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A PROOF OF LAZARSFELD'S THEOREM ON CURVES ON $K3$ SURFACES

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The Gieseker-Petri theorem, asserted by Petri [P] and proved by Gieseker in [G], states that for a general complex curve C of genus g the following condition (usually referred to as the *Brill-Noether-Petri condition*) holds:

For any line bundle A on C the multiplication map
$$\mu_{0,A}\colon H^0(A)\otimes H^0(\omega_C\otimes A^\vee)\to H^0(\omega_C)$$
is injective.

As is well known, the above theorem plays a central role in Brill-Noether theory (see for instance [ACGH]). Recently Lazarsfeld provided a new approach to the Gieseker-Petri theorem based on the observation that, in principle, there is no evident Brill-Noether theoretic obstruction to embed a smooth curve in a $K3$ surface. Lazarsfeld's result is the following

Theorem 1 ([L]). *Let S be a complex $K3$ surface and L a line bundle on S such that the linear system $|L|$ does not contain reducible or multiple curves. Then the general curve $C\in|L|$, if smooth, satisfies the Brill-Noether-Petri condition.*

This proves the Gieseker-Petri theorem since, as is well known, for any $g\geq 2$ there are $K3$ surfaces S such that $\mathrm{Pic}(S)=\mathbf{Z}\cdot[C]$, with C a smooth irreducible curve of genus g.

The purpose of this note is to give a proof of Theorem 1 which, while following some of Lazarsfeld's ideas, is essentially elementary and self-contained. The argument is infinitesimal in nature and goes as follows: for a line bundle A on $C\in|L|$ one considers its "μ_1 map with respect to the family $|L|$", i.e. the composition of the gaussian map $\mu_{1,A}\colon \ker(\mu_{0,A})\to H^0(\omega_C^{\otimes 2})$ with the transpose of the Kodaira-Spencer map $\delta_{C,S}^\vee\colon H^0(\omega_C^{\otimes 2})\to(T_C|L|)^\vee=H^1(N_{C|S}^\vee\otimes\omega_C)=H^1(\mathcal{O}_C)$. The point is that this map, denoted $\mu_{1,A,S}$, has a nice interpretation in terms of a certain vector bundle

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F_A on S canonically associated, following Lazarsfeld, to the pair (C, A). This allows one to show that if $|L|$ satisfies the hypothesis of the theorem then $\mu_{1,A,S}$ is always injective. From the well-known deformation-theoretic interpretation of the map μ_1 it follows that if C is general in $|L|$ then $\ker(\mu_{0,A})$ has to be zero.

The proof. To start with, let us establish the notation and recall a few basic facts. The deformation-theoretic meaning of the map $\mu_{1,A}$: $\ker(\mu_{0,A}) \to H^0(\omega_C^{\otimes 2})$ (see the proof of Lemma 1 below for a definition) is as follows: let W be a sufficiently small open neighbourhood of the point representing C in $\mathcal{M}_g$. Then there exist a finite cover $\mathcal{M}$ of W and a scheme $\mathcal{W}_d^r$ parametrizing couples (C', A'), where C' is parametrized by $\mathcal{M}$ and A' is a line bundle of C' such that $\deg(A') = d$ and $h^0(A') \geq r+1$ on C' [AC]. Let $\pi\colon \mathcal{W}_d^r \to \mathcal{M}$ be the projection and $d\pi_{(C,A)}\colon T_{(C,A)}\mathcal{W}_d^r \to H^1(T_C)$ the differential. Then, in case $h^0(A) = r+1$, we have that

$$\operatorname{Im}(d\pi_{(C,A)}) = \operatorname{Ann}(\operatorname{Im}(\mu_{1,A})),$$

where $H^0(\omega_C^{\otimes 2})$ is identified to $H^1(T_C)^\vee$ via Serre duality (see e.g. [CGGH, §2(c)]).

Now let S be a regular surface (i.e. $h^1(\mathcal{O}_S) = 0$) and L a line bundle on S. Let C be a smooth curve (if any) in $|L|$, U a suitable open neighborhood of C in $|L|$, and $\mathcal{W}_d^r(U)$ the scheme parametrizing pairs (C', A') where C' is a curve in U and A' is a line bundle on C' of degree d and $h^0(A') \geq r+1$. Moreover, let $\pi_S\colon \mathcal{W}_d^r(U) \to U$ be the projection map. Since S is regular, $T_C|L| = H^0(N_{C|S})$, where $N_{C|S}$ is the normal bundle of C in S. If in addition S is a $K3$ surface $N_{C|S} = \omega_C$. We will prove the following

Theorem 2. *Under the hypotheses of Lazarsfeld's theorem, let C be a smooth curve in $|L|$ and let A be a base-point-free line bundle on C such that* $\deg(A) = d$ *and* $h^0(A) = r+1$. *If the Petri map $\mu_{0,A}$ is not injective, then the derivative*

$$(d\pi_S)_{(C,A)}\colon T_{(C,A)}\mathcal{W}_d^r(U) \to T_C|L| = H^0(N_{C|S})$$

is not surjective.

As we are in characteristic 0, by Sard's lemma, Theorem 2 implies Theorem 1 (to consider base-point-free line bundles only is not restrictive since if Δ is the base locus of a line bundle A such that $\mu_{0,A}$ is not injective then also $\mu_{0,A(-\Delta)}$ is not injective).

In our situation let $\delta_{C,S}\colon H^0(N_{C|S}) \to H^1(T_C)$ be the Kodaira-Spencer map of the family U. Its transpose $\delta^\vee_{C,S}$ is (up to scalar coefficients) the coboundary map of the sequence

$$0 \to N^\vee_{C|S} \otimes \omega_C \to \Omega^1_S \otimes \omega_C \to \omega_C^{\otimes 2} \to 0$$

obtained by tensoring the cotangent sequence with ω_C. From standard first-order deformation theory one has that

$$\mathrm{Im}(d\pi_{S(C,A)}) \subset \delta^{-1}_{C,S}(\mathrm{Im}(\delta_{C,S}) \cap \mathrm{Im}(d\pi_{(C,A)})),$$

i.e., equivalently, that

$$\mathrm{Im}(d\pi_{S(C,A)}) \subset \mathrm{Ann}(\mathrm{Im}(\delta^\vee_{C,S} \circ \mu_{1,A})).$$

Set $\mu_{1,A,S} := \delta^\vee_C \circ \mu_{1,A}\colon \ker(\mu_{0,A}) \to H^1(N^\vee_{C|S} \otimes \omega_C)$, where $H^1(N^\vee_{C|S} \otimes \omega_C)$ is identified to $H^0(N_{C,S})^\vee$ via Serre duality. Theorem 2 is then implied by the following

Proposition. *Let S be a $K3$ surface and L a line bundle on S such that the linear system $|L|$ does not contain reducible or multiple curves. Then for any smooth curve $C \in |L|$ and for any base-point-free line bundle A on C the map $\mu_{1,A,S}$ is injective.*

Proof. Following Lazarsfeld, we associate canonically to A a vector bundle M_A on C and vector bundle F_A on S as follows: M_A is the kernel of the evaluation map $\mathrm{ev}_{C,A}\colon H^0(A) \otimes \mathcal{O}_C \to A$. Since A is base-point-free, we have that M_A is locally free of rank r, sitting in an exact sequence

$$0 \to M_A \to H^0(A) \otimes \mathcal{O}_C \to A \to 0. \tag{1}$$

Note that $\det(M_A) = A^\vee$.

Viewing A as a coherent sheaf on S, let us denote by F_A the kernel of the evaluation map $\mathrm{ev}_{S,A}\colon H^0(A) \otimes \mathcal{O}_S \to A$. Since A is base-point-free, we have that F_A is a locally free sheaf of rank $r+1$ on S, sitting in an exact sequence

$$0 \to F_A \to H^0(A) \otimes \mathcal{O}_S \to A \to 0. \tag{2}$$

By Porteous's formula one has that $\det(F_A) = \mathcal{O}_S(-C)$.

By construction there is a natural surjection $F_{A|C} \to M_A \to 0$. By determinant reasons the kernel is the line bundle $A \otimes N^\vee_{C|S}$. In such a way one obtains canonically an exact sequence

$$0 \to A \otimes N^\vee_{C|S} \to F_{A|C} \to M_A \to 0. \tag{3}$$

Tensoring (3) with $N_{C|S} \otimes A^\vee$ and using that, since S is a $K3$ surface, $N_{C|S} = \omega_C$ one gets an exact sequence

$$(4) \qquad 0 \to \mathscr{O}_C \to F_A \otimes \omega_C \otimes A^\vee \to M_A \otimes \omega_C \otimes A^\vee \to 0.$$

The point is

Lemma 1. *The coboundary map* $H^0(M_A \otimes \omega_C \otimes A^\vee) \to H^1(\mathscr{O}_C)$ *of sequence* (4) *is (up to scalar coefficients) the map* $\mu_{1,A,S}$.

Proof. We follow an argument of C. Voisin [V, Lemma 1.2(b)]. In the first place let us remark that, tensoring sequence (1) with $\omega_C \otimes A^\vee$ and taking cohomology, one has immediately that $\ker \mu_{0,A} = H^0(M_A \otimes \omega_C \otimes A^\vee)$.

Let us recall how the map $\mu_{1,A}$ is obtained. Tensoring the derivation operator $d\colon \mathscr{O}_C \to \omega_C$ with the evaluation map $\mathrm{ev}_{C,A}$ one obtains a map $H^0(A) \otimes \mathscr{O}_C \to \omega_C \otimes A$ whose restriction to M_A is $\mathscr{O}_C$-linear. Tensoring this map with $\omega_C \otimes A^\vee$ one obtains a map of $\mathscr{O}_C$-modules $s\colon M_A \otimes \omega_C \otimes A^\vee \to \omega_C^{\otimes 2}$ and $\mu_{1,A}$ is simply the corresponding map at the global sections level. As above, tensoring the derivation operator $d\colon \mathscr{O}_S \to \Omega^1_S$ with the map $\mathrm{ev}_{S,C}$ one gets a map $H^0(A) \otimes \mathscr{O}_S \to \Omega^1_S \otimes A$ whose restriction to F_A is $\mathscr{O}_S$-linear. Tensoring this map with $\omega_C \otimes A^\vee$ one gets a map of $\mathscr{O}_C$-modules $t\colon F_A \otimes \omega_C \otimes A^\vee \to \Omega^1_S \otimes \omega_C$, fitting in the commutative exact diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{O}_C & \longrightarrow & F_A \otimes \omega_C \otimes A^\vee & \longrightarrow & M_A \otimes \omega_C \otimes A^\vee & \longrightarrow & 0 \\
 & & \| & & \big\downarrow{\scriptstyle t} & & \big\downarrow{\scriptstyle s} & & \\
0 & \longrightarrow & \mathscr{O}_C & \longrightarrow & \Omega^1_S \otimes \omega_C & \longrightarrow & \omega_C^{\otimes 2} & \longrightarrow & 0
\end{array}$$

where the top row is sequence (4). Then the lemma follows immediately. q.e.d.

Let us continue with the proof of the proposition. Because of Lemma 1 to prove the proposition is equivalent to proving that $h^0(F_A \otimes \omega_C \otimes A^\vee) = 1$. To this end let us record some of the properties of the bundle F_A.

Lemma 2. (a) $h^0(F_A) = h^1(F_A) = 0$.

(b) $F_A^\vee$ *is generated by its global sections away from a finite set.*

(c) $h^0(F_A \otimes F_A^\vee) = h^0(F_A \otimes \omega_C \otimes A^\vee)$.

Proof. (a) is obtained taking cohomology in (2), since $h^1(\mathscr{O}_S) = 0$. Dualizing the sequence (2), since $\mathscr{E}xt^1_{\mathscr{O}_S}(\mathscr{O}_C, \mathscr{O}_S) = N_{C|S} = \omega_C$, one gets

$$(5) \qquad 0 \to H^0(A)^\vee \otimes \mathscr{O}_S \to F_A^\vee \to \omega_C \otimes A^\vee \to 0.$$

Since $h^1(\mathscr{O}_S) = 0$, (5) is exact at the global sections level and then (b) follows easily. Finally tensoring (5) with F_A one gets

$$0 \to H^0(A)^\vee \otimes F_A \to F_A \otimes F_A^\vee \to F_A \otimes \omega_C \otimes A^\vee \to 0.$$

Then (a) implies (c). q.e.d.

What we have proved so far is completely general. The hypothesis that $|L|$ contains only reduced and irreducible curves comes into play with the following key observation of Lazarsfeld's.

Lemma 3 [L_1, Lemma 1.3]. *If $h^0(F_A \otimes F_A^\vee) > 1$, i.e., if F_A has nontrivial endomorphisms, then the linear system $|C|$ contains a reducible or multiple curve.*

Then Lemma 3, together with Lemma 1 and Lemma 2(c), proves the proposition. q.e.d.

For the benefit of the reader, we reproduce here the outline of the proof of Lemma 3. Since there are nontrivial endomorphisms, there is some $f: F_A^\vee \to F_A^\vee$ dropping rank everywhere. To see this, one takes a nontrivial endomorphism g of $F_A^\vee$ and an eigenvalue λ of $g(x)$ for some $x \in S$. The determinant of $f := g - \lambda \cdot 1$ vanishes at x. Since $\det(f) \in H^0(\det(F_A \otimes F_A^\vee)) = \mathbf{C}$ it vanishes identically.

Let $N = \mathrm{Im}(f)$, $M_0 = \mathrm{coker}(f)$, and set $M = M_0/T(M_0)$, where $T(M_0)$ is the torsion subsheaf of M_0. Thus

$$[C] = c_1(E) = c_1(N) + c_1(M) + c_1(T(M_0))$$

in the Chow group $A_1(X) = \mathrm{Pic}(X)$. Now $c_1(T(M_0))$ is represented by a nonnegative linear combination of the codimension-one irreducible components of the support of $T(M_0)$. Therefore it is enough to prove that $c_1(N)$ and $c_1(M)$ are represented by nonzero effective curves. We have that N and M are torsion-free sheaves of positive rank and, as quotients of $F_A^\vee$, generated by their sections away from a finite set (Lemma 2(b)). Moreover, N and M are nontrivial, since $H^0(F_A) = 0$ (Lemma 2(a)). Let us assume for simplicity that N and M are locally free. Then, by the above, we already have that $c_1(N)$ and $c_1(M)$ are represented by effective (possibly zero) curves. Now such curves are in fact nonzero since, by Porteous's formula, $c_1(N) = 0$ (resp. $c_1(M) = 0$) if and only if N (resp. M) is trivial. If N, M are not locally free, one needs a little extra argument involving the double duals of N and M. q.e.d.

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References

[AC] E. Arbarello and M. Cornalba, *Su una congettura di Petri*, Comment. Math. Helv. **56** (1981) 1–38.

[ACGH] E. Arbarello, M. Cornalba, P. Griffiths, and J. Harris, *Geometry of algebraic curves.* I, Springer, Berlin, Heidelberg, and New York, 1985.

[CGGH] J. Carlson, M. Green, P. Griffiths, and J. Harris, *Infinitesimal variation of Hodge structure.* I, Compositio Math. **50** (1983) 109–205.

[EH] D. Eisenbud and J. Harris, *A simpler proof of the Gieseker-Petri theorem on special divisors*, Invent. Math. **74** (1983) 269–280.

[G] D. Gieseker, *Stable curves and special divisors: Petri's conjecture*, Invent. Math. **66** (1982) 251–275.

[L] R. Lazarsfeld, *Brill-Noether-Petri without degenerations*, J. Differential Geom. **23** (1986) 299–307.

[P] K. Petri, *Über Spezialcurven.* I, Math. Ann. **93** (1924) 182–209.

[V] C. Voisin, *L'application de Wahl des courbes satisfaisant la condition de Brill-Noether-Petri*, preprint.

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