i such that $\mathrm{codim}\, V^i_a(\mathcal{F}) = i$, i.e. such that equality is achieved in (a) of Theorem 1.4. This can be rephrased as follows

Corollary 1.9. *(a) If $\mathcal{F}$ is a non-zero GV-sheaf then there is a $i \geq 0$ such that* $\mathrm{codim}\, V^i(\mathcal{F}) = i$.

(b) Let $\mathcal{F}$ be any sheaf on X. Then either $V^i_a(\mathcal{F}) = \emptyset$ for all $i \geq 0$ or there is a $i \geq 0$ such that $\mathrm{codim}\, V^i_a(\mathcal{F}) \leq i$.

Proof. (a) follows immediately from the previous Lemma. (b) If not all of the cohomological support loci of $\mathcal{F}$ are empty then either $\mathcal{F}$ is GV, in which case part (a) applies, or there is a i such that $\mathrm{codim}\, V^i_a(\mathcal{F}) < i$. □

Lemma 1.8 is a particular instance of a wider and more precise picture. In fact the condition that the cohomological support loci $V^0_a(\mathcal{F})$ is a *proper* subvariety of $\mathrm{Pic}^0 A$ is equivalent, by Serre duality and base change, to the fact that the generic rank of $\widehat{R\Delta\mathcal{F}}$ is zero, i.e. $\widehat{R\Delta\mathcal{F}}$ is a torsion sheaf on $\mathrm{Pic}^0 A$. Under this condition Lemma 1.8 says that there is a $i > 0$ achieving the bound of Theorem 1.4(a), i.e. such that $\mathrm{codim}\, V^i(\mathcal{F}) = i$. There is the following converse (which will be in use in the proof of Hacon's characterization of theta-divisors in §5). In what follows we will say that a sheaf *has torsion* if it is not torsion-free.

Theorem 1.10 ([PaPo4] Cor. 3.2, [PaPo2] Prop. 2.8)**.** *Let $\mathcal{F}$ be a GV-sheaf on X. The following are equivalent*

[5]In *loc cit* this result appears with a unnecessary hypothesis

(a) there is a $i > 0$ such that $\operatorname{codim} V^i_a(\omega_X) = i$;
(b) $\widehat{R\Delta\mathcal{F}}$ has torsion.

Proof. We start by recalling a general commutative algebra result. Let $\mathcal{G}$ be a sheaf on a smooth variety Y. Then $\mathcal{G}$ is torsion-free if and only if

$$\operatorname{codim}_Y Supp(\mathcal{E}xt^i_Y(\mathcal{G}, \mathcal{O}_Y)) > i \qquad \text{for all } i > 0 \tag{5}$$

(see e.g. [PaPo4] Prop. 6.4 or [PaPo2] Lemma 2.9). We apply this to the sheaf $\widehat{R\Delta\mathcal{F}}$ on the smooth variety $\operatorname{Pic}^0 X$. From (5) and Prop. 1.6(b) it follows that: $\widehat{R\Delta\mathcal{F}}$ has torsion if and only if there exists a $i > 0$ such that

$$\operatorname{codim}_{\operatorname{Pic}^0(X)} Supp(R^i\Phi_{P_a}\mathcal{F}) = i. \tag{6}$$

By base change (see 1.3) condition (6) implies that there exists a $i > 0$ such that $\operatorname{codim} V^i_a(\mathcal{F}) \leq i$. Hence, since $\mathcal{F}$ is GV, $\operatorname{codim} V^i_a(\mathcal{F}) = i$. Conversely, the same argument of the indication of proof of Theorem 1.4 proves that, if (a) holds then in any case there is a $j \geq i$ such that $\operatorname{codim} Supp(R^j\Phi_{P_a}\mathcal{F}) = j$. Therefore, by (5), $\widehat{R\Delta\mathcal{F}}$ has torsion. □

1.3. **Mukai's equivalence of derived categories of abelian varieties, and non-vanishing.** Assume that X coincides with the abelian variety A (and the map a is the identity). In this special case, according to Notation 1.1, the Poincaré line bundle on $A \times \operatorname{Pic}^0 A$ is denoted by $\mathcal{P}$. A well known theorem of Mukai asserts that $R\Phi_{\mathcal{P}}$ is an equivalence of categories. More precisely, denoting $\mathbf{R}\Psi_{\mathcal{P}} : \mathbf{D}(\operatorname{Pic}^0 A) \to \mathbf{D}(A)$ the "converse" functor $\mathbf{Rp}_*(q^*(\cdot) \otimes \mathcal{P})$, and $q = \dim A$:

Theorem 1.11 ([Muk, Thm. 2.2])**.** *Let A be an abelian variety (over any algebraically closed field k). Then*

$$\mathbf{R}\Psi_{\mathcal{P}} \circ \mathbf{R}\Phi_{\mathcal{P}} = (-1_A)^*[-q], \qquad \mathbf{R}\Phi_{\mathcal{P}} \circ \mathbf{R}\Psi_{\mathcal{P}} = (-1_{\operatorname{Pic}^0 A})^*[-q].$$

Mukai's theorem can be used to provide non-vanishing criteria for spaces of global sections. Here are some immediate ones:

Lemma 1.12 (Non-vanishing)**.** *Let $\mathcal{F}$ be a non-zero sheaf on an abelian variety X.*

(a) If $\mathcal{F}$ is a GV-sheaf then $V^0(\mathcal{F})$ is non-empty.

(b) If $\operatorname{codim} V^i(\mathcal{F}) > i$ for all $i > 0$ is then $V^0(\mathcal{F}) = \operatorname{Pic}^0 X$.

Proof. (a) By base change, $\widehat{R\Delta\mathcal{F}} = R^d\Phi_{\mathcal{P}}R\Delta\mathcal{F}$ is supported at $-V^0(\mathcal{F})$. Therefore if $V^0(\mathcal{F}) = \emptyset$ then $R^d\Phi_{\mathcal{P}}R\Delta\mathcal{F}$ is zero, i.e., by Theorem 1.4, $\mathbf{R}\Phi_{\mathcal{P}}(R\Delta\mathcal{F})$ is zero. Then, by Mukai's theorem, $R\Delta\mathcal{F}$ is zero. Therefore $\mathcal{F}$ itself is zero, since $R\Delta$ is an involution on the derived category.

(b) If $V^0(\mathcal{F})$ is a *proper* subvariety of $\operatorname{Pic}^0 X$ then, by Lemma 1.8, there is at least a $i > 0$ such that $\operatorname{codim} V^i(\mathcal{F}) = i$. □

In the context of irregular varieties, Mukai's theorem is frequently used via the following relation, whose proof is an exercise

Proposition 1.13. *In the Notation of Terminology 1.1*

$$\mathbf{R}\Phi_{P_a} \cong \mathbf{R}\Phi_{\mathcal{P}} \circ \mathbf{R}a_*$$

Going back to Mukai's Theorem 1.11, the key point of its proof is the verification of the statement for the one-dimensional skyscraper sheaf at the identity point, namely that $\mathbf{R}\Phi_{\mathcal{P}}(\mathbf{R}\Psi_{\mathcal{P}}(k(\hat{0}))) = k(\hat{0})[-q]$. Since $\mathbf{R}\Psi_{\mathcal{P}}(k(\hat{0})) = \mathcal{O}_A$, this amounts to prove that $\mathbf{R}\Phi_{\mathcal{P}}(\mathcal{O}_A) = k(\hat{0})[-q]$, i.e.

$$R^i\Phi_{\mathcal{P}}(\mathcal{O}_A) = 0 \quad \text{for } i < q \qquad \text{and} \qquad R^q\Phi_{\mathcal{P}}(\mathcal{O}_A) = k(\hat{0}) \tag{7}$$

Since $V^i(\mathcal{O}_A) = \{\hat{0}\}$ for all i such that $0 \le i \le q$, the first part follows from easily from Theorem 1.4. Concerning the second part of (7), it does not follow from base change, and has to be proved with a different argument. Over the complex numbers this can be done easily using the explicit description of the Poincaré line bundle on an abelian variety ([Ke], Th.3.15 or [BiLa], Cor. 14.1.6). In arbitrary characteristic it is proved in [M], p.128. Another proof can be found in [Hu], p.202.

Let now X be an irregular variety of dimension d. The next proposition is a generalization of the second part of (7) to any smooth variety and is proved via an argument similar to Mumford's

Proposition 1.14 ([BLNPa] Prop. 6.1)**.** *Let X be a smooth projective variety (over any algebraically closed field k), equipped with a morphism to an abelian variety $a : X \to A$ such that the pullback map $a^* : \operatorname{Pic}^0 A \to \operatorname{Pic}^0 X$ is an embedding. Then*

$$R^d\Phi_{P_a}(\omega_X) \cong k(\hat{0})$$

Notes 1.15. (1) All results in this section work in any characteristic. They work also for compact Kähler manifolds as well.

(2) The implication $(b) \Rightarrow (a)$ of Theorem 1.4 and also Lemma 1.7 were already observed by Hacon ([H2] Th.1.2, Cor. 3.2). While the converse implication of Theorem 1.4 makes the picture conceptually more clear – and is also useful in various applications as Proposition 2.4 below – the careful reader will note that in the proof of the theorem of Chen-Hacon we are only using the implication $(b) \Rightarrow (a)$.

(3) Theorem 1.10 is in fact a particular case of a much more general statement: *the sheaf $\widehat{R\Delta\mathcal{F}}$ is not a k-syzygy sheaf if and only if there is a $i > 0$ such that* $\operatorname{codim} V^i(\mathcal{F}) = i + k$. An example of an application of this circle of ideas is the higher dimensional Castelnuovo - de Franchis inequality ([PaPo4]).

(4) The result mentioned in the previous note, together with Theorems 1.4 and 1.10, hold in a even more general setting concerning not only sheaves, but objects in the bounded derived category of X. Moreover, X doesn't need to be smooth, but only Cohen-Macaulay. Finally, all of this is not specific to transforms whose kernel is a line bundle but it works for all *integral transforms* between two given Cohen-Macaulay projective varieties X and Y. A thorough analysis of the implications at the derived category level of such results is carried out in [Po].

(5) The hypothesis of Lemma 1.12(b), also called *M-regularity*, has many applications to global generation properties. In fact more than the thesis of Lemma 1.12(b) holds: not only $V_0(\mathcal{F}) = \operatorname{Pic}^0 X$, but $\mathcal{F}$ is also *continuously globally generated.* A survey on M-regularity and its applications is [PaPo1]. A more recent development where the concept of M-regularity is relevant is the result of [BLNPa] on the bicanonical map of irregular varieties.

2. Background material II: generic vanishing theorems for the canonical sheaf and its higher direct images

2.1. Kollár's theorems on higher direct images of canonical sheaves.

The following theorems of Kollár's will be of ubiquitous use in what follows

Theorem 2.1 ([Ko1] Th.2.1, [Ko2], Th. 3.1). *Let X and Y be complex projective varieties of dimension d and $d-k$, with X smooth, and let $f : X \to Y$ a surjective map. Then*
(a) $R^i f_* \omega_X$ is torsion-free for all $i \geq 0$;
(b) $R^i f_* \omega_X = 0$ if $i > k$;
(c) Let L be an ample line bundle on Y. Then

$$H^j(L \otimes R^i f_* \omega_X) = 0 \qquad \textit{for all} \quad i \geq 0 \quad \textit{and} \quad j > 0;$$

(d) in the derived category of Y

$$\mathbf{R} f_* \omega_X \cong \bigoplus_{i=0}^{k} R^i f_* \omega_X[-i]$$

In the next section, Theorem 2.1 is used in the following variant due noted by Hacon and Pardini

Variant 2.2 ([HP2] Thm 2.1). *In the hypothesis and notation of Theorems 2.1, ω_X can be replaced by $\omega_X \otimes \beta$, where $[\beta]$ is a torsion point of* $\mathrm{Pic}^0 X$.

2.2. Generic vanishing theorems of Green-Lazarsfeld and Hacon.

According to the previous terminology, a *generic vanishing theorem* is the statement that a certain sheaf is a GV_{-k}-sheaf. Within such terminology, the Green-Lazarsfeld's generic vanishing theorem (([GL1]), arisen independently of the theory of Fourier-Mukai transforms, can be stated as follows

Theorem 2.3 (Green-Lazarsfeld, [GL1] Thm 1, [EL] Rem. 1.6). *Let $a : X \to A$ be a morphism from X to an abelian variety A, and let $k = \dim X - \dim a(X)$. Then ω_X is a GV_{-k}-sheaf (with respect to a). In particular, if a is generically finite onto its image then ω_X is a GV-sheaf.*

In fact Theorem 2.3 is sharp, as shown by the next Proposition.

Proposition 2.4 ([BLNPa] Prop. 2.7 (a similar result appears in [LPo], Prop.1.5)). *In the hypothesis and notation of Theorem 2.3 ω_X is a GV_{-k}-sheaf and it is not a $GV_{-(k-1)}$-sheaf.*

Proof. By Green-Lazarsfeld's generic vanishing theorem, ω_X is a GV_{-k}-sheaf. The last assertion means that there is a $j \geq 0$ (in fact $j \geq k$) such that

$$\operatorname{codim} V^j(\omega_X) = j - k. \tag{8}$$

By Theorem 2.1(b) and (d), $H^d(\omega_X) = \oplus_{i=0}^{k} H^{d-i}(R^i a_* \omega_X)$. Since $H^d(\omega_X) \neq 0$, it follows that $H^{d-k}(R^k a_* \omega_X) \neq 0$. Hence, by Corollary 1.9, there is a $\bar{i} \geq 0$ such that

$$\operatorname{codim} V^{\bar{i}}(R^k a_* \omega_X) \leq \bar{i}. \tag{9}$$

Again by Theorem 2.1(b),(d), and projection formula

$$H^i(X,\omega_X\otimes a^*\alpha)=\bigoplus_{h=0}^{min\{i,k\}} H^{i-h}(A,R^ha_*\omega_X\otimes\alpha) \tag{10}$$

Therefore $V_a^{\bar{i}+k}(X,\omega_X)\supseteq V^{\bar{i}}(A,R^ka_*\omega_X)$. Hence (9) yields that $\operatorname{codim} V_a^{\bar{i}+k}(\omega_X)\le\bar{i}=(\bar{i}+k)-k$. In fact equality holds, since ω_X is a GV_{-k}-sheaf. Therefore (8) is proved. □

In the argument of [GL1] the GV_{-k} condition is verified by proving condition (a) of Theorem 1.4, i.e. the bound on the codimension of the cohomological support loci $V_a^i(\omega_X)$. This is achieved via an infinitesimal argument, based on Hodge theory. In fact, the theorem of Green-Lazarsfeld holds, more generally, in the realm of compact Kähler varieties. Using the theory of Fourier-Mukai transforms, Hacon extended Theorem 2.3 to higher direct images of dualizing sheaves (in the case of projective varieties). Hacon's result can be stated in several slightly different variants. A simple one, which is enough for the application of the present paper, is the following

Theorem 2.5 (Hacon, [H2], Cor. 4.2)**.** *Let X be a smooth projective variety and let $a:X\to A$ be a morphism to an abelian variety. Then $R^ia_*\omega_X$ is a GV-sheaf on A for all $i\ge 0$.*

In fact, the Theorem of Green-Lazarsfeld follows from Hacon's via Kollár's Theorems:

Proof that Theorem 2.5 ⇒ Theorem 2.3. It follows from (10) that

$$V_a^i(X,\omega_X)=\bigcup_{h=0}^{\min\{i,k\}} V^{i-h}(A,R^ha_*\omega_X). \tag{11}$$

By Theorem 2.5, $\operatorname{codim} V^{i-h}(R^ha_*\omega_X)\ge i-h\ge i-k$. □

In the sequel, Hacon's generic vanishing theorem will also used in the following variant, noted by Hacon and Pardini

Variant 2.6 ([HP2] Thm 2.2)**.** *Theorem 2.5 (hence also Theorem 2.3) still holds if ω_X is replaced by $\omega_X\otimes\beta$, where $[\beta]$ is a torsion point of* $\operatorname{Pic}^0 X$.

Let us describe briefly the proof of Theorem 2.5, which is completely different from the arguments of Green and Lazarsfeld. Hacon's approach consists in reducing a generic vanishing theorem to a Kodaira-Nakano-type vanishing theorem. Interestingly enough this is done by verifying directly condition (b) of Theorem 1.4, rather than the dimensional bound for the cohomological support loci, i.e. condition (a). The argument is as follows. Let L be an ample line bundle on $\operatorname{Pic}^0 A$ and consider, for a positive n, the locally free sheaf on A obtained as the "converse" Fourier-Mukai transform of L^n:

$$\mathbf{R}\Psi_{\mathcal{P}}L^n=R^0\Psi_{\mathcal{P}}L^n$$

(see the notation at the beginning of Subsection 1.3), where the above equality follows from the vanishing of the higher cohomology of L^n, twisted by all numerically trivial line bundles. With an argument similar to the proof of Grauert-Riemenschneider vanishing, one proves that, given a sheaf $\mathcal{F}$ on A, the condition (b) of Theorem 1.4 (namely $R^i\Phi_{\mathcal{P}}(R\Delta\mathcal{F}) = 0$ for $0 \neq \dim X$) is equivalent to the fact that there exist an n_0 such that, for all $n \geq n_0$,

$$H^i(A, (R\Delta\mathcal{F}) \otimes R^0\Psi_{\mathcal{P}}L^n) = 0 \qquad \text{for all } \ i < q \qquad (\text{where } q = \dim A)$$

By Serre duality, this is equivalent to

$$H^i(\mathcal{F} \otimes (R^0\Psi_{\mathcal{P}}L^n)^*) = 0 \qquad \text{for all } \ i > 0 \tag{12}$$

On the other hand, it is well known that, up to an isogeny, $R^0\Psi_{\mathcal{P}}L^n$ is the direct sum of copies of negative line bundles. More precisely, let $\phi_{L^n} : \operatorname{Pic}^0 A \to A$ be the *polarization* associated to L^n. Then (see e.g. [Muk], Prop. 3.11(1))

$$\phi^*_{L^n}(R^0\Psi_{\mathcal{P}}L^n) \cong H^0(\operatorname{Pic}^0 A, L^n) \otimes L^{-n} \tag{13}$$

Therefore, putting together (12) and (13) it turns out that, to prove that $\mathcal{F}$ is a GV-sheaf, it is enough to prove that, for n big enough

$$H^i(\operatorname{Pic}^0 A, \phi^*_{L^n}(\mathcal{F}) \otimes L^n) = 0 \qquad \text{for all} \qquad i > 0. \tag{14}$$

Condition (14) is certainly satisfied by the sheaves $\mathcal{F}$ enjoying the following property: *for each isogeny $\pi : B \to A$, and for each ample line bundle N on B, $H^i(B, \pi^*\mathcal{F} \otimes N) = 0$ for all $i > 0$.* Such property is satisfied by a higher direct image of a canonical sheaf since, via étale base extension, its pullback via an étale cover is still a higher direct image of a canonical sheaf, so that Kollár' Theorem 2.1(c) applies.

Notes 2.7. In [PaPo3] it is shown that Hacon's approach works in greater generality, and does not need the ambient variety to be an abelian variety. This yields other "generic vanishing theorems". For example, Green-Lazarsfeld's Theorem 2.3 works also for line bundles of the form $\omega_X \otimes L$, with L nef (see *loc cit* Cor.5.2 and Thm. 5.8 for a better statement). In *loc cit* (Thm A) is also shown that a part of Hacon's approach works in the setting of arbitrary integral transforms.

2.3. The subtorus theorems of Green, Lazarsfeld and Simpson.

The geometry of the loci $V^i_a(\omega_X)$ is described by the Green-Lazarsfeld's subtorus theorem, with an important addition due to Simpson.

Theorem 2.8 ([GL2] Thm 0.1, [EL], Proof of Theorem 3, p.249, [S])**.** *Let X be a compact Kähler manifold, and W a component of $V^i(\omega_X)$ for some i. Then*

(a) There exists a torsion point $[\beta]$ and a subtorus B of $\operatorname{Pic}^0 X$ such that $W = [\beta] + B$.

(b) Let $g := \pi \circ alb : X \to \operatorname{Pic}^0 B$, where $\pi : \operatorname{Alb} X \to \operatorname{Pic}^0 B$ is the dual map of the embedding $B \hookrightarrow \operatorname{Pic}^0 X$. Then $\dim g(X) \leq \dim X - i$.

Simpson's result (conjectured by Beauville and Catanese) is that $[\beta]$ is a torsion point. Actually in [S] there are also, among other things, other different proofs of part (a) of the above theorem. It is worth to mention that, admitting part (a), the dimensional bound (b) is a direct consequence of the Generic Vanishing Theorem:

Proof of (b). By (a) of Theorem 2.8 it follows that $V^i_g(\omega_X \otimes \beta) = \operatorname{Pic}^0(\operatorname{Pic}^0 B) = B$. Therefore, by definition, $\omega_X \otimes \beta$ is a GV_{-h}-sheaf with $h \geq i$, with respect to g. Hence, by Variant 2.6, $\dim X - \dim g(X) \geq i$. □

Notes 2.9. (a) One defines, more generally, the loci $V^i_m(\mathcal{F}) = \{[\alpha] \in \mathrm{Pic}^0 A \mid h^i(X, \mathcal{F} \otimes \alpha) \geq m\}$ and the Green-Lazarsfeld-Simpson Theorem 2.8 holds more generally for these loci as well. As noted by Hacon-Pardini ([HP2] Thm 2.2.(b)), this implies that part (a) of Theorem 2.8 holds replacing ω_X with higher direct images of $R^i f_* \omega_X$, where f is a morphism $f : X \to Y$, where Y is a smooth irregular variety (for example, an abelian variety).

(b) An explicit description and classification of all possible (positive-dimensional) components of the loci $V^i(\omega_X)$ is known only for $i = \dim X - 1$, by the work of Beauville ([Be] Cor.2.3). Note that in this case, by Theorem 2.3(b), the image $g(X)$ is a curve.

3. Some results of Ein and Lazarsfeld

The content of this section is composed by four basic results of L. Ein and R. Lazarsfeld. We provide their proofs both for sake of self-containedness, and also because they are good examples of application of the general principles of the previous section. The first two of them will be basic steps in the proof of Chen-Hacon's theorem appearing in the next section (they appear also in the original argument), while the last two will be used in the characterization of desingularizations of theta-divisors (§5).

A theorem of Kawamata ([K]), conjectured by Ueno ([U]), asserts that the Albanese map of a complex projective variety X of Kodaira dimension is zero is: (a) surjective and (b) with connected fibres. As a consequence, one has Kawamata's characterization of abelian varieties as varieties of Kodaira dimension zero such that $q(X) = \dim X$. Subsequently Kollár addressed the problem of giving effective versions of such results, replacing the hypothesis on the Kodaira dimension with the knowledge of finitely many plurigenera ([Ko1],[Ko3],[Ko4]). In fact, Kollár proved that both part (a) of Kawamata's theorem and the characterization of abelian varieties held under the weaker assumption that $p_m(X) := h^0(\omega_X^m) = 1$ for some $m \geq 3$, and conjectured that $m = 2$ would suffice. Concerning part (a) of Kawamata's theorem, this was settled by Ein and Lazarsfeld in the result below (Theorem 3.1(b)), while the characterization of abelian varieties is the content of the theorem of Chen-Hacon (see the next section)[6].

Theorem 3.1 ([EL] Thm 4)**.** *Let X be a smooth projective variety such that $p_1(X) = p_2(X) = 1$.*

(a) There is no positive-dimensional subvariety Z of $\mathrm{Pic}^0 X$ *such that both Z and $-Z$ are contained in $V^0(\omega_X)$;*

(b) the Albanese map of X is surjective.

Proof. (a) (This is as in [Ko3], Th. 17.10, the proof is included here for sake of self-containedness). Assume that there is a positive-dimensional subvariety Z of $\mathrm{Pic}^0 X$ as in the statement. The images of the multiplication maps of global sections

$$H^0(\omega_X \otimes \gamma) \otimes H^0(\omega_X \otimes \gamma^{-1}) \to H^0(\omega_X^2)$$

are non-zero for all $\gamma \in T$. Therefore $h^0(\omega_X^2) > 1$, since otherwise the only effective divisor in $|\omega_X^2|$ would have infinitely many components.

[6]Concerning an effective version of part (b) of Kawamata's theorem, recently Zhi Jiang has proved that if $p_1(X) = p_2(X) = 1$ then the Albanese morphism has connected fibers ([J] Th.1.3). The proof uses also the theorem of Chen-Hacon.

(b) By hypothesis, the identity point $\hat{0}$ of $\mathrm{Pic}^0 X$ belongs to $V^0(\omega_X)$. It must be an isolated point, since otherwise, by Theorem 2.8, a positive-dimensional component Z of $V^0(\omega_X)$ containing $\hat{0}$ would be a subtorus, thus contradicting (a). Therefore $\hat{0}$ is also an isolated point of $V^0(\mathrm{Alb}\, X, alb_*\omega_X)$. Since $alb_*\omega_X$ is a GV-sheaf on $\mathrm{Alb}\, X$ (Theorem 2.5), $\hat{0}$ is also a isolated point of $V^{q(X)}(\mathrm{Alb}\, X, alb_*\omega_X)$ (Lemma 1.8). In particular $H^{q(X)}(\mathrm{Alb}\, X, alb_*\omega_X)$ is non-zero. Therefore alb, the Albanese map of X, is surjective. □

Note that the present argument is only partially different from the original one. The difference is in the proof that if $V^0(\omega_X)$ has some isolated points then the Albanese map of X is surjective. In [EL] this is proved via infinitesimal considerations connected with the Green-Lazarsfeld proof of the generic vanishing theorem, while here it follows form the general Lemma 1.8 about GV-sheaves $\mathcal{F}$ such that $V^0(\mathcal{F})$ is a proper subvariety. The present proof of part (b) is similar to the one provided in [PaPo3] Cor. 6.6.

Next, we provide a different proof of the following characterization of abelian varieties. The same type of argument will be applied to the characterization of theta-divisors of §5.

Theorem 3.2 (Ein-Lazarsfeld, [CH1] Thm 1.8)**.** *Let X be a smooth projective variety of maximal Albanese dimension such that* $\dim V^0(\omega_X) = 0$*. Then X is birational to an abelian variety.*

Proof. By Green-Lazarsfeld (Theorem 2.3) ω_X is a GV-sheaf (with respect to the Albanese morphism). By Lemma 1.8, the hypothesis yields that $\dim X = q(X)$ and $V^0(\omega_X) = \{\hat{0}\}$. Using Proposition 1.14, $\mathbb{C}(\hat{0}) = R^q\Phi_P(\omega_X)$. Therefore, by Lemma 1.6, $\mathcal{E}xt^q(\widehat{\mathcal{O}_X}, \mathcal{O}_{\mathrm{Pic}^0 X}) = \mathbb{C}(\hat{0})$. Moreover, since $\widehat{\mathcal{O}_X}$ is supported at a finite set, $\mathcal{E}xt^i(\widehat{\mathcal{O}_X}, \mathcal{O}_{\mathrm{Pic}^0 X}) = 0$ for $i < q$. Summarizing: $R\Delta\widehat{\mathcal{O}_X} = \mathbb{C}(\hat{0})[-q]$. Since the functor $R\Delta$ is an involution, $\widehat{\mathcal{O}_X} = \mathbb{C}(\hat{0})$. In conclusion

$$\mathbf{R}\Phi_P(\mathcal{O}_X) = \mathbb{C}(\hat{0})[-q].$$

By Proposition 1.13 this means that $\mathbf{R}\Phi_{\mathcal{P}}(\mathbf{R}alb_*\mathcal{O}_X) = \mathbb{C}(\hat{0})[-q]$. Then, by Mukai's inversion theorem 1.11, $\mathbf{R}alb_*\mathcal{O}_X = \mathcal{O}_{\mathrm{Alb}\, X}$. In particular $alb_*\mathcal{O}_X = \mathcal{O}_{\mathrm{Alb}\, X}$. Since alb is assumed to be generically finite, this means that it is birational onto $\mathrm{Alb}\, X$. □

The next result concerns varieties of maximal Albanese dimension and $\chi(\omega_X) = 0$. Similarly to Theorem 3.1, here the proof is only partially different from the original argument of [EL].

Theorem 3.3 ([EL] Thm 3)**.** *Let X be a smooth projective variety of maximal Albanese dimension such that $\chi(\omega_X) = 0$. Then the image of the Albanese map of X is fibred by translates of abelian subvarieties of* $\mathrm{Alb}\, X$*.*

Proof. Since ω_X is GV, the condition $\chi(\omega_X) = 0$ is equivalent to the fact that $V^0(\omega_X)$ is a proper subvariety of $\mathrm{Pic}^0 X$ (Lemma 1.6(a)). Hence, by Corollary 1.8, there is a positive k such that $V^k(\omega_X)$ has a component W of codimension k. At this point the proof is the one of Ein-Lazarsfeld: the Subtorus theorem 1.10 tells that $W = [\beta] + T$, where T is a subtorus of $\mathrm{Pic}^0 X$, and provides the following diagram

$$\begin{array}{ccc} X & \xrightarrow{alb} & \operatorname{Alb} X \\ & \searrow^{g} & \downarrow{\scriptstyle \pi} \\ & & B = \operatorname{Pic}^0 T \end{array} \quad ,$$

where: $\dim g(X) \leq \dim X - k$, π is surjective and the fibers of π are translates of k-dimensional subtori of $\operatorname{Alb} X$. Since *alb* is generically finite, it follows that $\dim g(X) = \dim X - k$ and that a generic fibre of g surjects onto a generic fibre of π. □

Notes 3.4. (a) We recall that Theorem 3.3 settled a conjecture of Kollár, asserting that a variety X of general type and maximal Albanese dimension should have $\chi(\omega_X) > 0$. Ein and Lazarsfeld in [EL] disproved the conjecture, producing a threefold X of general type, maximal Albanese dimension and $\chi(\omega_X) = 0$. But, at the same time, with Theorem 3.3, they showed that if $\chi(\omega_X) = 0$ then (a desingularization of) the Albanese image of X can't be of general type. However, the structure of varieties of general type and maximal Albanese dimension X with $\chi(\omega_X) = 0$ still remain mysterious. Results in this direction are due to Chen-Hacon (see [CH4], [CH5]).

(b) In [PaPo4] (Cor. 5.1) Theorem 3.3 has been extended as follows: let X be a variety of maximal Albanese dimension. Then the image of the Albanese map of X is fibred by h-codimensional subvarieties of subtori of $AlbX$, with $h \leq \chi(\omega_X)$ (see *loc cit* for a more precise statement). The proof uses k-syzygy sheaves and the Evans-Griffith syzygy theorem.

(c) Theorems 3.2 and 3.3, as well as the extension mentioned in (b) above, work also in the compact Kähler setting. Moreover, the present proof of Theorem 3.2 is algebraic, so that the results holds over any algebraically closed field.

We conclude Ein-Lazarsfeld's result on the singularities of theta-divisors. Here the difference with the original argument is that adjoint ideals are not invoked.

Theorem 3.5 ([EL] Th.1)**.** *Let Θ be an irreducible theta-divisor of a principally polarized abelian variety A. Then Θ is normal with rational singularities.*

Proof. Let $a : X \to \Theta$ be a resolution of singularities of X. It is well known that, under our hypotheses, the fact that Θ is normal with rational singularities is equivalent to the fact that the trace map $t : a_*\omega_X \to \omega_\Theta$ is an isomorphism. Since such map is in any case injective, it is enough to show that $\operatorname{coker} t$ is zero. We have that $a_*\omega_X$ is a GV-sheaf on A (this follows by Hacon's generic vanishing or, more simply, by Green-Lazarsfeld generic vanishing and the fact that, by Grauert-Riemenschneider vanishing, $V^i(X, \omega_X) = V^i(A, a_*\omega_X)$). Moreover, an immediate calulation with the adjunction formula shows that $V^i(\omega_\Theta) = \{\hat{0}\}$ for all $i > 0$. Thus, tensoring with α the exact sequence

$$0 \to a_*\omega_X \xrightarrow{t} \omega_\Theta \to \operatorname{coker} t \to 0 \tag{15}$$

and taking cohomology it follows that $\operatorname{codim} V^i(A, \operatorname{coker} t) > i$ for all $i > 0$. By non-vanishing (Lemma1.12(b)), it follows that if $\operatorname{coker} t \neq 0$ then $V^0(A, \operatorname{coker} t) = \operatorname{Pic}^0 A$, i.e., by Lemma 1.6(a), that $\chi(\operatorname{coker} t) > 0$. Then, since $\chi(\omega_\Theta) = 1$, from (15) it follows that $\chi(a_*\omega_X) = 0$, i.e. that $\chi(\omega_X) = 0$. At this point one concludes as in [EL]. In fact, by Theorem 3.3, Θ would be fibred by subtori of A, which is not the case since Θ is ample and irreducible. □

In [EL] there are also results on the singularities of pluri-theta divisors, extending previous seminal results of Kollár in [Ko3]. These, together with Theorem 3.5, has been extended to other polarizations of low degree – especially in the case of simple abelian varieties – by Debarre and Hacon ([DH]).

4. The Chen-Hacon birational characterization of abelian varieties

The goal of this section is to supply a new proof of the Chen-Hacon characterization of abelian varieties. We refer to the the previous section for a short history and motivation.

Theorem 4.1 (Chen-Hacon, [CH1])**.** *Let X be smooth complex projective variety. Then X is birational to an abelian variety if and only if $q(X) = \dim X$, and $h^0(\omega_X) = h^0(\omega_X^2) = 1$.*

Via the approach of Kollár and Ein-Lazarsfeld, the theorem of Chen-Hacon will be a consequence of the following

Lemma 4.2. *Let X be a projective variety of maximal Albanese dimension. If $\dim V^0(\omega_X) > 0$ then there exists a positive-dimensional subvariety Z of $\mathrm{Pic}^0 X$ such that both Z and $-Z$ are contained in $V^0(\omega_X)$.*

Proof. (of Theorem 4.1) Let X be a smooth projective variety such that $p_1(X) = p_2(X) = 1$ and $q(X) = \dim X$. By Theorem 3.1(b) the Albanese map of X is surjective, hence generically finite. By Lemma 4.2, combined with Theorem 3.1(a), it follows that $\dim V^0(\omega_X) = 0$. Therefore, thanks to the characterization provided by Theorem 3.2, X must be birational to an abelian variety. □

Proof. (of Lemma 4.2) Let W be a positive-dimensional component of $V^0(\omega_X)$. If W contains the identity point then it is a subtorus by Theorem 2.8(a), and the thesis of the Lemma is obviously satisfied. If W does not contain the identity point then, again by Theorem 2.8(a), $W = [\beta] + T$ where $[\beta]$ is a torsion point of $\mathrm{Pic}^0 X$ and T is a subtorus of $\mathrm{Pic}^0 X$ not containing $[\beta]$. To prove Lemma 4.2 it is enough to show that then there is a positive-dimensional subvariety Z of $[\beta^{-1}] + B$ which is is contained in $V^0(\omega_X)$.

Let $d = \dim X$, $q = q(X)$, and $k = \mathrm{codim}_{\mathrm{Pic}^0 X} W$. We have the diagram

$$\begin{array}{ccc} X & \xrightarrow{alb} & \mathrm{Alb}\, X \\ & \searrow^{g} & \downarrow{\scriptstyle \pi} \\ & & B = \mathrm{Pic}^0 T \end{array} ,$$

and, as in the proof of Theorem 3.3,

$$\dim g(X) = d - k. \tag{16}$$

Next, we claim that

$$R^k g_*(\omega_X \otimes \beta) \neq 0 \tag{17}$$

Indeed, by Kollár splitting (Variant 2.2(d)), and projection formula, we have (replacing $g(X)$ with B) that for all $\alpha \in T = \mathrm{Pic}^0 B$,

$$H^k(X, \omega_X \otimes \beta \otimes g^*\alpha) = \bigoplus_{i=0}^{k} H^{k-i}(B, R^i g_*(\omega_X \otimes \beta) \otimes \alpha) \tag{18}$$

We know that $H^k(X, \omega_X \otimes \beta \otimes g^*\alpha) > 0$ for all $\alpha \in \mathrm{Pic}^0 B$ (in other words: $[\beta] + g^* \mathrm{Pic}^0 B$ is contained in $V^k(X, \omega_X)$). By (18), this means that $\mathrm{Pic}^0 B = \cup_{i=0}^{k} V^{k-i}(B, R^i g_*(\omega_X \otimes \beta))$. But, by Hacon's generic vanishing theorem (as in Variant 2.6), all sheaves $R^i g_*(\omega_X \otimes \beta)$ are GV. In particular their $V^{k-i}(\,)$ are *proper* subvarieties of Pic^0 for $k - i > 0$. Therefore $T = \mathrm{Pic}^0 B$ must be equal to $V^0(B, R^k g_*(\omega_X \otimes \beta))$. This implies (17). By Kollár's torsion-freeness result (Variant 2.2(a)), $R^k g_*(\omega_X \otimes \beta)$ is torsion-free on $g(X)$.

Let $X \xrightarrow{f} Y \xrightarrow{a} g(X)$ be the Stein factorization of the morphism g. It follows from (17) that $R^k f_*(\omega_X \otimes \beta) \neq 0$. Therefore, denoting F a general fibre of f, $H^k(\omega_F \otimes \beta) > 0$. Since, by (16), the dimension of a general fibre F of f is k, $[\beta]$ must belong to the kernel of the restriction map $\mathrm{Pic}^0 X \to \mathrm{Pic}^0 F$. Hence so does $[\beta^{-1}]$. Therefore $R^k f_*(\omega_X \otimes \beta^{-1})$ is non-zero (in fact, again by Variant 2.2, it is torsion-free on Y). Hence

$$R^k g_*(\omega_X \otimes \beta^{-1}) \neq 0.$$

Finally, we claim that

$$\dim V^0(B, R^k g_*(\omega_X \otimes \beta^{-1})) > 0 \tag{19}$$

(in fact it turns out that $V^0(B, R^k g_*(\omega_X \otimes \beta^{-1}))$ has no isolated point). Granting (19) for the time being, we conclude the proof. By (18) with β^{-1} instead of β, the positive-dimensional subvariety $V^0(B, R^k g_*(\omega_X \otimes \beta^{-1}))$ induces a positive-dimensional subvariety, say Z, of $[\beta^{-1}] + B$, which is contained in $V^k(X, \omega_X)$[7]. By base change (Corollary 1.7), Z is contained in $V^0(X, \omega_X)$. This proves Lemma 4.2.

It remains to prove (19). Again by Hacon's generic vanishing (Variant 2.6), $R^k g_*(\omega_X \otimes \beta^{-1})$ is a GV-sheaf on B. Therefore, by non-vanishing (Corollary 1.12), the variety $V^0(B, R^k g_*(\omega_X \otimes \beta^{-1}))$ is non-empty. If $V^0(B, R^k g_*(\omega_X \otimes \beta^{-1}))$ had an isolated point, say $[\bar\alpha]$, then $[\bar\alpha]$ would belong also to $V^{q-k}(A, R^k g_*(\omega_X \otimes \beta^{-1}))$ (Corollary 1.8, recalling that $\dim B = q - k$). It would follow that $d - k = q - k$, i.e. $d = q$. Hence $H^{d-k}(A, R^k g_*(\omega_X \otimes \beta^{-1}) \otimes \bar\alpha) \neq 0$. Once again, by Kollár splitting as in (18), it would follow that

$$H^d(X, \omega_X \otimes \beta^{-1} \otimes g^*\bar\alpha) > 0$$

which is impossible, since, as β^{-1} does not belong to $B = g^* \mathrm{Pic}^0 A$, the line bundle $\beta^{-1} \otimes g^*\bar\alpha$ cannot be trivial. □

Finally, it is perhaps worth mentioning that slighly more has been proved

Corollary 4.3 (of the proof)**.** *Let X be a variety of maximal Albanese dimension such that $\dim V^0(\omega_X) > 0$. Given a positive-dimensional component $[\beta] + B$ of $V^0(\omega_X)$, where $[\beta]$ is of order $n > 1$ and B is a subtorus of $\mathrm{Pic}^0 X$, then, for all $k = 1, ..., n-1$ there is a positive-dimensional subtorus C_k of B such that $[\beta^k] + C_k$ is contained in $V^0(\omega_X)$.*

[7] In fact Z is a traslate of a subtorus, see Notes 2.9(a)

5. On Hacon's characterization of theta-divisors

Let Θ be an irreducible theta-divisor in a principally polarized abelian variety A, and let $X \to \Theta$ be a desingularization. Thanks to the fact that Θ has rational singularities (Theorem 3.5), $V^i(X, \omega_X) = V^i(\Theta, \omega_\Theta)$. Hence the following conditions hold:
(i) $V^i(X, \omega_X) = \{\hat{0}\}$ for all $i > 0$;
(ii) $h^0(\omega_X \otimes \alpha) = 1$ for all $[\alpha] \in \operatorname{Pic}^0 X$ such that $[\alpha] \neq \hat{0}$.
In particular, it follows that
(a) $\chi(\omega_X) = 1$;
(b) $\operatorname{codim} V^i(\omega_X) > i + 1$ for all i such that $0 < i < \dim X$.
The following result, slightly refining a theorem of Hacon's, shows that desingularizations of theta-divisors can be characterized – among varieties such that $\dim X < q(X)$ – by conditions *(a)* and *(b)*. The proof illustrates in a simple case a principle – already mentioned before Lemma 1.8 – often appearing in arguments based on generic vanishing theorems and Fourier-Mukai transform: the interplay between the size of the cohomological support loci $V^i(\mathcal{F})$, where $\mathcal{F}$ is a GV-sheaf, and the sheaf-theoretic properties of transform $\widehat{R\Delta\mathcal{F}}$. The statement and the argument provided here are modeled on Prop. 3.1 of [BLNPa].

Theorem 5.1. *Let X be a smooth projective variety such that: (a) $\chi(\omega_X) = 1$; (b) $\operatorname{codim} V^i(\omega_X) > i+1$ for all i such that $0 < i < \dim X$, (c) $\dim X < q(X)$. Then X is birational to a theta-divisor.*

Proof. Let us denote, as usual, $d = \dim X$ and $q = q(X)$. Conditions *(b)* and *(c)* imply that ω_X is a GV-sheaf. Therefore the Albanese map of X is generically finite (Proposition 2.4). Not only: *(b)* and *(c)* imply that $\operatorname{codim} V^i(\omega_X) > i$ for all $i > 0$. Therefore, by Theorem 1.10, the sheaf $\widehat{\mathcal{O}_X}$ is torsion-free. Since its generic rank is $\chi(\omega_X) = 1$, it has to be an ideal sheaf twisted by a line bundle on $\operatorname{Pic}^0 X$:

$$\widehat{\mathcal{O}_X} = \mathcal{I}_Z \otimes L.$$

Next, we claim that, for each (non-embedded) component W of Z

$$\operatorname{codim}_{\operatorname{Pic}^0 X} W = d + 1 \tag{20}$$

To prove this we note that, for $i > 1$,

$$\mathcal{E}xt^i(\widehat{\mathcal{O}_X}, \mathcal{O}_{\operatorname{Pic}^0 X}) = \mathcal{E}xt^i(\mathcal{I}_Z \otimes L, \mathcal{O}_{\operatorname{Pic}^0 X}) \cong \mathcal{E}xt^{i+1}(L \otimes \mathcal{O}_Z, \mathcal{O}_{\operatorname{Pic}^0 X}) \tag{21}$$

Let W be one such component of Z, and let $j + 1$ be its codimension. We have that the support of $\mathcal{E}xt^{j+1}(L \otimes \mathcal{O}_Z, \mathcal{O}_{\operatorname{Pic}^0 X})$ contains W. Hence, combining (21), Grothendieck duality (Lemma 1.6(b)), and the Auslander-Buchsbaum-Serre formula, it follows that $R^j\Phi_P(\omega_X)$ is supported in codimension $j+1$. This implies, by base-change, that $\operatorname{codim} V^j(\omega_X) \le j+1$. Because of condition *(c)*, it must be that $j = \dim X$. This proves (20). Next, we claim that

$$R\mathcal{H}om(\widehat{\mathcal{O}_X}, \mathcal{O}_{\operatorname{Pic}^0 X}) = R\mathcal{H}om(\mathcal{I}_Z \otimes L, \mathcal{O}_{\operatorname{Pic}^0 X}) = \mathbb{C}(\hat{0})[-d] \quad \text{and} \quad d = q - 1 \tag{22}$$

Indeed, arguing as in the proof of (20), from (20) it follows that $\mathcal{E}xt^i(\widehat{\mathcal{O}_X}, \mathcal{O}_{\operatorname{Pic}^0 X})$ is zero for $i < \operatorname{codim}_{\operatorname{Pic}^0 X} Z = d + 1$. By Lemma 1.6(b) again, this means that $R^i\Phi_P(\omega_X)$ is zero for $i < d + 1 = \operatorname{codim}_{\operatorname{Pic}^0 X} Z$. Since we know from Prop. 1.14 that $R^d\Phi_P(\omega_X) \cong \mathbb{C}(\hat{0})$, we get the first part of (22). Since the dualization functor is an involution, it follows that Z itself is the reduced point $\hat{0}$ and that $d = q-1$, completing the proof of (22). At this point the proof is exacly as in [BLNPa], Prop. 3.1. We report it here for the reader's convenience. By Proposition 1.13

$$R\Phi_P(\mathcal{O}_X) = R\Phi_{\mathcal{P}}(R\,alb_*\mathcal{O}_X) = \mathcal{I}_{\hat{0}} \otimes L[-q+1]$$

Therefore, by Mukai's Inversion Theorem 1.11

$$R\Psi_{\mathcal{P}}(\mathcal{I}_{\hat{0}} \otimes L) = (-1)^*_{\mathrm{Pic}^0 X} R\,alb_*\mathcal{O}_X[-1] \tag{23}$$

In particular:

$$R^0\Psi_{\mathcal{P}}(\mathcal{I}_{\hat{0}} \otimes L) = 0 \qquad \text{and} \qquad R^1\Phi_{\mathcal{P}}(\mathcal{I}_{\hat{0}} \otimes L) \cong alb_*\mathcal{O}_X \tag{24}$$

Applying $\Psi_{\mathcal{P}}$ to the standard exact sequence

$$0 \to \mathcal{I}_{\hat{0}} \otimes L \to L \to \mathcal{O}_{\hat{0}} \otimes L \to 0, \tag{25}$$

and using (24) we get,

$$0 \to R^0\Psi_{\mathcal{P}}(L) \to \mathcal{O}_{\mathrm{Alb}\,X} \to alb_*\mathcal{O}_X \tag{26}$$

whence $R^0\Psi_{\mathcal{P}}(L)$ is supported everywhere (since $alb_*\mathcal{O}_X$ is supported on a divisor). It is well known that this implies that L is *ample*. Therefore $R^i\Psi_{\mathcal{P}}(L) = 0$ for $i > 0$. Hence, by sequence (25), $R^i\Psi_{\mathcal{P}}(\mathcal{I}_{\hat{0}} \otimes L) = 0$ for $i > 1$. By (23) and (24), this implies that $R^i alb_*(\mathcal{O}_X) = 0$ for $i > 0$. Furthermore, (26) implies easily that $h^0(\mathrm{Pic}^0 X, L) = 1$, i.e. L is a *principal* polarization on $\mathrm{Pic}^0 X$. Therefore, via the identification $\mathrm{Alb}(X) \cong \mathrm{Pic}^0(X)$ provided by L, we have $R^0\Psi_{\mathcal{P}}(L) \cong L^{-1}$ (see [Muk, Prop. 3.11(1)]). Since the arrow on the right in (26) is onto, it follows that $alb_*\mathcal{O}_X = \mathcal{O}_\Theta$, where Θ is the only effective divisor in the linear series $|L|$. As we already know that alb is generically finite, this implies that alb is a birational morphism onto Θ. □

Notes 5.2. (1) The cohomological characterization of theta-divisors is due to Hacon ([H1]), who proved it under some extra hypotheses, subsequently refined in [HP1]. A further refinement was proved in [BLNPa], Prop. 3.1 and [LPo] Prop. 3.11. The present approach is the one of [BLNPa].

(2) Concerning the significance of the hypothesis of the above theorem, note that, removing the hypothesis $\dim X < q$ there are varieties non-birational to theta-divisors satisfying (a) and (b) (e.g., sticking to varieties of maximal Albanese dimension, the double cover of a p.p.a.v. ramified on a smooth divisor $D \in |2\Theta|$). Moreover products of (desingularized) theta-divisors are examples of varieties satisfying conditions (a) and (c), but not (b).

(3) Theorem 5.1 and its proof hold assuming, more generally, that X is compact Kähler. The argument works also for projective varieties over any algebraically closed field, except for the fact that Prop. 2.4 is used to ensure the maximal Albanese dimension. Therefore, up to adding the hypothesis that X is of maximal Albanese dimension and replacing condition (b) with the condition $\dim X < \dim \mathrm{Alb}\, X$, Theorem 5.1 holds in any characteristic.

(4) With the same argument one can prove the following characterization of abelian varieties, valid in any characteristic: *assume that X is a smooth projective variety of maximal Albanese dimension such that:* (a) $\chi(\omega_X) = 0$*; and* (b) $\mathrm{codim}\, V^i(\omega_X) > i$ *for all i such that* $0 < i < \dim X$. *Then X is birational to an abelian variety.*

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