GAUSSIAN MAPS AND GENERIC VANISHING I: SUBVARIETIES OF ABELIAN VARIETIES

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Dedicated to my teacher, Rob Lazarsfeld, on the occasion of his 60th birthday

Introduction

We work with irreducible projective varieties on an algebraically closed field of any characteristic, henceforth called *varieties*. The contents of this paper are:
(1) a general criterion expressing the vanishing of the higher cohomology of a line bundle on a Cohen-Macaulay variety in terms of a certain first-order conditions on hyperplane sections. Such conditions involve *gaussian maps* and the criterion is a generalization of well known results on hyperplane sections of K3 and abelian surfaces;
(2) using a relative version of the above criterion we prove the vanishing of higher direct images of Poincaré line bundles of normal Cohen-Macaulay subvarieties of abelian varieties[1]. As it is well known, this is equivalent to *generic vanishing*, a far-reaching result known to hold for compact Kahler manifolds thanks to a celebrated theorem of Green and Lazarsfeld. Our generic vanishing in turn implies a Kodaira-type vanishing for normal Cohen-Macaulay subvarieties of abelian varieties, holding for line bundles which are restrictions of ample line bundles on the abelian variety.

While part (1) consists of an essentially complete result, point (2) is an example, in a technically simple case, of a more general algebraic approach to generic vanishing applicable, in principle, to all varieties *mapping* to abelian varieties. This will be the object of forthcoming work. However, one should keep in mind the work of Hacon and Kovacs [HacKo] where – by exploiting the relation between *generic* and *Grauert-Riemenschneider* vanishing theorems – are shown examples of mildly singular varieties (over $\mathbb{C}$) and even smooth varieties (in characteristic $p > 0$) of dimension ≥ 3, with a (separable) generically finite map to an abelian variety where generic vanishing fails[2].

Now we turn to a more detailed presentation of the above topics.

0.1. **Motivation: gaussian maps on curves and vanishing on surfaces.** To introduce the first part, we start from a particular case: the vanishing of the H^1 of a line bundle on a surface in terms of gaussian maps on a sufficiently positive curve of the surface. To begin with, let us recall what classical gaussian maps are. Given a curve C and a line bundle A on C, denote M_A the kernel of the evaluation map $H^0(C, A) \otimes \mathcal{O}_C \to A$. It comes equipped with a natural differentiation map (of sections of A)

$$M_A \to \Omega^1_C \otimes A$$

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[1] by Poincaré line bundle of a subvariety X of an abelian variety A we mean the pullback to $X \times \mathrm{Pic}^0 A$ of a Poincaré line bundle on $A \times \mathrm{Pic}^0 A$

[2] this disproved an erroneous theorem of a previous preprint of the author

defined as

$$M_A = p_*(\mathcal{I}_\Delta \otimes q^*A) \to p_*((\mathcal{I}_\Delta \otimes A)_{|\Delta}) = \Omega^1_C \otimes A$$

where p, q and Δ are the projections and the diagonal of the product $C\times C$. Twisting with another line bundle B and taking global sections one gets the *gaussian map* (or *Wahl map*) of A and B:

$$\gamma_{A,B} : Rel(A,B) := H^0(C, M_A \otimes B) \to H^0(C, \Omega^1_C \otimes A \otimes B)$$ [3]

In our treatment it is more natural to set $A = N \otimes P$ and $B = \omega_C \otimes P^\vee$ for some line bundles N and P on the curve C, and to consider the dual map

$$g_{N,P} : \mathrm{Ext}^1_C(\Omega^1_C \otimes N, \mathcal{O}_C) \to \mathrm{Ext}^1_C(M_{N\otimes P}, P) \tag{0.1}$$

Note that $g_{N,P}$ can defined directly (even if ω_C is not a line bundle) as $\mathrm{Ext}^1_C(\cdot, P)$ of the differentiation map of $M_{N\otimes P}$.

The relation with the vanishing of the H^1 of line bundles on surfaces is in the following result, whose proof follows closely arguments contained in the papers of Beaville and Merindol [BM] and Colombo, Frediani and the author [CFP]. Let X be a Cohen-Macaulay surface and let Q a line bundle on X. Let L a base point-free line bundle on X such that also $L\otimes Q$ is base point-free, and let C a (reduced and irreducible) Cartier divisor in $|L|$, not contained in the singular locus of X. Let $N_C = L_{|C}$ be the normal bundle of C. We have the extension class

$$e \in \mathrm{Ext}^1_C(\Omega^1_C \otimes N_C, \mathcal{O}_C)$$

of the normal sequence

$$0 \to N_C^\vee \to (\Omega^1_X)_{|C} \to \Omega^1_C \to 0$$

We consider the (dual) gaussian map

$$g_{N_C, Q_{|C}} : \mathrm{Ext}^1_C(\Omega^1_C \otimes N_C, \mathcal{O}_C) \to \mathrm{Ext}^1_C(M_{N_C\otimes Q}, Q_{|C}) \tag{0.2}$$

Theorem. *(a) If $H^1(X,Q) = 0$ then $e \in \ker(g_{N_C,Q_{|C}})$.*
(b) If L is sufficiently positive[4] *then also the converse holds: if $e \in \ker(g_{N_C,Q_{|C}})$ then $H^1(X,Q) = 0$.*

Note that e is non-zero if L sufficiently positive. For example, if X is a smooth surface with trivial canonical bundle and $Q = \mathcal{O}_X$ then (a) says that if X is a K3 then $e \in \ker(g_{K_C,\mathcal{O}_C})$. This is a result of [BM]. Conversely, (b) says that if X is abelian and C is sufficiently positive then $e \notin \ker(g_{K_C,\mathcal{O}_C})$. This a result of [CFP].

The proof is a calculation with extension classes which can be explained more geometrically as follows. Suppose that C is a curve in a surface X and that C is embedded in an ambient variety Z. From the cotangent sequence

$$0 \to \mathcal{I}/\mathcal{I}^2 \to (\Omega^1_Z)_{|C} \to \Omega^1_C \to 0$$

(where $\mathcal{I}$ is the ideal of C in Z) one gets the long cohomology sequence

$$\cdots \to \mathrm{Hom}_C(\mathcal{I}/\mathcal{I}^2, N_C^\vee) \xrightarrow{H_Z} \mathrm{Ext}^1_C(\Omega^1_C, N_C^\vee) \xrightarrow{G_Z} \mathrm{Ext}^1_C((\Omega^1_Z)_{|C}, N_C^\vee) \to \cdots \tag{0.3}$$

The problem of extending the embedding $C \hookrightarrow Z$ to the surface X has an easy first-order obstruction: as it is well known, if the divisor $2C$ on X, seen as a scheme, is embedded in Z, then it lives, as embedded first-order deformation, in the Hom on the left[5]. The forgetful map H_Z, disregarding the

[3] the source is denoted $Rel(A,B)$ because it is the kernel of the multiplication of global sections of A and B

[4] this means that L is a sufficiently high multiple of a fixed ample line bundle on X. This condition can be made explicit.

[5] more precisely, the ideal of $2C$ in Z induces the morphism of $\mathcal{O}_Z$-modules $\mathcal{I}/\mathcal{I}^2 \to N_C$ whose kernel is $\mathcal{I}_{2C/Z}/\mathcal{I}^2$, see e.g. [BaF] or [Fe]

embedding, takes it to the class of the normal sequence $e \in \mathrm{Ext}^1_C(\Omega^1_C, N_C^\vee)$. Therefore $e \in \ker(G_Z)$. Now we specialize to the case when the ambient variety is a projective space, specifically:

$$Z = \mathbb{P}(H^0(C, N_C \otimes Q)^\vee) := \mathbb{P}_Q$$

(in this informal discussion we are assuming, for simplicity, that the line bundle $L \otimes Q$ is very ample). By the Euler sequence the map $G_{\mathbb{P}_Q}$ is the dual gaussian map $g_{N_C,Q_{|C}}$ of (0.2). Notice that in this case there is the special feature that our extension problem can be relaxed to the problem of extending the embedding of C in $\mathbb{P}_Q$ to an embedding of the surface X in a possibly bigger projective space $\mathbb{P}$, containing $\mathbb{P}_Q$ as a linear subspace. Since the restriction of $\Omega^1_{\mathbb{P}}$ to $\mathbb{P}_Q$ splits, this has the same first-order obstruction, namely $e \in \ker(g_{N_C,\mathcal{Q}_{|C}})$.

The relation of all that with vanishing of the H^1 is classical: the embedding of C in $\mathbb{P}_Q$ can be extended (in the relaxed sense above) to an embedding of X if and only if the restriction map $\rho_X : H^0(X, L \otimes Q) \to H^0(C, N_C \otimes Q)$ is surjective. This is implied by the vanishing of $H^1(X, Q)$, so we get (a). The converse is more complicated: by Serre vanishing, if L is sufficiently positive the vanishing of $H^1(X, Q)$ is *equivalent* to the surjectivity of the restriction map ρ_X, and also to the surjectivity of the restriction map $\rho_{2C} : H^0(2C, (L \otimes Q)_{|2C}) \to H^0(C, N_C \otimes Q)$, hence to the fact that $2C$ lives in $\mathrm{Hom}_C(\mathcal{I}/\mathcal{I}^2, N_C^\vee)$. Now if e is in the kernel of $g_{N_C,Q_{|C}} = G_{\mathbb{P}_Q}$ then e comes from some embedded deformations in $\mathrm{Hom}_C(\mathcal{I}/\mathcal{I}^2, N_C^\vee)$. However these do not necessarily include $2C$. A more refined analysis proves that this indeed the case as soon as L is sufficiently positive.

0.2. **Gaussian maps on hyperplane sections and vanishing.** Extending the Theorem to higher dimension is a bit more complicated. The relevant case deals with the vanishing of the H^n of line bundle on variety of dimension $n+1$ [6]. To this purpose, we consider "hybrid" gaussian maps as follows: let C be a curve in a n-dimensional variety Y and let A_C be a line bundle on C. The *Lazarsfeld sheaf* ([L]) is the kernel $F^Y_{A_C}$ of the evaluation map of A_C, *seen as a sheaf on* Y:

$$H^0(A_C) \otimes \mathcal{O}_Y \to A_C$$

(note that $F^Y_{A_C}$ is never locally free if $\dim Y \geq 3$). As above, it comes equipped with a differentiation map

$$F^Y_{A_C} \to \Omega^1_Y \otimes A_C$$

If B is a line bundle on Y, we define the *gaussian map of* A_C *and* B as

$$\gamma^Y_{A_C,B} : Rel(A_C, B) = H^0(Y, F_{A_C} \otimes B) \to H^0(\Omega^1_Y \otimes A_C \otimes B)$$

As above, we will rather use the dual map

$$g^Y_{M_C,R} : \mathrm{Ext}^n_Y(\Omega^1_Y \otimes M_C, \mathcal{O}_Y) \to \mathrm{Ext}^n_Y(F_{M_C \otimes R}, R)$$

where M_C is a line bundle on C and R a line bundle on Y such that $B = \omega_Y \otimes R^\vee$ and $A_C = M_C \otimes R$. Again, this map can defined directly (even if ω_C is not a line bundle) as $\mathrm{Ext}^1_C(\cdot, R)$ of the differentiation map of $M_{M_C \otimes R}$. The case $n = 1$ is recovered taking $Y = C$.

These maps can be extended to a relative flat setting. In this paper we consider only the simplest case, namely a family of line bundles on a fixed variety Y, as this is the only one needed in the subsequent applications. In the notation above, let T be a another projective CM variety (or

[6] If $0 < k < n$ then, by Serre vanishing, $H^k(X, Q) \cong H^k(Y, Q_{|Y})$, where Y is a sufficiently positive $k+1$-dimensional hyperplane section. Hence this case can be reduced to the previous one.

Note: one could think to consider the equality $h^n(X, Q) = h^1(X, \omega_X \otimes Q^\vee)$ and then reduce, as above, to a surface. However, this is not possible in the relative case, since in general there is no Serre duality isomorphism of the direct images. Even in the non-relative case the resulting criterion is usually more difficult to apply scheme), and let $\mathcal{R}$ be a line bundle on $Y \times T$. Let ν and π denote the two projections, respectively on Y and T. We can consider the *relative Lazarsfeld sheaf* $\mathcal{F}^Y_{M_C,\mathcal{R}}$, kernel of the relative evaluation map

$$\pi^*\pi_*(\mathcal{R}\otimes \nu^* M_C) \to \mathcal{R}\otimes \nu^* M_C$$

where, as above, we see M_C as a sheaf on Y. The $\mathcal{O}_{Y\times T}$-module $\mathcal{F}$ is a equipped with the differentiation map (see Subsection 1.1 below)

$$\mathcal{F}^Y_{M_C,\mathcal{R}} \to \nu^*(\Omega^1_Y\otimes M_C)\otimes \mathcal{R} \tag{0.4}$$

Applying $\mathrm{Ext}^n_{Y\times T}(\,\cdot\,,\mathcal{R})$ and restricting to the direct summand $\mathrm{Ext}^n_Y(\Omega^1_Y\otimes M_C,\mathcal{O}_Y)$ we get the (dual) gaussian map

$$g^Y_{M_C,\mathcal{R}} : \mathrm{Ext}^n_Y(\Omega^1_Y\otimes M_C,\mathcal{O}_Y)\to \mathrm{Ext}^n_{Y\times T}(\mathcal{F}^Y_{M_C,\mathcal{R}}\,,\mathcal{R})$$

The announced generalization of the above Theorem is as follows. Let X be a $n+1$-dimensional Cohen-Macaulay variety, let T be another CM variety, and let $\mathcal{Q}$ be a line bundle on $X\times T$. Let be L a line bundle on X, with n divisors $Y_1,\dots,Y_n \in |L|$ such that their intersection is an integral curve C not contained in the singular locus of X. We assume also that the line bundle $\nu^*L^{\otimes n}\otimes\mathcal{Q}$ is relatively base point-free, namely the relative evaluation map $\pi^*\pi_*(\nu^*L^{\otimes n}\otimes \mathcal{Q})\to \nu^*L^{\otimes n}\otimes\mathcal{Q}$ is surjective. Let N_C denote the line bundle $L_{|C}$. We choose a divisor among $Y_1,\dots,Y_n$, say $Y=Y_1$, such that C is not contained in the singular locus of Y. We consider the "restricted normal sequence"

$$0\to N_C^\vee \to (\Omega^1_X)_{|C}\to (\Omega^1_Y)_{|C}\to 0 \tag{0.5}$$

Since

$$\mathrm{Ext}^1_C(\Omega^1_Y\otimes N_C,\mathcal{O}_C)\cong \mathrm{Ext}^n_Y(\Omega^1_Y\otimes N_C^{\otimes n},\mathcal{O}_Y) \tag{0.6}$$

(see Subsection 1.1 below) we can see the class e of (0.5) as belonging to $\mathrm{Ext}^n_Y(\Omega^1_Y\otimes N_C^{\otimes n},\mathcal{O}_Y)$. Finally, we consider the dual gaussian map

$$g_{N_C^{\otimes n},\mathcal{Q}_{|Y\times T}} : \mathrm{Ext}^n_Y(\Omega^1_Y\otimes N_C^{\otimes n},\mathcal{O}_Y)\to \mathrm{Ext}^n_{Y\times T}(\mathcal{F}^Y_{N_C^{\otimes n},\mathcal{Q}_{|Y\times T}}\,,\mathcal{Q}_{|Y\times T}) \tag{0.7}$$

Then we have the following result, recovering part (b) the previous theorem as the case $n=1$ and $T=\{point\}$

Theorem A. *If L is sufficiently positive and $e\in\ker(g^Y_{N_C^{\otimes n},\mathcal{Q}_{|Y\times T}})$ then $R^n\pi_*\mathcal{Q}=0$.*

The relative case will be especially interesting for us because the vanishing of $R^i\pi_*\mathcal{Q}$ for $i\le n$ means *generic* vanishing (see below).

Concerning the other implication what we can prove is

Proposition B. *(a) Assume that $T=\{point\}$. If $H^n(X,\mathcal{Q})=0$ then $e\in\ker(g_{N_C^{\otimes n},\mathcal{Q}_{|Y}})$.*
(b) In general, assume that $R^i\pi_*(\mathcal{Q}_{|Y\times T})=0$ for $i<n$. If $R^n\pi_*\mathcal{Q}=0$ then $e\in\ker(g^Y_{N_C^{\otimes n},\mathcal{Q}_{|Y\times T}})$.

To motivate these statements, let us go back to the previous informal discussion. We assume for simplicity that $T=\{point\}$. Let X be a $n+1$-dimensional variety and C a curve in X as above. By the Koszul resolution of the ideal of C and Serre vanishing, the vanishing of $H^n(X,Q)$ implies the surjectivity of the restriction map $\rho_X : H^0(X,L^{\otimes n}\otimes Q)\to H^0(C,N_C^{\otimes n}\otimes Q)$, and it is in fact equivalent to that as soon as L is sufficiently positive. Hence it is natural to look for first order obstructions to extending an embedding of the curve C (a 1-dimensional complete intersection of linearly equivalent divisors of X) into $\mathbb{P}_Q=\mathbb{P}(H^0(C,N_C^{\otimes n}\otimes Q)^\vee)$ to X. More generally, we can consider the same problem for any given ambient variety Z, rather than projective space. However, to find a first-order obstruction one cannot anymore replace X with the first order neighborhood of C in X. We rather have to pick a divisor in $|L|$ containing C, say $Y=Y_1$ and replace X with the scheme $2Y\cap Y_2\cap\cdots\cap Y_n$. In analogy with the case of curves on surfaces, it is natural to consider the long cohomology sequence

$$\cdots\to \mathrm{Hom}_C(\mathcal{I}_Y/\mathcal{I}_Y^2,N_C^\vee)\stackrel{H_Z^Y}{\to}\mathrm{Ext}^1_C((\Omega^1_Y)_{|C},N_C^\vee)\stackrel{G_Z^Y}{\to}\mathrm{Ext}^1_C((\Omega^1_Z)_{|C},N_C^\vee)\to\cdots \tag{0.8}$$

(where $\mathcal{I}_Y$ is the ideal of Y in Z). As above, a necessary condition for the lifting to X of the embedding of $C\hookrightarrow Z$ is that the "restricted normal class" e of (0.5) belongs to $\ker(G_Z^Y)$.

However note that, differently from the case when X is a surface, when $Z=\mathbb{P}_Q$ the map $g_{N_C^{\otimes n},Q_{|Y}}$ is not $G^Y_{\mathbb{P}_Q}$, but rather a more complicated "hybrid" version of gaussian map. The reason is that, looking for sufficient conditions (the most interesting for us) for the lifting to X of the embedding $C\hookrightarrow\mathbb{P}_Q$, one cannot assume that the divisor Y is already embedded in $\mathbb{P}_Q$.

The following version of the vanishing criterion provided by Theorem A is technically easier to apply

Corollary C. *Keeping the notation of Theorem A, if the line bundle L is sufficiently positive then the kernel of the map $g^Y_{N_C^{\otimes n},\mathcal{Q}_{|Y\times T}}$ is at most 1-dimensional (spanned by e). Therefore if $g^Y_{N_C^{\otimes n},\mathcal{Q}_{|Y\times T}}$ is non-injective then $R^n\pi_*\mathcal{Q}=0$.*

Again, in the particular case $T=\{point\}$, and X is a surface with trivial canonical bundle this says that the dual gaussian map $g_{\omega_C,\mathcal{O}_C}$ is not injective if and only if X is a K3 surface, which is well known by the result of Wahl (see [W]) and [CFP] Thm B.

0.3. **Generic vanishing for subvarieties of abelian varieties.** Although difficult – if not impossible – to use in most cases, the above results can be applied in some very special circumstances. For example, in analogy with the literature on curves sitting on K3 surfaces and Fano 3-folds, or Enriques surfaces and Enriques-Fano 3-folds (see for example [W], [BM],[V], [CLM], [KLM]) Proposition B can supply non-trivial necessary conditions for a n-dimensional variety to sit in some very special $n+1$-dimensional varieties.

However in this paper we rather focus on the sufficient condition for vanishing provided by Theorem A and Corollary C, as it provides an algebraic approach to *generic vanishing*, a far-reaching concept introduced by Green and Lazarsfeld in the papers [GL1] and [GL2]. Namely we consider a variety X with a map to an abelian variety, generically finite onto its image

$$a:X\to A \tag{0.9}$$

Denoting $\mathrm{Pic}^0A=\widehat{A}$ the dual variety, we consider the pullback to $X\times\widehat{A}$ of a Poincaré line bundle $\mathcal{P}$ on $A\times\widehat{A}$:

$$\mathcal{Q}=(a\times\mathrm{id}_{\widehat{A}})^*\mathcal{P} \tag{0.10}$$

We keep the notation of the previous section. In particular we denote ν and π the projections of $X\times\widehat{A}$. A first way of expressing generic vanishing is the vanishing of higher direct images

$$R^i\pi_*\mathcal{Q}=0\quad\text{for}\quad i<\dim X \tag{0.11}$$

For smooth varieties over the complex numbers (0.11) was proved (as a particular case of a more general statement) by Hacon ([Hac]), settling a conjecture of Green and Lazarsfeld. Another way of expressing the generic vanishing condition is the original one of Green and Lazarsfeld [GL1] and [GL2]. This involves the *cohomological support loci*

$$V_a^i(X) = \{\alpha \in \mathrm{Pic}^0 A \mid h^i(X, a^*\alpha) > 0\} \quad .$$

Green-Lazarsfeld's theorem is that, if the map a is generically finite, then

$$\operatorname{codim}_{\widehat{A}} V_a^i(X) \geq \dim X - i \ ^7. \tag{0.12}$$

It is easy to see that (0.11) implies (0.12). Subsequently, it has been observed in [PP1] and [PP2] that (0.11) is *equivalent* to (0.12)[8]. The heart of Hacon's proof of (0.11) consists in a clever reduction to Kodaira-Kawamata-Viehweg vanishing. The argument of Green and Lazarsfeld for (0.12) uses Hodge theory. On the other hand a characteristic-free example of both (0.11) and (0.12) is given by abelian varieties themselves ([M] p.127). Here we show that (0.11) (and, therefore, (0.12)) holds for normal Cohen-Macaulay subvarieties of abelian varieties on an algebraically closed field of any characteristic

Theorem D. *In the above notation, assume that X is normal Cohen-Macaulay and the morphism a is an embedding. Then $R^i\pi_*\mathcal{Q} = 0$ for all $i < \dim X$.*

The strategy of the proof consists of course in applying Theorem A to the Poincaré line bundle Q. In order to do so we take a general complete intersection $C = Y \cap Y_2 \cap \cdots \cap Y_n$ of X, with $Y_i \in |L|$, where, as above, L is a sufficiently positive line bundle on X and $n+1 = \dim X$. The main issue of the argument consists in comparing two spaces of first-order deformations: the first is the kernel of the gaussian map $g_{N_C^{\otimes n}, \mathcal{Q}_{Y\times\widehat{A}}}$. The second is the kernel of the map G_Z^Y of (0.8) with $Z = A$ [9] (by (0.6) the two maps have the same source). As in the discussion following Theorem A, the variety $X \subset A$ induces naturally, via the restricted normal extension class e, a non trivial element of $\ker G_A^Y$. In view of Corollary C, to get the vanishing of $R^n\pi_*\mathcal{Q}$ it is enough to prove that the intersection of $\ker G_A^Y$ and $\ker g_{N_C^{\otimes n}, \mathcal{Q}_{|Y\times\widehat{A}}}$ is non-zero. This analysis is accomplished by means of the Fourier-Mukai transform associated to the Poincarè line bundle[10]. In doing this we were inspired by the classical papers [Mu1] and [Ke] where it is solved the conceptually related problem of comparing the first-order embedded deformations of a curve in its jacobian and the first-order deformations of the Picard bundle on the dual. The vanishing of $R^i\pi_*\mathcal{Q}$ for $i < n$ follows from this step, by reducing to a sufficiently positive $i+1$-dimensional hyperplane section.

Note that conditions (0.12) can be expressed dually as

$$\operatorname{codim}_{\mathrm{Pic}^0 A}\{\alpha \in \mathrm{Pic}^0 A \mid h^i(\omega_X \otimes \alpha) > 0\,\} \geq i \qquad \text{for all} \quad i > 0$$

According to the terminology of [PP2], this is stated saying that the dualizing sheaf ω_X is a GV-sheaf. As a first application we note that, combining with Proposition 3.1 of [PP3] ("GV tensor $IT_0 = IT_0$") we get the following Kodaira-type vanishing

Corollary E. *Let X be a normal Cohen-Macaulay subvariety of an abelian variety A, and let L be an ample line bundle on A. Then $H^i(X, \omega_X \otimes L) = 0$ for all $i > 0$.*

[7] In general, if the map a is not generically finite, Hacon's and Green-Lazarsfeld's theorems are respectively $R^i\pi_*\mathcal{Q} = 0$ for $i < \dim a(X)$ and $\operatorname{codim}_{\mathrm{Pic}^0 A} V_a^i(X) \geq \dim a(X) - i$. However this can be reduced to the case of generically finite a by taking sufficiently positive hyperplane sections of dimension equal to the rank of a

[8] in [PP2] this is stated only in the smooth case, but this hypothesis is unnecessary

[9] this is simply the dual of the multiplication map $V \otimes H^0(N_C \otimes \omega_C) \to H^0(\Omega^1_Y \otimes N_C \otimes \omega_C)$, where V is the cotangent space of A at the origin

[10] we remark, incidentally, that (0.11) for abelian varieties is key point assuring that the Fourier-Mukai transform is an equivalence of categories

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1. Proof of Theorem A and Proposition B

1.1. **Preliminaries.** The proof consists in a computation with extension classes, similar to the one of [CFP] Lemma 3.1. The geometric explanation of the argument is outlined in the Introduction (Subsections 0.1 and 0.2). In the first place, some warning about the notation:

Notation 1.1. We have the three varieties $C \subset Y \subset X$ (respectively of dimension 1, n and $n+1$). The projections of $X \times T$ on X and T are denoted respectively ν and π. It will make a difference to consider the relative evaluation maps of a sheaf $\mathcal{A}$ on $C \times T$ seen as a sheaf on $Y \times T$, or on $X \times T$, or on $C \times T$ itself: their kernels are the various different Lazarsfeld sheaves attached to $\mathcal{A}$ in different ambient varieties. Therefore we denote

$$\pi_Y = \pi_{|Y\times T} \qquad \pi_C = \pi_{|C\times T}$$

For example, on $Y \times T$ we have

$$0 \to \mathcal{F}^Y_{A,\mathcal{Q}_{|Y\times T}} \to \pi_Y^*\pi_*(\mathcal{Q}\otimes\nu^*A) \to \mathcal{Q}\otimes\nu^*A \tag{1.1}$$

and on $X \times T$

$$0 \to \mathcal{F}^X_{A,\mathcal{Q}} \to \pi^*\pi_*(\mathcal{Q}\otimes\nu^*A) \to \mathcal{Q}\otimes\nu^*A \tag{1.2}$$

Next, we clarify a few points appearing in the statements.

The differentiation map (0.4). We describe explicitly the differentiation map (0.4) mentioned in the Introduction. We keep the notation there: M_C is a line bundle on the curve C while $\mathcal{R}$ is a line bundle on $Y \times T$. Now let p,q and $\widetilde{\Delta}$ denote the two projections and the diagonal of the fibred product $(Y \times T) \times_T (Y \times T)$. Concerning the Lazarsfeld sheaf $\mathcal{F}^Y_{M_C,\mathcal{R}}$ we claim that there is a canonical isomorphism

$$\mathcal{F}^Y_{M_C,\mathcal{R}} \cong p_*(\mathcal{I}_{\widetilde{\Delta}} \otimes q^*(\mathcal{R}\otimes\nu^*M_C)) \tag{1.3}$$

Admitting the claim, the differentiation map (0.4) is defined as usual, as p_* of the restriction to $\widetilde{\Delta}$. The isomorphism (1.3): in the first place $p_*(\mathcal{I}_{\widetilde{\Delta}} \otimes q^*(\mathcal{R}\otimes\nu^*M_C)$ is the kernel of the map (p_* of the restriction map)

$$r : p_*q^*(\mathcal{R}\otimes\nu^*M_C) \to p_*q^*((\mathcal{R}\otimes\nu^*M_C)_{|\widetilde{\Delta}}) \cong \mathcal{R}\otimes\nu^*M_C$$

(it is easily seen that the sequence $0 \to \mathcal{I}_{\widetilde{\Delta}} \to \mathcal{O}_{Y\times_T Y} \to \mathcal{O}_{\widetilde{\Delta}} \to 0$ remains exact when restricted to $(Y \times T) \times_T (C \times T)$). To prove (1.3) we note that, by flat base change,

$$\pi^*_{|Y}\pi_*(\mathcal{Q}\otimes\nu^*M_C) \cong p_*q^*(\mathcal{Q}\otimes\nu^*M_C)$$

and, via such isomorphism, the map r is identified to the relative evaluation map.

The isomorphism (0.6). This follows from the spectral sequence

$$\mathrm{Ext}^i_C(\Omega^1_Y\otimes N_C^{\otimes n}, \mathcal{E}xt^j_Y(\mathcal{O}_C,\mathcal{O}_Y)) \Rightarrow \mathrm{Ext}^{i+j}_Y(\Omega^1_Y\otimes N_C^{\otimes n},\mathcal{O}_Y)$$

using that, being C the complete intersection of $n-1$ divisors in $|L_{|Y}|$, $\mathcal{E}xt^j_Y(\mathcal{O}_C,\mathcal{O}_Y) = N^{\otimes n-1}_{|\mathcal{C}}$ if $j=n-1$ and zero otherwise. Seeing the elements of Ext-groups as higher extension classes with their natural multiplicative structure (Yoneda Ext's, see e.g. [E] p. 652–655), we denote

$$\mathcal{K}^Y_C \in \mathrm{Ext}^{n-1}_Y(\mathcal{O}_C, L^{\otimes -(n-1)}_{|Y}) \tag{1.4}$$

the extension class of the Koszul resolution of $\mathcal{O}_C$ as $\mathcal{O}_Y$-module

$$0 \to L^{\otimes -(n-1)}_{|Y} \to \cdots \to (L^{\otimes -1}_{|Y})^{\oplus n-1} \to \mathcal{O}_Y \to \mathcal{O}_C \to 0\ . \tag{1.5}$$

Then the multiplication with $\mathcal{K}^Y_C$

$$\mathrm{Ext}^1_C(\Omega^1_Y \otimes N_C, \mathcal{O}_C) \xrightarrow{\cdot\mathcal{K}^Y_C} \mathrm{Ext}^n_Y(\Omega^1_Y \otimes N^{\otimes n}_C, \mathcal{O}_Y)$$

is an isomorphism (coinciding, up to scalar, with (0.6)).

1.2. Statement of first step and proof of Proposition B.

Notation 1.2. From this point we will adopt the hypotheses and the notation of Theorem A. We also adopt the following typographical abbreviations:

$$\mathcal{F}^Y = \mathcal{F}^Y_{N^{\otimes n}_C, \mathcal{Q}_{|Y\times T}} \qquad \mathcal{F}^X = \mathcal{F}^X_{N^{\otimes n}_C, \mathcal{Q}} \qquad g = g_{N^{\otimes n}_C, \mathcal{Q}_{|Y\times T}}\ .$$

The first, and most important, step of the proof consists of an explicit calculation of the class $g(e)$ of the statement of Theorem A and Proposition B. This is the content of Lemma 1.3 below. The strategy is as follows. Applying $\mathrm{Ext}^n_{Y\times Y}(\,\cdot\,, \mathcal{Q}_{|Y\times T})$ to the basic sequence

$$0 \to \mathcal{F}^Y \to \pi^*_Y\pi_*(\mathcal{Q}\otimes\nu^* N^{\otimes n}_C) \to \mathcal{Q}\otimes\nu^*N^{\otimes n}_C \to 0$$

(namely (1.1) for $A = N^{\otimes n}_C$ and $\mathcal{R} = \mathcal{Q}_{|Y\times T}$) we get the following diagram with exact row
(1.6)

$$\begin{array}{ccccc}
\mathrm{Ext}^n_{Y\times T}(\mathcal{Q}\otimes\nu^*N^{\otimes n}_C, \mathcal{Q}_{|Y\times T}) & \xrightarrow{h} & \mathrm{Ext}^n_{Y\times T}(\pi^*_{|Y}\pi_*(\mathcal{Q}\otimes\nu^*N^{\otimes n}_C), \mathcal{Q}_{|Y\times T}) & \xrightarrow{f} & \mathrm{Ext}^n_{Y\times T}(\mathcal{F}^Y, \mathcal{Q}_{|Y\times T}) \\
 & & & & \uparrow g \\
 & & & & \mathrm{Ext}^n_Y(\Omega^1_Y\otimes N^{\otimes n}_C, \mathcal{O}_Y)
\end{array}$$

We will produce a certain class b in the source of f, namely

$$b \in \mathrm{Ext}^n_{Y\times T}(\pi^*_{|Y}\pi_*(\mathcal{Q}\otimes\nu^*N^{\otimes n}_C), \mathcal{Q}_{|Y\times T}) \tag{1.7}$$

such that its coboundary map

$$\delta_b : \pi_*(\mathcal{Q}\otimes\nu^*\mathcal{N}^{\otimes n}_C) \to R^n\pi_*(\mathcal{Q}_{|Y\times T})$$

is the composition

$$\begin{array}{ccc}
\pi_*(\mathcal{Q}\otimes\nu^*\mathcal{N}^{\otimes n}_C) & \xrightarrow{\alpha} & R^n\pi_*(\mathcal{Q}) \\
 & \searrow^{\delta_b} & \downarrow \beta \\
 & & R^n\pi_*(\mathcal{Q}_{|Y\times T})
\end{array} \tag{1.8}$$

where the horizontal map α is the coboundary map of the natural extension of $\mathcal{O}_{X\times T}$-modules

$$0 \to \mathcal{Q} \to \cdots \to (\mathcal{Q}\otimes\nu^*L^{\otimes n-1})^{\oplus n} \to \mathcal{Q}\otimes\nu^*L^{\otimes n} \to Q\otimes\nu^*N^{\otimes n}_C \to 0 \tag{1.9}$$

(ν^* of the Koszul resolution of $\mathcal{O}_C$ as $\mathcal{O}_X$-module, twisted by $\mathcal{Q}\otimes\nu^*L^{\otimes n}$) and the vertical map β is simply $R^n\pi_*$ of the restriction map:

$$R^n\pi_*(\mathcal{Q})\to R^n\pi_*(\mathcal{Q}_{|Y\times T})$$

Then main Lemma is

Lemma 1.3. $f(b)=g(e)$.

Note that this will already prove Proposition B. Indeed, if $T=\{point\}$ then the space of (1.7) is

$$\mathrm{Ext}^n_Y(H^0(C,\mathcal{Q}\otimes\nu^*N_C^{\otimes n})\otimes\mathcal{O}_Y,\mathcal{Q}_{|Y})\cong\mathrm{Hom}_k(H^0(C,\mathcal{Q}\otimes\nu^*N_C^{\otimes n}),H^n(Y,\mathcal{Q}_{|Y}))\tag{1.10}$$

hence the class b coincides, up to scalar, with its coboundary map δ_b. If $H^n(X,\mathcal{Q})=0$ then $\delta_b=0$. Then the Lemma says that $g(e)=0$, proving Proposition B in this case. If $\dim T>0$ we consider the spectral sequence

$$\mathrm{Ext}^i_T(\pi_*(\mathcal{Q}\otimes N_C^{\otimes n})\,,\,R^j\pi_*(\mathcal{Q}_{|Y\times T}))\Rightarrow\mathrm{Ext}^{i+j}_{Y\times T}(\pi_Y^*\pi_*(p^*(\mathcal{Q}\otimes\nu^*N_C^{\otimes n}))\,,\,\mathcal{Q}_{|Y\times T}).$$

coming from the isomorphism

$$\mathbf{R}\mathrm{Hom}_T(\pi_*(\mathcal{Q}\otimes N_C^{\otimes n})\,,\,\mathbf{R}\pi_*(\mathcal{Q}_{|Y\times T}))\cong\mathbf{R}\mathrm{Hom}_{Y\times T}(\pi_Y^*\pi_*(p^*(\mathcal{Q}\otimes\nu^*N_C^{\otimes n}))\,,\,\mathcal{Q}_{|Y\times T})$$

Since we are assuming that $R^i\pi_*(\mathcal{Q}_{|Y\times T})=0$ for $i<n$ then the spectral sequence degenerates providing an isomorphism as (1.10) and Proposition B follows in the same way. □

Definition of the class b of (1.7). We consider the exact complex of $\mathcal{O}_{X\times T}$-modules (1.9). Composing with the relative evaluation map of $\mathcal{Q}\otimes\nu^*N_C^{\otimes n}$ (seen as a sheaf on $X\times T$)

$$\pi^*\pi_*(\mathcal{Q}\otimes\nu^*N_C^{\otimes n})\to\mathcal{Q}\otimes\nu^*N_C^{\otimes n}$$

we get the exact complex

$$0\to\mathcal{Q}\to\cdots\to(\mathcal{Q}\otimes\nu^*L^{\otimes n-1})^{\oplus n}\to\mathcal{E}\to\pi^*\pi_*(\mathcal{Q}\otimes\nu^*N_C^{\otimes n})\to 0\tag{1.11}$$

where $\mathcal{E}$ is a $\mathcal{O}_{X\times T}$-module. Since $tor^i_{X\times T}(\pi^*\pi_*(\mathcal{Q}\otimes\nu^*N_C^{\otimes n}),\nu^*\mathcal{O}_Y)=0$ for $i>0$, restricting (1.11) to $Y\times T$ we get an *exact* complex of $\mathcal{O}_{Y\times T}$-modules

$$0\to\mathcal{Q}_{|Y\times T}\to\cdots\to(\mathcal{Q}\otimes\nu^*L^{\otimes n-1})^{\oplus n}_{|Y\times T}\to\mathcal{E}_{|Y\times T}\to\pi_Y^*\pi_*((\mathcal{Q}\otimes\nu^*N_C^{\otimes n}))\to 0\tag{1.12}$$

We define the class b of (1.7) as the extension class of the exact complex (1.12). The assertion about its coboundary map follows from its definition.

1.3. **Proof of Lemma 1.3.** We first compute $g(e)$ [11]. The exact sequences defining $\mathcal{F}^X$ and $\mathcal{F}^Y$ (see Notation 1.2) fit into the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F}^X & \longrightarrow & \pi^*\pi_*(\mathcal{Q}\otimes\nu^*N_C^{\otimes n}) & \longrightarrow & \mathcal{Q}\otimes\nu^*N_C^{\otimes n} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow{=} & & \\
0 & \longrightarrow & \mathcal{F}^Y & \longrightarrow & \pi_Y^*\pi_*(\mathcal{Q}\otimes\nu^*N_C^{\otimes n}) & \longrightarrow & \mathcal{Q}\otimes\nu^*N_C^{\otimes n} & \longrightarrow & 0
\end{array}$$

yielding, after restricting the top row to $Y\times T$, the exact sequence

$$0\to\mathcal{Q}\otimes\nu^*N_C^{\otimes n-1}\to(\mathcal{F}^X)_{|Y\times T}\to\mathcal{F}^Y\to 0\tag{1.13}$$

where the sheaf on the left is $tor_1^{\mathcal{O}_{X\times T}}(\mathcal{Q}\otimes\nu^*N_C^{\otimes n},\mathcal{O}_{Y\times T})$.

[11] This argument follows [V] p. 252)

This sequence in turn fits into the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{Q}\otimes\nu^*N_C^{\otimes n-1} \arrow[r] \arrow[d, "="] & (\mathcal{F}^X)_{|Y\times T} \arrow[r] \arrow[d] & \mathcal{F}^Y \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{Q}\otimes\nu^*N_C^{\otimes n-1} \arrow[r] & \mathcal{Q}\otimes\nu^*(\Omega^1_X\otimes N_C^{\otimes n}) \arrow[r] & \mathcal{Q}\otimes\nu^*(\Omega^1_Y\otimes N_C^{\otimes n}) \arrow[r] & 0
\end{tikzcd}$$

where the class of the bottow row is $e\in\nu^*\mathrm{Ext}^1_C(\Omega^1_Y\otimes N_C,\mathcal{O}_C)$ It follows that $g(e)$ (where now e is seen in $\nu^*(\mathrm{Ext}^n_Y(\Omega^1_Y\otimes N_C^{\otimes n},\mathcal{O}_Y))$, see (0.6) and Subsection 1.1) is the class of the sequence (1.13) with ν^* of the Koszul resolution of $\mathcal{O}_C$, as $\mathcal{O}_Y$-module, (tensored with $\mathcal{Q}\otimes\nu^*N_C^{\otimes n}$) attached on the left:

$$0\to\mathcal{Q}_{|Y\times T}\to\cdots\to(\mathcal{Q}\otimes\nu^*L_{|Y}^{\otimes n-2})^{\oplus n-1}\to(\mathcal{Q}\otimes\nu^*L_{|Y}^{\otimes n-1})\to(\mathcal{F}^X)_{|Y\times T}\to\mathcal{F}^Y\to 0\ . \tag{1.14}$$

Next, we compute $f(b)$. The exact complex (1.11) is the middle row of the commutative exact diagram

(1.15)

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
 & \mathcal{F}^X \arrow[r, "="] \arrow[d] & \mathcal{F}^X \arrow[d] & \\
0\to\mathcal{Q}\to\cdots\to(\mathcal{Q}\otimes\nu^*L^{\otimes n-1})^{\oplus n} \arrow[r] \arrow[d, "="] & \mathcal{E} \arrow[r] \arrow[d] & \pi^*\pi_*(\mathcal{Q}\otimes\nu^*N_C^{\otimes n}) \arrow[r] \arrow[d] & 0 \\
0\to\mathcal{Q}\to\cdots\to(\mathcal{Q}\otimes\nu^*L^{\otimes n-1})^{\oplus n} \arrow[r] & \mathcal{Q}\otimes\nu^*L^{\otimes n} \arrow[r] \arrow[d] & \mathcal{Q}\otimes\nu^*N_C^{\otimes n} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 &
\end{tikzcd}$$

This provides us with the commutative exact diagram
(1.16)

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
0\to\mathcal{Q}_{|Y\times T}\to\cdots\to(\mathcal{Q}\otimes\nu^*L_{|Y}^{\otimes n-2})^{\oplus n-1}\to\mathcal{Q}\otimes\nu^*L_{|Y}^{\otimes n-1} \arrow[r] & (\mathcal{F}^X)_{|Y\times T} \arrow[r] \arrow[d] & \mathcal{F}^Y \arrow[r] \arrow[d] & 0 \\
 & \mathcal{E}_{|Y\times T} \arrow[r] & \pi_Y^*\pi_*(\mathcal{Q}\otimes\nu^*N_C^{\otimes n}) \arrow[r] & 0
\end{tikzcd}$$

where the long row is (1.14), whose class is $g(e)$.

Next we claim that the exact complex (1.12) defining the class b is equivalent, *as extension,* to the exact complex:
(1.17)

$$0\to\mathcal{Q}_{|Y\times T}\to\cdots\to(\mathcal{Q}\otimes\nu^*L_{|Y}^{\otimes n-2})^{\oplus n-1}\to\mathcal{Q}\otimes\nu^*L_{|Y}^{\otimes n-1}\to\mathcal{E}_{|Y\times T}\to\pi_Y^*\pi_*(\mathcal{Q}\otimes\nu^*N_C^{\otimes n})\to 0$$

Admitting this for the time being, we can complete diagram (1.16) as follows
(1.18)

$$
\begin{tikzcd}
0 \to \mathcal{Q}_{|Y\times T} \cdots \to (\mathcal{Q}\otimes \nu^* L_{|Y}^{\otimes n-2})^{\oplus n-1} \arrow[d, "="] \to \mathcal{Q}\otimes \nu^* L_{|Y}^{\otimes n-1} \arrow[r] & (\mathcal{F}^X)_{|Y\times T} \arrow[r] \arrow[d] & \mathcal{F}^Y \arrow[r] \arrow[d] & 0 \\
0 \to \mathcal{Q}_{|Y\times T} \cdots \to (\mathcal{Q}\otimes \nu^* L_{|Y}^{\otimes n-2})^{\oplus n-1} \to \mathcal{Q}\otimes \nu^* L_{|Y}^{\otimes n-1} \arrow[r] & \mathcal{E}_{|Y\times T} \arrow[r] & \pi_Y^*\pi_*(\mathcal{Q}\otimes \nu^* N_C^{\otimes n}) \arrow[r] & 0
\end{tikzcd}
$$

The class of the bottow row is b. By definition, the class of the top row is $f(b)$, and it is equal to $g(e)$. This proves Lemma 1.3.

Finally, we prove the claim. For $n = 1$, i.e. $C = Y$ there is nothing to prove. For $n > 1$, since $C = Y_1 \cap \cdots \cap Y_n$, with $Y_i \in |L|$ and $Y = Y_1$, restricting the ideal sheaf $\mathcal{I}_{C/X}$ to Y one gets a sheaf with a torsion part, namely $\mathcal{I}_{C/Y} \oplus N_C^{-1}$. Accordingly the Koszul resolution of $\mathcal{I}_{C/X}$, $0 \to K_X^\bullet \to \mathcal{I}_{C/X} \to 0$, restricted to Y, splits as the direct sum sum of the Koszul resolution of $\mathcal{I}_{C/Y}$:

$$0 \to K_Y^\bullet \to \mathcal{I}_{C/Y} \to 0$$

and of the twisted Koszul resolution

$$0 \to K_Y^\bullet \otimes L_{|Y}^{-1} \to L_{|Y}^{-1} \to N_C^{-1} \to 0$$

shifted by one. At this point the claim follows from the fact that, restricting the exact complex (1.11) to Y, in the "tail" of (1.12) namely the exact complex $0 \to \mathcal{Q}_{|Y\times T} \to \cdots \to (\mathcal{Q}\otimes \nu^* L^{\otimes n-1})^{\oplus n}_{|Y\times T}$ one can eliminate the part corresponding to the resolution of $\mathcal{I}_{C/Y}$. $\square$

1.4. **Conclusion of the Proof of Theorem A.** The last step is

Lemma 1.4. *We keep the notation and setting of Lemma 1.3. Assume that the line bundle L on X is sufficiently positive. If $f(b) = 0$ then $b = 0$.*

Assuming this, Theorem A follows: if $g(e) = 0$ then, by Lemmas 1.3 and 1.4 it follows that $b = 0$, hence its coboundary map $\delta_b = \beta \circ \alpha$ is zero (see (1.8)). Taking L sufficiently positive, it follows easily from relative Serre vanishing that α is surjective and β is injective. Therefore the map δ_b is zero if and only if its target, namely $R^n\pi_*(\mathcal{Q})$, is zero. $\square$

Proof. (of Lemma 1.4) The proof is a somewhat tedious repeated application of Serre vanishing. From the basic exact sequence of diagram (1.6) we have that if $f(b) = 0$ then there is a

$$c \in \mathrm{Ext}^n_{Y\times T}(\mathcal{Q}\otimes \nu^* N_{|C}^{\otimes n}, \mathcal{Q}_{|Y\times T})$$

such that

$$h(c) = b \ . \tag{1.19}$$

Now we consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}^n_{Y\times T}(\mathcal{Q}\otimes\nu^*N_C^{\otimes n},\mathcal{Q}_{|Y\times T}) \arrow[r, "r"] \arrow[d, "h"] & \mathrm{Ext}^n_{X\times T}(\mathcal{Q}\otimes\nu^*L^{\otimes n},\mathcal{Q}_{|Y\times T}) \arrow[d, "h'"] \\
\mathrm{Ext}^n_{Y\times T}(\pi_Y^*\pi_*(\mathcal{Q}\otimes\nu^*N_C^{\otimes n}),\mathcal{Q}_{|Y\times T}) \arrow[r, "s"] \arrow[d, "\mu"] & \mathrm{Ext}^n_{X\times T}(\pi^*\pi_*(\mathcal{Q}\otimes\nu^*L^{\otimes n}),\mathcal{Q}_{|Y\times T})) \arrow[d, "\mu'"] \\
\mathrm{Hom}_T(\pi_*(\mathcal{Q}\otimes\nu^*N_C^{\otimes n}),R^n\pi_*(\mathcal{Q}_{Y\times T})) \arrow[r, "t"] & \mathrm{Hom}_T(\pi_*(\mathcal{Q}\otimes\nu^*L^{\otimes n})),R^n\pi_*(\mathcal{Q}_{|Y\times T}))
\end{tikzcd} \tag{1.20}$$

where:
(a) h is as above and h' is the analogous map $\mathrm{Ext}^n_X(ev_X,\mathcal{Q}_{|Y\times T})$, where ev_X is the relative evaluation map on $X\times T$: $\pi^*\pi_*(\mathcal{Q}\otimes\nu^*L^{\otimes n})\to\mathcal{Q}\otimes\nu^*L^{\otimes n}$;
(b) μ is the map taking an extension to its coboundary map. Consequently the map $\mu\circ h$ takes an extension class $e\in\mathrm{Ext}^n_{Y\times T}(\mathcal{Q}\otimes\nu^*N_C^{\otimes n})\,\mathcal{Q}_{Y\times T})$ to its coboundary map

$$\pi_*(\mathcal{Q}\otimes\nu^*N_C^{\otimes n})\to R^n\pi_*(\mathcal{Q}_{|Y\times T})$$

The map $\mu'\circ h'$ operates in the same way;
(c) notice that the target of r is simply $H^n(\nu^*L^{\otimes -n}_{|Y\times T})$, i.e. $\mathrm{Ext}^n_{Y\times T}(\nu^*L^{\otimes n}_{|Y},\mathcal{O}_{Y\times T})$. Therefore r is the natural map $\mathrm{Ext}^n_{Y\times T}((\nu^*L^{\otimes n})_{|C},\mathcal{O}_{Y\times T})\to\mathrm{Ext}^n_{Y\times T}(\nu^*L^{\otimes n}_{|Y},\mathcal{O}_{Y\times T})$;
(d) s and t are the natural maps.

We know that the coboundary map of b, factorizes through the natural coboundary map $\alpha:\pi_*(\mathcal{Q}\otimes\nu^*N_C^{\otimes n})\to R^n\pi_*(\mathcal{Q})$. This precisely implies that $t\circ\mu(b)=0$. Therefore (1.19) implies that $\mu'\circ h'\circ r(c)=0$. The Lemma follows from the fact that r and $\mu'\circ h'$ are injective.

Injectivity of r: if $n=1$, i.e. $\mathcal{Y}=\mathcal{C}$, r is just the identity (compare (c) above). If $n>1$, from chasing in the Koszul resolution of $\mathcal{O}_C$ as $\mathcal{O}_Y$-module it follows that the injectivity of r holds as soon as $\mathrm{Ext}^{n-i}_{Y\times T}(\nu^*L^{\otimes n-i}_{|Y},\mathcal{O}_{Y\times T})=0$ for $i=1,\dots,n-1$. But these are simply $H^{n-i}(Y\times T,L^{\otimes i-n}_{|Y}\boxtimes\mathcal{O}_T)$ and the result follows easily from Künneth decomposition, Serre vanishing and Serre duality.

Injectivity of $\mu'\circ h'$: We have that $\mathrm{Ext}^n_{X\times T}(\mathcal{Q}\otimes\nu^*L^{\otimes n},\mathcal{Q}_{|Y\times T})\cong H^n(Y\times T,L^{-n}_{|Y}\boxtimes\mathcal{O}_T)$.
If L is a sufficiently positive, it follows as above that this is isomorphic to
$H^n(Y,L^{-n}_{|Y})\otimes H^0(T,\mathcal{O}_T)$. Therefore $\mu'\circ h'$ is identified to the H^0 of the following map of $\mathcal{O}_T$-modules

$$H^n(Y,L^{-n}_{|Y})\otimes\mathcal{O}_T\to\mathcal{H}om_T(\pi_*(\mathcal{Q}\otimes\nu^*L^{\otimes n}),R^n\pi_*(\mathcal{Q}_{|Y\times T})) \tag{1.21}$$

Since the source is torsion-free, the injectivity of $\mu'\circ h'$ holds as soon as (1.21) is injective at a general fiber. For a closed point $t\in T$, let $\mathcal{Q}_t=\mathcal{Q}_{|X\times\{t\}}$. By base change, for t general, the map (1.21) at the fiber over t is

$$H^n(Y,L^{\otimes -n}_{|Y})\to H^0(X,Q_t\otimes L^{\otimes n})^\vee\otimes H^n(Y,\mathcal{Q}_{t|Y}). \tag{1.22}$$

that is the the Serre-dual of the multiplication map of global sections

$$H^0(X,\mathcal{Q}_t\otimes L^{\otimes n})\otimes H^0(Y,(\omega_X\otimes\mathcal{Q}_t^{-1}\otimes L)_{|Y})\to H^0(Y,(\omega_X\otimes L^{\otimes n+1})_{|Y}). \tag{1.23}$$

At this point a standard argument with Serre vanishing shows that (1.23) is surjective as soon as L is sufficiently positive. This will proves the injectivity of $\mu'\circ h'$. This concludes also the proof of the Lemma. $\square$

2. Proof of Corollary C

The deduction of Corollary C from Theorem A is a standard argument with Serre vanishing. However, there are some complications due to the weakness of the assumptions on the singularities of the variety X.

Definition of the gaussian map on the ambient variety $\mathbf{X} \times \mathbf{T}$. The argument makes use of a (dual) gaussian map defined on the ambient variety $X \times T$ itself. Namely, for a line bundle A on X we define $\mathcal{M}^X_{A,\mathcal{Q}}$ as the kernel of the relative evaluation map

$$\pi^*\pi_*(\mathcal{Q}\otimes\nu^*A)\to\mathcal{Q}\otimes\nu^*A$$

As in (0.4) and Subsection 1.1 there is the isomorphism

$$\mathcal{M}^X_{A,\mathcal{Q}}\cong p_*(\mathcal{I}_{\widetilde{\Delta}_X}\otimes q^*(\mathcal{Q}\otimes\nu^*A)) \tag{2.1}$$

and the differentiation map $\mathcal{M}^X_{A,\mathcal{Q}}\to\mathcal{Q}\otimes\nu^*(\Omega^1_X\otimes A)$.

Now, taking as $A=L^{\otimes n}$ and taking $\mathrm{Ext}^{n+1}_{X\times T}(\,\cdot\,,\mathcal{Q}\otimes\nu^*L^\vee)$ we get the desired dual gaussian map on X:

$$g_X:\mathrm{Ext}^{n+1}_X(\Omega^1_X\otimes L^{\otimes n+1},\mathcal{O}_X)\to\mathrm{Ext}^{n+1}_{X\times T}(\mathcal{M}_{L^{\otimes n},\mathcal{Q}},Q\otimes\nu^*L^\vee)\ .$$

First step. We consider the commutative diagram

$$\begin{array}{ccc}\mathrm{Ext}^n_Y(\Omega^1_Y\otimes N_C^{\otimes n},\mathcal{O}_Y) & \xrightarrow{g} & \mathrm{Ext}^n_{Y\times T}(\mathcal{F}^Y_{N_C^{\otimes n},\mathcal{Q}_{|Y\times T}},\mathcal{Q}_{Y\times T}) \\ \Big\downarrow{\scriptstyle\mu} & & \Big\downarrow{\scriptstyle\eta} \\ \mathrm{Ext}^{n+1}_X(\Omega^1_X\otimes L^{\otimes n+1},\mathcal{O}_X) & \xrightarrow{g_X} & \mathrm{Ext}^{n+1}_{X\times T}(\mathcal{M}^X_{L^{\otimes n},\mathcal{Q}},\mathcal{Q}\otimes\nu^*L^\vee)\end{array} \tag{2.2}$$

where, as in the previous section, g denotes the main character, namely the (dual) gaussian map $g_{N_C^{\otimes n},\mathcal{Q}_{Y\times T}}$. The maps μ and η are the natural ones, and the definition is left to the reader. However such maps are most easily seen by considering the commutative diagram

$$\begin{array}{ccc}H^q(X\times T,\mathcal{M}^X_{L^{\otimes n},\mathcal{Q}}\otimes\mathcal{Q}^\vee\otimes\nu^*(\omega_X\otimes L)) & \xrightarrow{g'_X} & H^0(X,\Omega^1_X\otimes L^{\otimes n+1}\otimes\omega_X) \\ \Big\downarrow{\scriptstyle\eta'} & & \Big\downarrow{\scriptstyle\mu'} \\ H^q(Y\times T,\mathcal{F}^Y_{N_C^{\otimes n},\mathcal{Q}}\otimes\mathcal{Q}^\vee\otimes\nu^*\omega_Y) & \xrightarrow{g'} & H^0(Y,\Omega^1_Y\otimes L^{\otimes n}\otimes\omega_Y)\end{array} \tag{2.3}$$

where $q=\dim T$. (Note that, since X is Cohen-Macaulay, adjunction formulas for dualizing sheaves do hold. Incidentally we note also that, if X is Gorenstein, then (2.3) is the dual of diagram (2.2).

As it is easy to see, after tensoring with $\omega_C\otimes N_C$ the restricted normal sequence (0.5) remains exact:

$$0\to\omega_C\to\Omega^1_X\otimes L\otimes\omega_C\to\Omega^1_Y\otimes N_C\otimes\omega_C\to 0 \tag{2.4}$$

Therefore e defines naturally a linear functional on $H^0(\Omega^1_Y\otimes N_C\otimes\omega_C)$ (compare also (2.9) below), still denoted by e. We have

Claim 2.1. *If L is sufficiently positive, then the map g'_X is surjective, while cokerμ' is one-dimensional since (cokerμ')$^\vee$ is spanned by e.*

Proof. The Claim for μ' follows immediately from the exact sequence (2.4). This is because Serre vanishing ensures the surjectivity of the restriction

$$H^0(\Omega^1_X \otimes L^{\otimes n} \otimes \omega_X) \to H^0(\Omega^1_X \otimes L \otimes \omega_C).$$

Concerning the surjectivity of the map g'_X, we first note that by Serre vanishing,

$$R^i p_*(\mathcal{I}_{\widetilde{\Delta}_X} \otimes q^*(\nu^* L^{\otimes n} \otimes \mathcal{Q})) = \begin{cases} 0 & \text{for } i > 0 \\ \text{locally free} & \text{for } i = 0 \end{cases} \tag{2.5}$$

Now we project on T. A standard computation using (2.5), base-change and Serre vanishing, Leray spectral sequence and Künneth decomposition show that the map g'_X is identified to

(2.6)

$$H^q\big(T, \pi_*\big(p_*\big(\mathcal{I}_{\widetilde{\Delta}_X} \otimes q^*(\nu^* L^{\otimes n} \otimes \mathcal{Q})\big) \otimes \nu^* L \otimes \mathcal{Q}^{-1} \otimes \omega_{X\times T}\big)\big) \to H^q(T, H^0(X, \Omega^1_X \otimes L^{\otimes n+1}) \otimes \omega_T))$$

This the H^q of a map of coherent sheaves on the q-dimensional variety T. Hence the surjectivity of (2.6) is implied by the generic surjectivity of the map of sheaves itself. By base change, at a generic fibre $X \times t$ the map of sheaves is the gaussian map

$$\gamma_t : H^0(X, p_*(\mathcal{I}_{\Delta_X} \otimes q^*(L^{\otimes n} \otimes \mathcal{Q}_t)) \otimes L \otimes \mathcal{Q}_t^{-1} \otimes \omega_X) \to H^0(X, \Omega^1_X \otimes L^{\otimes n+1} \otimes \omega_X)$$

The map γ_t is defined by restriction to the diagonal in the usual way. Once again it follows from relative Serre vanishing (on $(X \times T) \times_T (X \times T)$) that, as soon as L is sufficiently positive, γ_t is surjective for all t. This proves the surjectivity of the g'_X and concludes the proof of the Claim. $\square$

Last step. If C is Gorenstein, Claim 2.1 achieves the proof of Corollary C. Indeed, diagram (2.2) is dual to diagram (2.3) and it follows that the kernel of our map $g = g^Y_{N_C^{\otimes n}, \mathcal{Q}}$ is at most one-dimensional, spanned by e. In the general case, Corollary C follows in the same way once proved the following

Claim 2.2. *As soon as L is sufficiently positive, the maps g'_X and μ' are respectively Serre-dual of the maps g_X and μ.*

Proof. To prove this assertion for g'_X we note that, concerning its source, by (2.5) the sheaf $\mathcal{M}^X_{L^{\otimes n}, \mathcal{Q}}$ is locally free. Therefore

$$\mathrm{Ext}^{n+1}_{X\times T}(\mathcal{M}^X_{L^{\otimes n}, \mathcal{Q}}, \mathcal{Q} \otimes \nu^* L^\vee) \cong H^q(\mathcal{M}^X_{L^{\otimes n}, \mathcal{Q}} \otimes \mathcal{Q} \otimes \nu^* \omega_X)$$

Next, we show the Serre duality

$$H^0(X, \Omega^1_X \otimes \omega_X \otimes L^{\otimes n+1})^\vee \cong \mathrm{Ext}^{n+1}_X(\Omega^1_X \otimes L^{\otimes n}, L^{-1}) \tag{2.7}$$

Indeed, since, by definition of dualizing complex ([H], Ch.V,§2, Prop. 2.1 at p. 258) in the derived category of X, $\mathcal{O}_X = \mathbf{R}\mathcal{H}om(\omega_X, \omega_X)$, we have that

$$\mathbf{R}\mathrm{Hom}_X(\Omega^1_X \otimes L^{\otimes n+1}, \mathcal{O}_X) = \mathbf{R}\mathrm{Hom}_X(\Omega^1_X \otimes L^{\otimes n+1}, \mathbf{R}\mathcal{H}om(\omega_X, \omega_X)) =$$
$$= \mathbf{R}\mathrm{Hom}_X((\Omega^1_X \otimes \mathcal{O}_X \otimes L^{\otimes n+1}) \underline{\otimes}^{\mathbf{L}} \omega_X, \omega_X)$$

By Serre-Grothendieck duality, this is isomorphic to

$$\mathbf{R}\mathrm{Hom}_k(\mathbf{R}\Gamma(X, (\Omega^1_X \otimes L^{\otimes n+1}) \underline{\otimes}^{\mathbf{L}} \omega_X[n+1]), k) \tag{2.8}$$

The spectral sequence computing $\mathbf{R}\Gamma(X, (\Omega^1_X \otimes L^{\otimes n+1}) \underline{\otimes}^{\mathbf{L}} \omega_X)$ degenerates to the isomorphisms

$$H^i(X, \Omega^1_X \otimes L^{\otimes n+1} \underline{\otimes}^{\mathbf{L}} \omega_X) \cong \bigoplus_i H^0(X, tor^X_i(\Omega^1_X, \omega_X) \otimes L^{\otimes n+1})$$

(if L is sufficiently positive, by Serre vanishing there are only H^0's). Therefore (2.7) follows. This proves the Claim for g'_X.

Concerning μ', at this point it is enough to prove the Serre duality

$$\mathrm{Ext}^n_Y(\Omega^1_{X|C}\otimes L^n,\mathcal{O}_Y)\cong \mathrm{Ext}^1_C(\Omega^1_{Y|C}\otimes L,\mathcal{O}_C)\cong H^0(\Omega^1_C\otimes N\otimes\omega_C)^\vee \tag{2.9}$$

where the first isomorphism is (0.6). Arguing as above, it is enough to prove that the $\mathcal{O}_C$-modules

$$tor^i_C((\Omega^1_Y)_{|C},\omega_C)\otimes N_C$$

have vanishing higher cohomology for all i. For $i>0$ this follows simply because they are supported on points. For $i=0$ note that, by the exact sequence (2.4), it is enough to show that

$$H^1(\Omega^1_X\otimes N_C\otimes\omega_C)=0 \tag{2.10}$$

To prove this, we tensor the Koszul resolution of $\mathcal{O}_C$ as $\mathcal{O}_X$-module with $\Omega^1_X\otimes\omega_X\otimes L^{\otimes n+1}$ getting a complex (exact at the last step on the right)

$$0\to\Omega^1_X\otimes\omega_X\otimes L\to\cdots\to(\Omega^1_X\otimes\omega_X\otimes L^{\otimes n})^{\oplus n}\to\Omega^1_X\otimes\omega_X\otimes L^{\otimes n+1}\to\Omega^1_X\otimes\omega_C\otimes N_C\to 0$$

Since C is not contained in the singular locus of X, the homology sheaves are supported on points. Therefore the required vanishing (2.10) follows from Serre vanishing via a diagram-chase. This concludes the proof of Claim 2.2 and of Corollary C. $\square$

3. Gaussian maps and Fourier-Mukai transform

We begin the proof of Theorem D. This section serves to establish the notation and the setup. The main goal is to show that, when the parameter variety T is an abelian variety, the (dual) gaussian map $g_{N_C^{\otimes n},\mathcal{Q}_{|Y\times T}}$ of the Introduction can be naturally intepreted as a piece of a certain relative version of the classical Fourier-Mukai transform associated to the Poincarè line bundle, applied to a certain space of morphisms.

Assumptions/Notation 3.1. We will keep all the notation and hypotheses of the Introduction. Explicitly:

- let X be a $n+1$-dimensional normal Cohen-Macaulay subvariety of an abelian variety A. As usual we choose an ample line bundle L on X such that we can find n divisors $Y=Y_1,\ldots,Y_n\in X$ such that their intersection in an irreducible curve C. The line bundle $L_{|C}$ is denoted N_C.
- Let $\mathcal{P}$ be a Poincaré line bundle on $A\times\widehat{A}$. We denote

$$\mathcal{Q}=\mathcal{P}_{|X\times\widehat{A}}\qquad\text{and}\qquad\mathcal{R}=\mathcal{P}_{|Y\times\widehat{A}}$$

- ν and π are the projections of $Y\times\widehat{A}$.
- We assume that the line bundle $\nu^*L^{\otimes n}\otimes\mathcal{Q}$ is relatively base point-free, namely the evaluation map $\pi^*\pi_*(\nu^*L^{\otimes n}\otimes\mathcal{Q})\to\nu^*L^{\otimes n}\otimes\mathcal{Q}$ is surjective.
- p, q and $\widetilde{\Delta}$ are the projections and the diagonal of $(Y\times\widehat{A})\times_{\widehat{A}}(Y\times\widehat{A})$
- The gaussian map of the Introduction (see (0.7)) is

(3.1)
$$g=g_{N_C^{\otimes n},\mathcal{R}}:\mathrm{Ext}^1_C(\Omega^1_Y\otimes N_C,\mathcal{O}_C)\cong\mathrm{Ext}^n_Y(\Omega^1_Y\otimes N_C,\mathcal{O}_Y)\to\mathrm{Ext}^n_{Y\times\widehat{A}}(p_*(q^*(\mathcal{I}_{\widetilde{\Delta}}\otimes\mathcal{R}\otimes\nu^*N_C^{\otimes n})),\mathcal{R})$$

obtained as (the restriction to the relevant Künneth direct summand) of $\mathrm{Ext}^n_{Y\times\widehat{A}}(\,\cdot\,,\mathcal{R})$ of the differentiation (i.e. restriction to the diagonal) map (see also Subsection 1.1).
- The projections of $Y\times A$ will be denoted p_1 and p_2.

Warning 3.2. Since the variety X is assumed to be smooth in codimension 1, and our arguments will concern a sufficiently positive line bundle L, we could have assumed from the beginning that the curve C is smooth and the divisor Y is smooth along Y. This would simplify some points of the argument. For example, under that assumption the restricted equisingular normal sheaf of the next section becomes a honest locally free restricted normal bundle, making much easier some of the steps (e.g. Prop. 4.2(b) below). However, in view of possible future generalizations, we preferred to develop the arguments in greater generality as far as we could, making the smoothness asumption only at the end of the proof. See also Remarks 4.1 and 5.5 below.

mFourier-Mukai transform. Now we consider the trivial abelian scheme $Y \times A \to Y$ and its dual $Y \times \widehat{A} \to Y$. The Poincaré line bundle $\mathcal{P}$ induces naturally a Poincaré line bundle $\widetilde{\mathcal{P}}$ on $(Y \times A) \times_Y (Y \times \widehat{A})$ (namely the pullback of $\mathcal{P}$ to $Y \times A \times \widehat{A}$) and we consider the functors

$$\mathbf{R}\Phi : \mathbf{D}(Y \times A) \to \mathbf{D}(Y \times \widehat{A}) \qquad \text{and} \qquad \mathbf{R}\Psi : \mathbf{D}(Y \times \widehat{A}) \to \mathbf{D}(Y \times A)$$

defined respectively by $\mathbf{R}\pi_{Y\times\widehat{A}*}(\pi^*_{Y\times A}(\,\cdot\,) \otimes \tilde{P})$ and $\mathbf{R}\pi_{Y\times A*}(\pi^*_{Y\times\widehat{A}}(\,\cdot\,) \otimes \tilde{P})$. By Mukai's theorem ([Mu2] Thm 1.1) they are equivalences of categories, more precisely

$$\mathbf{R}\Psi \circ \mathbf{R}\Phi \cong (-1)^*[-q] \qquad \text{and} \qquad \mathbf{R}\Phi \circ \mathbf{R}\Psi \cong (-1)^*[-q] \tag{3.2}$$

It follows that, if $\mathcal{F}$ and $\mathcal{G}$ are sheaves on $Y \times A$ then we have the functorial isomorphism

$$FM_i : \mathrm{Ext}^i_{Y\times A}(\mathcal{F}, \mathcal{G}) \xrightarrow{\cong} \mathrm{Ext}^i_{Y\times\widehat{A}}(\mathbf{R}\Phi(\mathcal{F}), \mathbf{R}\Phi(\mathcal{G})) \tag{3.3}$$

(note that the Ext-spaces on the right are usually hyperexts).

The gaussian map. Now we focus on the target of the gaussian map (3.1). Let $\Delta_Y \subset Y \times A$ be the graph of the embedding $Y \hookrightarrow A$. In other words, Δ_Y is the diagonal of $Y \times Y$, seen as subscheme of $Y \times A$. It follows from the definitions that

$$\mathbf{R}\Phi_{\tilde{\mathcal{P}}}(\mathcal{O}_{\Delta_Y}) = \mathcal{P}_{|Y\times\widehat{A}} = \mathcal{R} \tag{3.4}$$

Moreover, we have that

$$p_*(q^*(\mathcal{I}_{\widetilde{\Delta}} \otimes \mathcal{R} \otimes \nu^* N_C^{\otimes n})) \cong R^0\Phi(\mathcal{I}_{\Delta_Y} \otimes p_2^* N_C^{\otimes n}) \tag{3.5}$$

This is because

$$(Y \times Y) \times_Y (Y \times \widehat{A}) \cong Y \times Y \times \widehat{A} \cong (Y \times \widehat{A}) \times_{\widehat{A}} (Y \times \widehat{A})$$

and, via such isomorphisms, $\tilde{\mathcal{P}}_{|(Y\times Y)\times_Y(Y\times\widehat{A})} \cong q^*(\mathcal{P}_{|Y\times\widehat{A}}) = q^*(\mathcal{R})$.

Moreover, for any sheaf $\mathcal{F}$ supported on $Y \times C$ (as $\mathcal{I}_{\Delta_Y} \otimes \nu^* N_C^{\otimes n}$) we have that $R^i\Phi(\mathcal{F}) = 0$ for $i > 1$. Therefore the fourth quadrant spectral sequence

$$\mathrm{Ext}^p_{Y\times\widehat{A}}(R^q\Phi(\mathcal{F}), \mathcal{R}) \Rightarrow \mathrm{Ext}^{p-q}(\mathbf{R}\Phi(\mathcal{F}), \mathcal{R})$$

reduces to a long exact sequence

$$\cdots \to \mathrm{Ext}^{i-1}_{Y\times\widehat{A}}(R^0\Phi(\mathcal{F}), \mathcal{R}) \to \mathrm{Ext}^{i+1}_{Y\times\widehat{A}}(R^1\Phi(\mathcal{F}), \mathcal{R}) \to \mathrm{Ext}^i_{Y\times\widehat{A}}(\mathbf{R}\Phi(\mathcal{F}), \mathcal{R}) \to \mathrm{Ext}^i_{Y\times\widehat{A}}(R^0\Phi(\mathcal{F}), \mathcal{R}) \to \cdots \tag{3.6}$$

Putting together all that we are led to the following diagram, with right column exact in the middle

$$\begin{tikzcd}
\mathrm{Ext}^n_Y(\Omega^1_Y\otimes N_C^{\otimes n},\mathcal{O}_Y) \arrow[d, "="] & \\
\mathrm{Ext}^n_{\Delta_Y}((\mathcal{I}_{\Delta_Y}\otimes p_2^*N_C^{\otimes n})_{|\Delta_Y},\mathcal{O}_{\Delta_Y}) \arrow[d, "u"] & \mathrm{Ext}^{n+1}_{Y\times\widehat{A}}(R^1\Phi_{\widetilde{\mathcal{P}}}(\mathcal{I}_{\Delta_Y}\otimes p_2^*N_C^{\otimes n}),\mathcal{R}) \arrow[d, "\alpha"] \\
\mathrm{Ext}^n_{Y\times A}(\mathcal{I}_{\Delta_Y}\otimes p_2^*N_C^{\otimes n},\mathcal{O}_{\Delta_Y}) \arrow[r, "FM_n", "\cong"'] & \mathrm{Ext}^n_{Y\times\widehat{A}}(\mathbf{R}\Phi_{\widetilde{\mathcal{P}}}(\mathcal{I}_{\Delta_Y}\otimes p_2^*N_C^{\otimes n}),\mathcal{R}) \arrow[d, "\beta"] \\
 & \mathrm{Ext}^n_{Y\times\widehat{A}}(R^0\Phi_{\widetilde{\mathcal{P}}}(\mathcal{I}_{\Delta_Y}\otimes p_2^*N_C^{\otimes n}),\mathcal{R})
\end{tikzcd} \tag{3.7}$$

where u is the natural map (see also (4.13) below). In conclusion, the kernel of the gaussian map can be described as follows

Lemma 3.3. *The gaussian map $g = g_{N_C^{\otimes n},\mathcal{R}}$ of (3.1) is the composition $\beta\circ FM_n\circ u$. Therefore*

$$\ker(g)\cong Im(FM_n\circ u)\cap Im(\alpha)$$

Proof. The identification of the two maps follows from (3.5) and simply because they are defined in the same way. □

4. Cohomological computations on $Y\times A$ and a reduction of Theorem D

In this section we describe the source of the Fourier-Mukai map FM_n of diagram (3.7) above, and related cohomology spaces of sheaves $\mathcal{F}$ on $Y\times A$ which are supported on $Y\times C$. We will use the Grothendieck duality (or change of rings) spectral sequence

$$\mathrm{Ext}^i_{Y\times C}(\mathcal{F},\mathcal{E}xt^j_{Y\times A}(\mathcal{O}_{Y\times C},\mathcal{O}_{\Delta_Y}))\Rightarrow \mathrm{Ext}^{i+j}_{Y\times A}(\mathcal{F},\mathcal{O}_{\Delta_Y}) \tag{4.1}$$

With this in mind, we compute the sheaves $\mathcal{E}xt^i_{Y\times A}(\mathcal{O}_{Y\times C},\mathcal{O}_{\Delta_Y})$ in Proposition 4.2 below.

4.1. **Preliminaries.**

The following standard identifications will be useful

$$\bigoplus_i \mathcal{E}xt^i_{Y\times A}(\mathcal{O}_{\Delta_Y},\mathcal{O}_{\Delta_Y})\cong\bigoplus_i \Lambda^i T_{A,0}\otimes\mathcal{O}_{\Delta_Y} \tag{4.2}$$

(as graded algebras), where $T_{A,0}$ is the tangent space of A at 0. This holds because Δ_Y is the preimage of 0 via the difference map $Y\times A\to A$, $(y,x)\mapsto y-x$ (which is flat), and $\mathcal{E}xt^\bullet_A(k(0),k(0))$ is $\Lambda^\bullet T_{A,0}\otimes k(0)$.

Moreover, letting $\Delta_C\subset Y\times C\subset Y\times A$ the diagonal of $C\times C$ and

$$\delta_C : C\xrightarrow{\cong}\Delta_C\hookrightarrow Y\times C$$

the natural embedding, we have

$$\bigoplus_i \mathcal{E}xt^i_{Y\times A}(\mathcal{O}_{\Delta_C},\mathcal{O}_{\Delta_Y})\cong\bigoplus_i \Lambda^{i-n+1}T_{A,0}\otimes\delta_{C*}N_C^{\otimes n-1} \tag{4.3}$$

(as graded modules on the algebra above). This is seen as follows: since C is the complete intersection of $n-1$ divisors of Y, all of them in $|L_{|Y}|$, $\mathcal{E}xt^j_{\Delta_Y}(\mathcal{O}_{\Delta_C},\mathcal{O}_{\Delta_Y})=0$ if $j\neq n-1$ and $\mathcal{E}xt^{n-1}_{\Delta_Y}(\mathcal{O}_{\Delta_C},\mathcal{O}_{\Delta_Y})=\delta_{C*}N_C^{n-1}$. Therefore (4.3) follows from (4.2) and the spectral sequence

$$\mathcal{E}xt^h_{\Delta_Y}(\mathcal{O}_{\Delta_C},\mathcal{E}xt^j_{Y\times A}(\mathcal{O}_{\Delta_Y},\mathcal{O}_{\Delta_Y}))\Rightarrow \mathcal{E}xt^{h+j}_{Y\times A}(\mathcal{O}_{\Delta_C},\mathcal{O}_{\Delta_Y}).$$

4.2. **The (equisingular) restricted normal sheaf.** We consider the $\mathcal{O}_C$-module $\mathcal{N}'$ defined by the sequence

$$0\to(\mathcal{T}_Y)_{|C}\to(\mathcal{T}_A)_{|C}\to\mathcal{N}'\to 0 \tag{4.4}$$

When $Y=C$ the sheaf $\mathcal{N}'$ is usually called the *equisingular normal sheaf* ([S] Prop. 1.1.9). Therefore we will refer to $\mathcal{N}'$ as the restricted equisingular normal sheaf.

Remark 4.1. Note that, since X is non-singular in codimension one then the curve C can be taken to be smooth and the divisor Y smooth along C so that $\mathcal{N}'$ is locally free and it is the restriction to C of the normal sheaf of Y. Eventually we will make this assumption in the last section. However the computations of the present section work in the more general setting.

The sheaves $\mathcal{E}xt^j_{Y\times A}(\mathcal{O}_{Y\times C},\mathcal{O}_{\Delta_Y})$ appearing in (4.1) are described as follows

Proposition 4.2. (a)

$$\bigoplus_i\mathcal{E}xt^i_{Y\times A}(\mathcal{O}_{Y\times C},\mathcal{O}_{\Delta_Y})\cong\bigoplus_i\delta_{C*}(\Lambda^{i-n+1}\mathcal{N}'\otimes N_C^{\otimes n-1})$$

(as graded modules on the algebra (4.2)). In particular the left hand side it is zero for $i<n-1$.
(b)

$$\mathcal{E}xt^{q-1}_{Y\times A}(\mathcal{O}_{Y\times C},\mathcal{O}_{\Delta_Y})\cong\omega_C\ .$$

Proof. (a) We apply $\mathcal{H}om_{Y\times A}(\,\cdot\,,\mathcal{O}_{\Delta_Y})$ to the basic exact sequence

$$0\to\mathcal{I}_{\Delta_C/Y\times C}\to\mathcal{O}_{Y\times C}\to\mathcal{O}_{\Delta_C}\to 0 \tag{4.5}$$

where $\mathcal{I}_{\Delta_C/Y\times C}$ denotes the ideal of Δ_C in $Y\times C$. Since Δ_C is the intersection (in $Y\times A$) of Δ_Y and $Y\times C$, the resulting long exact sequence is chopped in short exact sequences (where we plug the isomorphism (4.3))

$$0\to\mathcal{E}xt^{i-1}_{Y\times A}(\mathcal{I}_{\Delta_C/Y\times C},\mathcal{O}_{\Delta_Y})\to\Lambda^{i-n+1}T_{A,0}\otimes N_C^{\otimes n-1}\to\mathcal{E}xt^i_{Y\times A}(\mathcal{O}_{Y\times C},\mathcal{O}_{\Delta_Y})\to 0\ . \tag{4.6}$$

This proves that

$$\mathcal{E}xt^i_{Y\times A}(\mathcal{O}_{Y\times C},\mathcal{O}_{\Delta_Y})=\begin{cases}0 & \text{if } i<n-1\\ \delta_{C*}N_C^{\otimes n-1} & \text{if } i=n-1\end{cases}\ . \tag{4.7}$$

For $i=n$ it follows from (4.7) and the spectral sequence (4.1) applied to $\mathcal{I}_{\Delta_C/Y\times C}$ that

$$\mathcal{E}xt^{n-1}_{Y\times A}(\mathcal{I}_{\Delta_C/Y\times C},\mathcal{O}_{\Delta_Y})\cong\mathcal{H}om_{Y\times C}(\mathcal{I}_{\Delta_C/Y\times C},\mathcal{E}xt^{n-1}_{Y\times A}(\mathcal{O}_{Y\times C},\Delta_Y)\cong\delta_{C*}(\mathcal{T}_Y\otimes N_C^{\otimes n-1})$$

and that (4.6) is identified to (4.4), tensored with $N^{n-1}_{|C}$, i.e.

$$0\to\mathcal{T}_Y\otimes N_C^{\otimes n-1}\to T_{A,0}\otimes N_C^{\otimes n-1}\to\mathcal{N}'\otimes N_C^{\otimes n-1}\to 0 \tag{4.8}$$

This proves the statement for $i=n$. For $i>n$, Proposition 4.2 follows by induction. Indeed $\mathcal{E}xt^\bullet_{Y\times A}(\mathcal{O}_{Y\times C},\mathcal{O}_{\Delta_Y})$ is naturally a graded-module over the exterior algebra $\mathcal{E}xt^\bullet_{Y\times A}(\mathcal{O}_\Delta,\mathcal{O}_{\Delta_Y})\cong\big(\Lambda^\bullet T_{A,0}\big)\otimes\mathcal{O}_{\Delta_Y}$ (see (4.2)). Assume that the statement of the present Proposition holds for the positive integer $i-1$. Because of the action of the exterior algebra the sequences (4.6) and (4.8) yield that the kernel of the surjection

$$\Lambda^{i-n+1}T_{A,0}\otimes N_C^{\otimes n-1}\to \mathcal{E}xt^i_{Y\times A}(\mathcal{O}_{Y\times C},\mathcal{O}_\Delta)\to 0$$

is surjected (up to twisting with $N_C^{\otimes n-1}$) by $\Lambda^{i-n}T_{A,0}\otimes(\mathcal{T}_Y)_{|C}$. Therefore $\mathcal{E}xt^i_{Y\times A}(\mathcal{O}_{Y\times C},\mathcal{O}_\Delta)$ is equal to $(\Lambda^{i-n+1}\widetilde{\mathcal{N}}')\otimes N_C^{\otimes n-1}$. This proves (a).

(b) If Y is smooth along C then $\mathcal{N}'$ is locally free (coinciding with the restricted normal bundle) (see Remark 4.1). In this case (b) follows at once from (a). In the general case the proof is as follows. We claim that for each i the left hand side of (a) can be alternatively described as follows

$$\mathcal{E}xt^i_{Y\times A}(\mathcal{O}_{Y\times C},\mathcal{O}_{\Delta_Y})\cong \mathcal{T}or^{Y\times A}_{q-1-i}(p_2^*\omega_C,\mathcal{O}_{\Delta_Y})$$

This is proved by means of the isomorphism of functors

$$\mathbf{R}\mathrm{Hom}_{Y\times A}(\mathcal{O}_{Y\times C},\mathcal{O}_{\Delta_Y})\cong \mathbf{R}\mathrm{Hom}_{Y\times A}(\mathcal{O}_{Y\times C},\mathcal{O}_{Y\times A})\underline{\otimes}^{\mathbf{L}}_{Y\times A}\mathcal{O}_{\Delta_Y}$$

and the corresponding spectral sequences. In fact, since C is Cohen-Macaulay, we have that $\mathcal{E}xt^i_{Y\times A}(\mathcal{O}_{Y\times C},\mathcal{O}_{Y\times A})=0$ for $i\neq q-1$ and equal to $p_2^*\omega_C$ for $i=n-1$. The spectral sequence computing the right hand side degenerates, proving the claim.

In particular, for $i=q-1$ we have $\mathcal{E}xt^{q-1}_{Y\times A}(\mathcal{O}_{Y\times C},\mathcal{O}_{\Delta_Y})\cong(p_2^*\omega_C)\otimes\mathcal{O}_{\Delta_Y}\cong\delta_{C*}\omega_C$ $\square$

4.3. **Reduction of the statement of Theorem D.** As a first application of Prop. 4.2, we reduce the statement of Theorem D – in the equivalent formulation provided by Lemma 3.3 – to a simpler statement. This will involve the issue of "comparing two spaces of first-order deformations" mentioned in the Introduction (after the statement of Theorem D). All this will the content of Proposition 4.4 below.

We consider the first spectral sequence (4.1) applied to $\mathcal{F}=p_2^*N_C^{\otimes n}$, rather than to $\mathcal{I}_\Delta\otimes p_2^*N_C^{\otimes n}$. Plugging the identification provided by Lemma 4.2 we get

$$H^j(C,\Lambda^{i-n+1}\mathcal{N}'\otimes N_C^{-1})\Rightarrow \mathrm{Ext}^{j+i}_{Y\times A}(p_2^*N_C^{\otimes n},\mathcal{O}_{\Delta_Y})$$

Since the H^i's on the left are zero for $i\neq 0,1$, the spectral sequence is reduced to short exact sequences

$$0\to H^1(C,\Lambda^{i-n}\mathcal{N}'\otimes N_C^{-1})\xrightarrow{v_i}\mathrm{Ext}^i_{Y\times A}(p_2^*N_C^{\otimes n},\mathcal{O}_{\Delta_Y})\xrightarrow{w_i}H^0(C,\Lambda^{i-n+1}\mathcal{N}'\otimes N_C^{-1})\to 0 \tag{4.9}$$

At the beginning, for $i=n$, we have in fact the exact sequence

$$0\to H^1(C,N_C^{-1})\xrightarrow{v_n}\mathrm{Ext}^n_{Y\times A}(p_2^*N_C^{\otimes n},\mathcal{O}_{\Delta_Y})\xrightarrow{w_n}H^0(C,\mathcal{N}'\otimes N_C^{-1})\to 0$$

Combining with the exact sequence coming from the spectral sequence (3.6), applied to $\mathcal{F}=p_2^*N_C^{\otimes n-1}$ we get

$$\begin{tikzcd}
H^1(C,N_C^{-1}) \arrow[d, "v_n"] & \mathrm{Ext}^{n+1}_{Y\times\widehat{A}}(R^1\Phi(p_2^*N_C^{\otimes n}),\mathcal{R}) \arrow[d, "a_n"] \\
\mathrm{Ext}^n_{Y\times A}(p_2^*N_C^{\otimes n},\mathcal{O}_{\Delta_Y}) \arrow[r, "FM_n", "\cong"'] \arrow[d, "w_n"] & \mathrm{Ext}^i_{Y\times\widehat{A}}(\mathbf{R}\Phi(p_2^*L_C^{\otimes n}),\mathcal{R}) \arrow[d, "b_n"] \\
H^0(C,\mathcal{N}'\otimes N_C^{-1}) & \mathrm{Ext}^i_{Y\times\widehat{A}}(R^0\Phi(p_2^*N_C^{\otimes n}),\mathcal{R})
\end{tikzcd} \tag{4.10}$$

Remark 4.3. Note that, as shown by the exact sequence (4.4) defining the restricted equisingular normal sheaf, we get that

$$H^0(C,\mathcal{N}'\otimes N_C^\vee)=\ker(H^1(C,\mathcal{T}_Y\otimes N_C^\vee)\xrightarrow{G}H^1(C,\mathcal{T}_A\otimes N_C^\vee))$$

This map G is the (restriction to $H^1(C,\mathcal{T}_Y\otimes N_C^\vee)$) of the map G_A^Y of the Introduction (see also Remark 5.3 below).

Proposition 4.4. *If the map a_n is non-zero then the map $w_n\circ FM_n^{-1}\circ a_n$ is non-zero and its image is contained in the kernel of the gaussian map (3.1).*

Proof. We apply $\mathrm{Ext}^n_{Y\times A}(\cdot\,,\mathcal{O}_{\Delta_Y})$ to the usual exact sequence

$$0\to\mathcal{I}_{\Delta_C/Y\times C}\otimes p_2^*N_C^{\otimes n}\to p_2^*N_C^{\otimes n}\to\delta_{C*}N_C^{\otimes n}\to 0 \tag{4.11}$$

Using the spectral sequence (4.1) and the isomorphisms provided by Prop. 4.2 we get the commutative exact diagram

$$\begin{tikzcd}
H^1(C,N_C^{-1})\otimes\Lambda^0T_{A,0} \arrow[r, "="] \arrow[d, "\|"] & H^1(C,N_C^{-1})\otimes\Lambda^0T_{A,0} \arrow[d] & \\
H^1(C,N_C^{-1})\otimes\Lambda^0T_{A,0} \arrow[r, tail, "v_n"] & \mathrm{Ext}^n_{Y\times A}(p_2^*N_C^{\otimes n},\mathcal{O}_{\Delta_Y}) \arrow[r, two heads, "w_n"] \arrow[d, "f"] & H^0(C,\mathcal{N}'\otimes N_C^{-1}) \arrow[d, tail] \\
& \mathrm{Ext}^n_{Y\times A}(\mathcal{I}_{\Delta_C/Y\times C}\otimes p_2^*N_C^{\otimes n},\mathcal{O}_{\Delta_Y}) \arrow[d] & H^1(T_Y\otimes N_C^{-1}) \arrow[l, tail, "u"'] \arrow[d, "G_A^Y"] \\
& H^1(C,N_C^{-1})\otimes T_{A,0} \arrow[r, "="] & H^1(C,N_{|C}^{-1})\otimes T_{A,0}
\end{tikzcd} \tag{4.12}$$

where:
- we have used Lemma 4.3 to compute

$$\mathrm{Ext}^i_{Y\times A}(\delta_{C*}N_C^{\otimes n},\mathcal{O}_{\Delta_Y})\cong H^1(C,N_C^{-1})\otimes\Lambda^{i-n+1}T_{A,0}$$

and the map

$$u:H^1(\mathcal{T}_Y\otimes N_C^{-1})\to\mathrm{Ext}^n_{Y\times A}(\mathcal{I}_{\Delta_C/Y\times C}\otimes p_2^*N_C^{\otimes n},\mathcal{O}_{\Delta_Y}) \tag{4.13}$$

is the composition of the natural inclusion

$$H^1(\mathcal{T}_Y\otimes N_C^{-1})\hookrightarrow\mathrm{Ext}^1_C(\Omega^1_Y\otimes N_C,\mathcal{O}_C)\cong\mathrm{Ext}^1_{Y\times C}(\mathcal{I}_{\Delta_C/Y\times C}\,,\,\delta_{C*}N_C^{-1})\cong$$
$$\cong\mathrm{Ext}^1_{Y\times C}(\mathcal{I}_{\Delta_C/Y\times C}\otimes p_2^*N_C^{\otimes n},\mathcal{E}xt^{n-1}_{Y\times A}(\mathcal{O}_{Y\times C},\mathcal{O}_{\Delta_Y}))$$

(see Prop. 4.2) and the natural injective map, arising in the beginning of the spectral sequence (4.1),

$$\mathrm{Ext}^1_{Y\times C}(\mathcal{I}_{\Delta_C/Y\times C}\otimes p_2^*N_C^{\otimes n},\mathcal{E}xt^{n-1}_{Y\times A}(\mathcal{O}_{Y\times C},\mathcal{O}_{\Delta_Y}))\hookrightarrow\mathrm{Ext}^n_{Y\times A}(\mathcal{I}_{\Delta_C/Y\times C}\otimes p_2^*N_C^{\otimes n},\mathcal{O}_{\Delta_Y})\ .$$

Next, we look at the Fourier-Mukai image of the central column of (4.12). In order to do so, we first apply the Fourier-Mukai transform $\mathbf{R}\Phi$ to sequence (4.11). We claim that we get the exact sequence

$$0\to R^0\Phi(\mathcal{I}_{\Delta_C/Y\times C}\otimes p_2^*N_C^{\otimes n})\to R^0\Phi(p_2^*N_C^{\otimes n})\to\nu^*(N_C^{\otimes n})\otimes\mathcal{R}\to 0$$

and the isomorphism

$$R^1\Phi(\mathcal{I}_{\Delta_C/Y\times C}\otimes p_2^*N_C^{\otimes n})\xrightarrow{\sim}R^1\Phi(p_2^*N_C^{\otimes n}) \tag{4.14}$$

Indeed we have that $R^i\Phi(\delta_{C*}(N_C^{\otimes n})) = \nu^*(N_C^{\otimes n}) \otimes \mathcal{R}$ for $i = 0$ and zero otherwise. The map $R^0\Phi(p_2^*N_C^{\otimes n}) \to \nu^*(N_C^{\otimes n} \otimes \mathcal{R})$ is nothing else but the relative evaluation map

$$\pi^*\pi_*(\nu^*(N_C^{\otimes n}) \otimes \mathcal{R}) \to N_C^{\otimes n} \otimes \mathcal{R}$$

whose surjectivity follows from the assumptions (see Notation/Assumptions 3.1). This proves what claimed.

Now we apply to that $\mathbf{R}\mathrm{Hom}_{Y\times\widehat{A}}(\cdot\,, \mathcal{R})$ and the spectral sequence on the $Y \times \widehat{A}$-side, namely (3.6). We get the exact diagram

(4.15)

$$\begin{tikzcd}
 & H^1(N_C^{-1}) \otimes H^0(\mathcal{O}_{\widehat{A}}) \arrow[r, "="] \arrow[d] & H^1(N_{|C}^{-1}) \otimes H^0(\mathcal{O}_{\widehat{A}}) \arrow[d] \\
\mathrm{Ext}^{n+1}(R^1\Phi(p_2^*N_C^{\otimes n}), \mathcal{R}) \arrow[r, "a_n"] \arrow[d, "\cong"] & \mathrm{Ext}^n(\mathbf{R}\Phi(p_2^*N_C^{\otimes n}), \mathcal{R}) \arrow[r, "b_n"] \arrow[d, "FM_n(f)"] & \mathrm{Ext}^n(R^0\Phi(p_2^*N_C^{\otimes n}), \mathcal{R}) \arrow[d] \\
\mathrm{Ext}^{n+1}(R^1\Phi(p_2^*N_C^{\otimes n}), \mathcal{R}) \arrow[r, "\alpha"] & \mathrm{Ext}^n(\mathbf{R}\Phi(\mathcal{I}_{\Delta^Y_C/Y\times C} \otimes p_2^*N_C^{\otimes n}), \mathcal{R}) \arrow[r, "\beta"] \arrow[d] & \mathrm{Ext}^n(R^0\Phi(\mathcal{I}_{\Delta^Y_C/Y\times C} \otimes p_2^*N_C^{\otimes n}), \mathcal{R}) \arrow[d] \\
 & H^1(N_C^{-1}) \otimes H^1(\mathcal{O}_{\widehat{A}}) \arrow[r, "="] & H^1(N_C^{-1}) \otimes H^n(\mathcal{O}_{\widehat{A}})
\end{tikzcd}$$

where, for brevity, at the place on the left of the second row we have plugged the isomorphism (4.14) into $\mathrm{Ext}^n(\mathbf{R}\Phi_{\widetilde{\mathcal{Q}}}(\mathcal{I}_{\Delta^Y_C/Y\times C} \otimes p_2^*N_C^n), \mathcal{R})$. It follows, in particular, that the map $FM_n(f)$ induces the isomorphism of the images of a_n and α;

$$FM_n(f) : \mathrm{im}(a_n) \xrightarrow{\cong} \mathrm{im}(\alpha) \tag{4.16}$$

A diagram-chase in (4.12) in (4.15) proves the first part of the Proposition, namely that if the map a_n is non-zero then the map $w_n \circ FM_n^{-1} \circ a_n$ is non-zero. The second part follows from Proposition 3.3. $\square$

5. Proof of Theorem D

The strategy of proof of Theorem D now proceeds seeing the exact sequences of diagram (4.10) (the one on the left column is induced by the spectral sequence (4.1) on $Y \times A$ and the one on the right column is induced by the spectral sequence (3.6) on $Y \times \widehat{A}$) as the first homogeneous pieces of two exact sequences of graded modules over the exterior algebra. Namely, for each $i \geq n$ we have

$$\begin{tikzcd}
\bigoplus_i \mathrm{Ext}^1_C(L_{|C}^{\otimes n}, \Lambda^{i-n}\mathcal{N}' \otimes N_C^{\otimes n-1}) \arrow[d, "v_i"] & \bigoplus_i \mathrm{Ext}^{i+1}_{Y\times\widehat{A}}(R^1\Phi(p_2^*N_C^{\otimes n}), \mathcal{R}) \arrow[d, "a_i"] \\
\bigoplus_i \mathrm{Ext}^i_{Y\times A}(p_2^*N_C^{\otimes n}, \mathcal{O}_{\Delta_Y}) \arrow[r, "FM_i", "\cong"'] \arrow[d, "w_i"] & \bigoplus_i \mathrm{Ext}^i_{Y\times\widehat{A}}(\mathbf{R}\Phi(p_2^*N_C^{\otimes n}), \mathcal{R}) \arrow[d, "b_i"] \\
\bigoplus_i \mathrm{Hom}_C(N_C^{\otimes n}, \Lambda^{i-n+1}\mathcal{N}' \otimes N_C^{\otimes n-1}) & \bigoplus_i \mathrm{Ext}^i_{Y\times\widehat{A}}(R^0\Phi(p_2^*N_C^{\otimes n}), \mathcal{R})
\end{tikzcd} \tag{5.1}$$

The exterior algebra acts on the left-hand side as $\Lambda^\bullet T_{A,0} \cong \mathrm{Ext}^\bullet_{Y\times A}(\mathcal{O}_{\Delta_Y}, \mathcal{O}_{\Delta_Y})$ (see (4.2) and (4.3)) and on the right hand side as $\Lambda^\bullet H^1(\mathcal{O}_{\widehat{A}}) \hookrightarrow \mathrm{Ext}^\bullet_{Y\times\widehat{A}}(\mathcal{R},\mathcal{R})$.

5.1. **Computations in degree $\mathbf{q-1}$.** In this subsection we will perform some explicit calculations in degree $q-1$. Although not strictly necessary for the proof of Theorem D, they render the argument more explicit. In degree $q-1$ we have the special feature that the Hom space at the bottom of the left column is naturally isomorphic to $\mathrm{Hom}(N_C^{\otimes n}, \omega_C)$ (Prof. 4.2(b)). The following Proposition shows that what we want to prove in degree n, namely that the map $w_n \circ FM_n^{-1} \circ a_n$ is non-zero, is true, in strong form, in degree $q-1$.

Proposition 5.1. *The map w_{q-1} has a canonical (up to scalar) section σ and the injective map $\tau = (FM_{q-1})_{|Im(\sigma)}$ factorizes trough a_{q-1}. Summarizing, in degree $i = q-1$ diagram (5.1) specializes to*

(5.2)
$$\begin{tikzcd}
\mathrm{Ext}^1_C(L^{\otimes n}_{|C}, \Lambda^{q-1-n}\mathcal{N}' \otimes N_C^{\otimes n-1}) \arrow[d, "v_{q-1}"] & \mathrm{Ext}^q_{Y\times\widehat{A}}(R^1\Phi(p_2^* N_C^{\otimes n}), \mathcal{R}) \arrow[d, "a_{q-1}"] \\
\mathrm{Ext}^{q-1}_{Y\times A}(p_2^* N_C^{\otimes n}, \mathcal{O}_{\Delta_Y}) \arrow[d, "w_{q-1}"] \arrow[r, leftrightarrow, "FM_{q-1}", "\cong"'] \arrow[ur, bend left, "\tau"] & \mathrm{Ext}^{q-1}_{Y\times\widehat{A}}(\mathbf{R}\Phi(p_2^* N_C^{\otimes n}), \mathcal{R}) \arrow[d, "b_{q-1}"] \\
\mathrm{Hom}_C(N_C^{\otimes n}, \omega_C) \arrow[u, bend left, "\sigma"] & \mathrm{Ext}^i_{Y\times\widehat{A}}(R^0\Phi(p_2^* N_C^{\otimes n}), \mathcal{R})
\end{tikzcd}$$

Proof. The section σ is given (up to scalar) by the product map

(5.3)
$$\mathrm{Ext}^{q-1}_{Y\times A}(p_2^* N_C^{\otimes n}, \mathcal{O}_{Y\times A}) \otimes \mathrm{Hom}_{Y\times A}(\mathcal{O}_{Y\times A}, \mathcal{O}_{\Delta_Y}) \xrightarrow{\sigma} \mathrm{Ext}^{q-1}_{Y\times A}(p_2^* N_C^{\otimes n}, \mathcal{O}_{\Delta_Y})$$

In fact, note that,

$$\mathrm{Ext}^{q-1}_{Y\times A}(p_2^* N_C^{\otimes n}, \mathcal{O}_{Y\times A}) \cong p_1^* H^0(\mathcal{O}_Y) \otimes p_2^* \mathrm{Ext}^{q-1}_A(N_C^{\otimes n}, \mathcal{O}_A) \cong p_1^* H^0(\mathcal{O}_Y) \otimes p_2^* \mathrm{Hom}_C(N_C^{\otimes n}, \omega_C)$$

The fact that s is a section of w_{q-1} is clear, as the latter is the natural map $\mathrm{Ext}^{q-1}_{Y\times A}(p_2^* N_C^{\otimes n}, \mathcal{O}_{\Delta_Y}) \to H^0(\mathcal{E}xt^{q-1}_{Y\times A}(p_2^* N_C^{\otimes n}), \mathcal{O}_{\Delta_Y}) \cong H^0(\mathcal{E}xt^{q-1}_{Y\times A}(p_2^* N_C^{\otimes n}, \mathcal{O}_{Y\times A}) \otimes \mathcal{O}_{\Delta_Y}) \cong p_2^* \mathrm{Ext}^{q-1}_A(N_C^{\otimes n}, \mathcal{O}_A) \otimes H^0(\mathcal{O}_{\Delta_Y})$.

Next, we prove the second part of the statement. On the $Y \times \widehat{A}$-side, we consider the following product map

(5.4)
$$\begin{tikzcd}
\mathrm{Hom}_{Y\times\widehat{A}}(R^1\Phi(p_2^* N_C^{\otimes n}), \mathcal{O}_{Y\times\hat{0}}) \otimes \mathrm{Ext}^q_{Y\times\widehat{A}}(\mathcal{O}_{Y\times\hat{0}}, \mathcal{R}) \arrow[r] \arrow[d, "\cong"] & \mathrm{Ext}^q_{Y\times\widehat{A}}(R^1\Phi(p_2^* N_C^{\otimes n}), \mathcal{R}) \arrow[d, "a_{q-1}"] \\
\mathrm{Ext}^{-1}_{Y\times\widehat{A}}(\mathbf{R}\Phi(p_2^* N_C^{\otimes n}), \mathcal{O}_{Y\times\hat{0}}) \otimes \mathrm{Ext}^q_{Y\times\widehat{A}}(\mathcal{O}_{Y\times\hat{0}}, \mathcal{R}) \arrow[r] & \mathrm{Ext}^{q-1}_{Y\times\widehat{A}}(\mathbf{R}\Phi(p_2^* N_C^{\otimes n}), \mathcal{R})
\end{tikzcd}$$

where the vertical isomorphism comes from the usual spectral sequence (3.6). By (3.2) the inverse of the Fourier-Mukai transform is $(-1)^*_A \circ \mathbf{R}\Psi[q]$. We have that

$$(-1)^*_A \circ \mathbf{R}\Psi(\mathcal{O}_{Y\times\hat{0}}) = (-1)^*_A \circ R^0\Psi(\mathcal{O}_{Y\times\hat{0}}) \cong \mathcal{O}_{Y\times A}$$

and, by (3.4),

$$(-1)^*_A \circ \mathbf{R}\Psi(\mathcal{R}) \cong (-1)^*_A \circ R^q\Psi(\mathcal{R}) = \mathcal{O}_{\Delta_Y}$$

Therefore, thanks to Mukai's inversion theorem (3.2), the Fourier-Mukai transform identifies – on the $Y \times A$-side – the source of the bottom row in diagram (5.4) to

$$\mathrm{Ext}^{-1}_{Y\times A}(p_2^* N_C^{\otimes n}[-q], \mathcal{O}_{Y\times A}) \otimes \mathrm{Ext}^q(\mathcal{O}_{Y\times A}, \mathcal{O}_{\Delta_Y}[-q]) \cong$$

$$\mathrm{Ext}^{q-1}_{Y\times A}(p_2^* N_C^{\otimes n}, \mathcal{O}_{Y\times A}) \otimes \mathrm{Hom}_{Y\times A}(\mathcal{O}_{Y\times A}, \mathcal{O}_{\Delta_Y})$$

This concludes the proof of the Proposition. □

5.2. **Proof of Theorem D.**

Notation 5.2. We introduce the following typographical abbreviations on diagram (5.1): the isomorphic (via the Fourier-Mukai transform) spaces of the central row of diagram (5.1) are identified to vector spaces E_i, and we denote V_i, E_i, W_i the spaces appearing in the left column of diagram (5.1) (from top to down), and A_i, E_i, B_i the spaces appearing in the right column (from top to down). We denote also $\Lambda^\bullet T$ the acting exterior algebra. The structure of $\Lambda^\bullet T$-graded modules induces natural maps (we focus on degrees n and $q-1$ as they are the relevant ones in our argument)

$$\begin{tikzcd} & A_n \arrow[d, "a_n"] & \\ V_n \arrow[r, tail, "v_n"] & E_n \arrow[r, two heads, "w_n"] \arrow[d, "b_n"] & W_n \\ & B_n & \end{tikzcd}$$

$$\downarrow \phi$$

(5.5)

$$\begin{tikzcd} & \Lambda^{q-1-n}T^\vee \otimes A_{q-1} \arrow[d, "\tilde a_{q-1}"] & \\ \Lambda^{q-1-n}T^\vee \otimes V_{q-1} \arrow[r, tail, "\tilde v_{q-1}"] & \Lambda^{q-1-n}T^\vee \otimes E_{q-1} \arrow[r, two heads, "\tilde w_{q-1}"] \arrow[d, "\tilde b_{q-1}"] & \Lambda^{q-1-n}T^\vee \otimes W_{q-1} \\ & \Lambda^{q-1-n}T^\vee \otimes B_{q-1} & \end{tikzcd}$$

where we have denoted $\tilde v_{q-1} = \mathrm{id} \otimes v_{q-1}$ and so on. We denote also

$$\phi_{A_n} : A_n \to \Lambda^{q-1-n}T^\vee \otimes A_{q-1}$$

and so on.

At this point we make the following assumption

(*) *the extension class e of the restricted cotangent sequence*

$$0 \to N_C^\vee \to (\Omega^1_X)_{|C} \to (\Omega^1_Y)_{|C} \to 0 \tag{5.6}$$

belongs to the subspace $H^1(\mathcal{T}_Y \otimes N_C^\vee)$ of $\mathrm{Ext}^1_C(\Omega^1_Y \otimes N_C, \mathcal{O}_C)$. Note that if C is smooth and Y is smooth along C this is obvious, since the two spaces coincide.

Remark 5.3. Note that, if (*) holds then e belongs to the subspace $H^0(\mathcal{N}'\otimes N_C^\vee)$ of $H^1(\mathcal{T}_Y\otimes N_C^\vee)$. In essence, this follows from the deformatic-theoretic interpretation: as mentioned in Remark 4.3 from the exact sequence defining the restricted equisingular normal sheaf (4.4) we get that

$$H^0(C,\mathcal{N}'\otimes N_C^\vee)=\ker\big(\ H^1(C,\mathcal{T}_Y\otimes N_C^\vee)\xrightarrow{G}H^1(C,\mathcal{T}_A\otimes N_C^\vee)\ \big)$$

The fact that e belongs to $H^0(\mathcal{N}'\otimes N_C^\vee)$ follows from deformation-theoretic interpretation of this map G (it is the (restriction to $H^1(C,\mathcal{T}_Y\otimes N_C^\vee)$ of the map G_A^Y of the Introduction). More formally, one sees that e belongs to $\ker G$ because the target of G is $\mathrm{Hom}_k(\Omega^1_{A,0},H^1(C,N_C^\vee))$ and G takes an extension class f to the map $\Omega^1_{A,0}\to H^1(N_C^{-1})$ obtained by composing the coboundary map of f with the map $\Omega^1_{A,0}\to H^0((\Omega^1_Y)_{|C})$. If the extension class is (5.6) then this map factorizes through $H^0((\Omega^1_X)_{|C})$, hence $e\in\ker G$.

From diagram (5.5) we have the map

$$\phi_{W_n}:H^0(C,\mathcal{N}'\otimes N_C^\vee)=\ker(G)\to\mathrm{Hom}(\Lambda^{q-1-n}T,H^0(\omega_C\otimes N_C^{-\otimes n}))$$

Lemma 5.4. $\phi_{W_n}(e)\neq 0$.

Proof. Recall that $T=T_{A,0}$. We make the identification $\Lambda^{q-1-n}T_{A,0}\cong\Lambda^{n+1}\Omega^1_{A,0}$. Accordingly $\phi_{W_n}(e)$ is identified to a map

$$\phi_{W_n}(e):\Lambda^{n+1}\Omega^1_{A,0}\to H^0(\omega_C\otimes N_C^\vee)$$

We consider the map

$$\Lambda^{n+1}\Omega^1_{A,0}\to H^0((\Lambda^{n+1}\Omega_X)_{|C})\tag{5.7}$$

obtained as Λ^{n+1} of the co-differential $\Omega^1_{A,0}\otimes\mathcal{O}_C\to(\Omega^1_X)_{|C}$. Since the co-differential is surjective the map (5.7) is non-zero. If C is smooth and X and Y are smooth along C then the target of (5.7) is $H^0((\omega_X)_{|C})=H^0(\Omega_C\otimes N_C^{-n})$. Via the above identifications, the map $\phi_{W_n}(e)$ coincides, up to scalar, with (5.7). The Lemma follows in this case, Even if X is not smooth along C the $\phi_{W_n}(e)$ is the composition of the map (5.7) and the H^0 of the canonical map $\Lambda^{n+1}((\Omega^1_X)_{|C})\to(\omega_X)_{|C}\cong\omega_C\otimes N_C^{-1}$. Such composition is clearly non-zero and the Lemma follows as above. □

At this point, the line of the argument is clear. Since $\phi_{W_n}(e)$ is non-zero, by Lemma 5.4 and by Proposition 5.1 it belongs to the image of $\tilde a_{q-1}$. Thus it must come from a non-zero element of the image of a_n. By Lemma 3.3 and Proposition 4.4 we can apply Corollary C. This will prove that $R^n\pi_*\mathcal{Q}=0$.

However, some verification still has to be made. The point is that, since the maps in diagram (5.5) at degree n does not seem to have any natural splitting compatible with the multiplicative structure, it is not a priori clear that the fact that $\phi_{W_n}(e)$ is non-zero, and belongs to $Im(\tilde a_{q-1})$ implies that there is a $d\in Im(a_n)$ such that $\phi(d)=\phi_{W_n}(e)$. (In fact, by Proposition 4.4, it is enough to show that the map a_n is non-zero). An argument ensuring that this is in fact the case is as follows.

Again, we consider the vector space $\bigoplus_i E_i$ namely

$$\bigoplus_i\mathrm{Ext}^i_{Y\times A}(p_2^*N_C^{\otimes n},\mathcal{O}_{\Delta_Y})\cong\bigoplus_i\mathrm{Ext}^i_{Y\times\widehat A}(\mathbf{R}\Phi(p_2^*N_C^{\otimes n}),\mathcal{R})$$

Besides being a graded module over the exterior algebra $\Lambda^\bullet T$, it is in fact a graded module over the bigraded algebra $H^\bullet(\mathcal{O}_C)\otimes\Lambda^\bullet T$. This follows, for example, from the natural action of $p_2^*H^\bullet(\mathcal{O}_A)$

on $\bigoplus_i \mathrm{Ext}^i_{Y\times A}(p_2^*N_C^{\otimes n}, \mathcal{O}_{\Delta_Y})$, and this action factorizes trough $p_2^*H^\bullet(\mathcal{O}_C)$[12]. The action of $H^\bullet(\mathcal{O}_C)$ on the left column of diagram (5.1) is by cup-product with $H^1(\mathcal{O}_C)$, which is zero for elements of V_i, and belongs to V_{i+1} for elements of W_i (see Notation 5.2). On the right column of (5.1), the action is given by the cup-product $R^0\Phi(N_C^{\otimes n})\otimes H^1(\mathcal{O}_C) \to R^1\Phi(N_C^{\otimes n})$ and taking Ext's. Therefore the product of elements of A_i with $H^1(\mathcal{O}_C)$ is in B_{i+1}, while the product of elements of B_i with $H^1(\mathcal{O}_C)$ is zero.

Now we claim that this cup-product induces an injective map

$$W_{q-1} \to H^1(\mathcal{O}_C)^\vee \otimes V_q \tag{5.8}$$

namely

$$H^0(\omega_C \otimes N_C^{-n}) \to H^1(\mathcal{O}_C)^\vee \otimes H^1(\omega_C \otimes N_C^{-n})$$

Indeed such map is the dual of

$$H^0(N_C^{\otimes n}) \otimes H^1(\mathcal{O}_C) \to H^1(N_C^{\otimes n})$$

which is surjective, since it is the H^1 of the evaluation map $H^0(N_C^{\otimes n})\otimes \mathcal{O}_C \to N_C^{\otimes n}$ and N_C is a line bundle on a curve. This proves (5.8).

Now we are ready for the conclusion of the argument. We consider the commutative diagram

$$\begin{tikzcd}
E_n \arrow[r, "\phi_{E_n}"] \arrow[d] \arrow[dr, "\psi"] & \Lambda^{q-1-n}T^\vee \otimes E_{q-1} \arrow[d, "\eta"] \\
H^1(\mathcal{O}_C)^\vee \otimes E_{n+1} \arrow[r] & \Lambda^{q-1-n}T^\vee \otimes H^1(\mathcal{O}_C)^\vee \otimes E_q
\end{tikzcd}$$

From Lemma 5.4 (and Proposition 5.1) $\phi_{W_n}(e)$ is a non zero element of $e_{q-1} \in \Lambda^{q-1-n}T^\vee \otimes A_{q-1} \subset \Lambda^{q-1-n}T^\vee \otimes E_{q-1}$. By the injectivity of (5.8) $\eta(e_{q-1})$ is non-zero. It follows that there is a $\tilde{e}_n \in E_n$ such that $\psi(\tilde{e}_n) = \eta(e_{q-1})$ is non-zero. Knowing how multiplication with $H^1(\mathcal{O}_C)$ works, this implies that there is a non-zero $d \in Im(a_n)$ such that $\psi(d) = \psi(e_n)$. As explained above, this proves that the gaussian map g is non-injective, hence that $R^n\pi_*\mathcal{Q} = 0$, as soon as assumption *(*)* can be made. This is certainly the case if the ambient variety X is normal, since in this case, for sufficiently positive L, we can take C in the smooth locus of X, C smooth and Y smooth along C.

To prove the vanishing of $R^i\pi_*\mathcal{Q}$ for $i < n$ one takes a sufficiently positive ample line bundle M on X and a $i+1$-dimensional complete intersection of divisors in $|M|$, say X'. It follows easily from relative Serre vanishing that $R^i\pi_*(\mathcal{Q}) = R^1\pi_*(\mathcal{Q}_{|X'\times\widehat{A}})$. Therefore the desired vanishing follows by induction. This concludes the proof of Theorem D.

Remark 5.5. The hypothesis that X is smooth in codimension one is used to ensure that assumption *(*)* can be made.

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