PENCILS OF HERMITIAN FORMS

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Abstract. We consider Hermitian symmetric matrices H under the standard action a^*Ha of the complex linear group.

Rank and signature give a complete set of discrete invariants for a single Hermitian symmetric matrix.

For a pair (H_1, H_2) of Hermitian symmetric matrices, a complete set of invariants consists of the rank r, defined as the complement of the dimension of the intersection of their two kernels; of a set of minimal indices, that are positive integers that appear only when no linear combination $aH_1 + bH_2$ has rank r; and by some relative eigenvalues, with multiplicities, of H_2 with respect to H_1, that can be real, complex or infinity (they are points of the complex projective line). This description is related to a canonical biorthogonal decomposition. The case of larger sets of linearly independent Hermitian matrices is an open question in algebra.

The linear groups act naturally on the space of matrices. We shall consider in the following the question of finding canonical forms for pairs of general and of Hermitian matrices. The use of a large group in the first case, and the special structure of the Hermitian matrices in the second, allow to reduce part of the discussion to the Jordan form of endomorphisms. However, an extra feature appears, which is related to the Hilbert resolutions of polynomial modules, when we deal with *singular* pencils.

I used canonical forms for pairs of Hermitian matrices in the study of a class or real submanifolds of complex manifolds (see [4]).

We give two different proofs of the main Theorem 2.8. The first utilize the canonical form for pencils of linear maps of [2], that we rehearse in §1.

In §4 we give an independent direct proof of Theorem 2.8, which allows to better understand the geometrical significance.

1. Pencils of linear maps

Denote by $\mathbb{M}_{m\times n}(\mathbb{C})$ the space of complex $m\times n$ matrices. The group $\mathbf{G}_{m,n}(\mathbb{C}) = \mathbf{GL}_m(\mathbb{C}) \times \mathbf{GL}_n(\mathbb{C})$ acts on $\mathbb{M}_{m\times n}(\mathbb{C})$ by

$$\mathbb{M}_{m\times n}(\mathbb{C}) \times \mathbf{GL}_{m,n}(\mathbb{C}) \ni (A, (\alpha, \beta)) \longrightarrow \alpha A \beta^{-1} \in \mathbb{M}_{m\times n}(\mathbb{C}).$$

Canonical forms, through the associated invariants, are a tool to discuss the congruence of the subsets of $\mathbb{M}_{m\times n}(\mathbb{C})$ under the action of $\mathbf{G}_{m,n}(\mathbb{C})$.

Remark 1.1. The only invariant of a single matrix $A \in \mathbb{M}_{m\times n}(\mathbb{C})$ is its rank. A matrix A of rank r is congruent to

$$\begin{pmatrix} I_r & 0_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{pmatrix}.$$

The next natural question, that we shall discuss in the following, is to find a canonical form for a pair of linearly independent linear maps. To describe the result, we shall essentially follow [2].

Since canonical forms for matrices correspond to canonical forms of their transposes, we can and we shall assume, in the following, that $m \leq n$.

A *pencil* of complex matrices is a linear plane[1]

$$\langle A_0, A_1\rangle = \{k_0A_0 + k_1A_1 \mid k_0, k_1 \in \mathbb{C}\} \subset \mathbb{M}_{m\times n}(\mathbb{C}),$$

where A_1 and A_2 are linearly independent matrices.

The *pair* A_0, A_1 can be conveniently associated to the first degree polynomial $A_0 + \lambda A_1 \in \mathbb{M}_{m\times n}(\mathbb{C})[\lambda]$, or to the corresponding affine line

$$\{A(\lambda) = A_0 + \lambda A_1 \mid \lambda \in \mathbb{C}\} \subset \mathbb{M}_{m\times n}(\mathbb{C}).$$

Definition 1.2. Let $A_0, A_1 \in \mathbb{M}_{m\times n}(\mathbb{C})$ be complex $m\times n$ matrices, with $m \leq n$. We say that the pencil $\langle A_0, A_1\rangle$ is *nondegenerate* if

$$\ker A_0 \cap \ker A_1 = 0, \quad \operatorname{Im} A_0 + \operatorname{Im} A_1 = \mathbb{C}^m. \tag{1.1}$$

Remark 1.3. By Grassmann intersection formula, a necessary condition for $\langle A_0, A_1\rangle \subset \mathbb{M}_{m\times n}(\mathbb{C})$ being nondegenerate, is that $n \leq 2m$.

We can always reduce the study of a pencil of complex matrices to the non degenerate case. We have indeed

Proposition 1.4. *Let $\langle A_0, A_1\rangle \subset \mathbb{M}_{m\times n}(\mathbb{C})$ be a degenerate pencil.*

If $k = \dim\ker A_0 \cap \ker A_1 > 0$, $\ell = m - \dim(\operatorname{Im} A_0 + \operatorname{Im} A_1)$, then we can find $\alpha \in \mathbf{GL}(m,\mathbb{C}), \beta \in \mathbf{GL}(n,\mathbb{C})$ and $A'_0, A'_1 \in \mathbb{M}_{(m-\ell)\times(n-k)}(\mathbb{C})$ such that $\langle A'_0, A'_1\rangle$ is nondegenerate and

$$\alpha(A_0 + \lambda A_1)\beta = \begin{pmatrix} 0_{\ell\times k} & 0_{\ell\times(n-k)} \\ 0_{(m-\ell)\ell\times k} & A'_0 + \lambda A'_1 \end{pmatrix}. \tag{1.2}$$

Definition 1.5. Let $A_0, A_1 \in \mathbb{M}_{m\times n}(\mathbb{C})$. The rank of the pencil $\langle A_0, A_1\rangle$ is

$$\operatorname{rank}\langle A_0, A_1\rangle = \sup_{a,b\in\mathbb{C}} \operatorname{rank}(k_0A_0 + k_1A_1). \tag{1.3}$$

If either $m \neq n$, or $\operatorname{rank}\langle A_0, A_1\rangle < n$ the pencil $\langle A_0, A_1\rangle$ is called *singular*.

If $\operatorname{rank}\langle A_0, A_1\rangle < n$, we say that the pencil $\langle A_0, A_1\rangle$ is *singular for columns*.

If $\operatorname{rank}\langle A_0, A_1\rangle < m$, we say that the pencil $\langle A_0, A_1\rangle$ is *singular for rows*.

If $m = n$ and $\operatorname{rank}\langle A_0, A_1\rangle = n$ the pencil $\langle A_0, A_1\rangle$ is called *regular*.

Remark 1.6. The rank of the pencil $\langle A_0, A_1\rangle$ is the *generic* rank of $A(\lambda) = A_0 + \lambda A_1$. This means that the set of $\lambda \in \mathbb{C}$ for which $\operatorname{rank} A(\lambda) < \operatorname{rank}\langle A_0, A_1\rangle$ is finite.

[1]We may consider the pencil as a line in the projective space $\mathbb{PM}_{m\times n}(\mathbb{C}) \simeq \mathbb{CP}^{mn-1}$.

Example 1.7. Let

$$A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then

$$A(\lambda)\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \lambda z_3 \\ z_3 \\ z_1 + \lambda z_2 \end{pmatrix}.$$

Thus $\langle A_0, A_1 \rangle$ is nondegenerate, but

$$\operatorname{Im} A(\lambda) = \{z \in \mathbb{C}^3 \mid \lambda z_2 = z_1\}$$

has dimension 2 for all $\lambda \in \mathbb{C}$. Therefore $\operatorname{rank}\langle A_0, A_1 \rangle = 2$, and the pencil is singular.

1.1. **Regular pencils.**

Proposition 1.8. *Assume that $m = n$ and that the pencil $\langle A_0, A_1 \rangle$ is regular. Then we can find $\alpha, \beta \in \mathbf{GL}(n, \mathbb{C})$ such that*

$$\alpha(A_0 + \lambda A_1) = \begin{pmatrix} J_1(\lambda) & & & \\ & J_2(\lambda) & & \\ & & \ddots & \\ & & & J_s(\lambda) \end{pmatrix} \tag{1.4}$$

where the matrices $J_h(\lambda)$ have either the form

$$J_{\lambda_h}^{r_h}(\lambda) = \begin{pmatrix} \lambda + \lambda_h & 1 & & & \\ & \lambda + \lambda_h & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda + \lambda_h & 1 \\ & & & & \lambda + \lambda_h \end{pmatrix} \in \mathbb{H}_{r_h \times r_h}(\mathbb{C}), \tag{1.5}$$

with $\lambda_h \in \mathbb{C}$, or

$$J_{\infty}^{r_h}(\lambda) = \begin{pmatrix} 0 & \lambda & & & \\ & 0 & \lambda & & \\ & & \ddots & \ddots & \\ & & & 0 & \lambda \\ & & & & 0 \end{pmatrix} \in \mathbb{H}_{r_h \times r_h}(\mathbb{C}). \tag{1.6}$$

Proof. By the assumption, we can fix a $\lambda_0 \in \mathbb{C}$ for which $B = A(\lambda_0) = A_0 + \lambda_0 A_1$ is invertible.

Then $L_0 = A_0 B^{-1}$ and $L_1 = A_1 B^{-1}$ are matrices corresponding to commuting endomorphisms. Indeed, $L_0 = I_n - \lambda_0 L_1$.

Consider the decomposition

$$\mathbb{C}^n = W_0 \oplus W_1, \text{ with } W_0 = \bigcup\nolimits_{h \in \mathbb{N}} \ker L_1^h, \; W_1 = \bigcap_{h \in \mathbb{N}} L_1^h(\mathbb{C}^n). \tag{1.7}$$

Then L_1 is invertible on W_1 and nilpotent on W_0. We claim that L_0 is invertible on W_0.

Indeed, since $[L_0, L_1] = L_0L_1 - L_1L_0 = 0$, we have $L_0(W_0) \subset W_0$. If L_0 and L_1 were both nilpotent on W_0, then $\ker L_0 \cap \ker L_1 \cap W_0 \neq 0$ by Engel's theorem. But this implies that $\ker A_0 \cap \ker A_1 \neq 0$, contradicting the assumption that $\langle A_0, A_1\rangle$ is regular.

Consider

$$B_0 = A_1A_0^{-1}|_{W_0} : W_0 \xrightarrow{L_0^{-1}} W_0 \xrightarrow{L_1} W_0 \in \mathrm{End}_{\mathbb{C}}(W_0),$$
$$B_1 = A_0A_1^{-1}|_{W_1} : W_1 \xrightarrow{L_1^{-1}} W_0 \xrightarrow{L_0} W_0 \in \mathrm{End}_{\mathbb{C}}(W_1).$$

We note that B_0 is nilpotent. We can find a basis $e_1, \dots, e_q$ of W_0 and a basis $e_{q+1}, \dots, e_n$ of W_1 in which the matrices of B_0 and B_1 are in Jordan form. Then $A_0e_1, \dots, A_0e_q, A_1e_{q+1}, \dots, A_1e_n$ is also a basis of $\mathbb{C}^n$ and, in these bases, the matrices $A(\lambda)$ have the desired form. □

Definition 1.9. The string

$$((\lambda_1, r_1), \dots, (\lambda_{s'}, r_{s'}), \underbrace{(\infty, r_{s'+1}), \dots, (\infty, r_s)}_{r-r'}),$$

where $\lambda_1, \dots, \lambda_s'$ are the eigenvalues of B_1, repeated with their multiplicity, $r_1, \dots, r_{s'}$ the size of their Jordan blocks, and $r_{s'+1}, \dots, r_s$ the size of the Jordan blocks of B_0, is said to be *associated* to the pair (A_0, A_1).

1.2. **Singular pencils.** We turn now to the case of a singular pencil. Recall that we assumed $m \leq n$.

Lemma 1.10. *If the pencil $\langle A_0, A_1\rangle$ is singular for columns, then there is some non zero polynomial*

$$z(\lambda) = z_0 + z_1\lambda + \cdots + z_d\lambda^d \in \mathbb{C}^d[\lambda]$$

such that

$$(A_0 - \lambda A_1)z(\lambda) = 0. \tag{1.8}$$

Proof. Let $\mathbb{C}(\lambda)$ be the field of complex rational function of one variable. By the assumption, $A_0 - \lambda A_1$ defines a matrix in $\mathbb{M}_{m\times n}(\mathbb{C}(\lambda))$ having rank r less than n. Then $\dim_{\mathbb{C}(\lambda)} \ker(A_0 - \lambda A_1) = n - r > 0$ and there exists $Z(\lambda) \in [\mathbb{C}(\lambda)]^n \setminus \{0\}$ such that $(A_0 - \lambda A_1)Z(\lambda) = 0$. We can write $Z(\lambda) = (1/f(\lambda))z(\lambda)$ with $f(\lambda) \in \mathbb{C}[\lambda] \setminus \{0\}$ and $z(\lambda) \in \mathbb{C}^n[\lambda] \setminus \{0\}$. Clearly $z(\lambda)$ solves (1.8). □

Remark 1.11. We note that (1.8) is equivalent to

$$\begin{cases} A_0z_0 = 0, \\ A_0z_j = A_1z_{j-1}, \quad \text{for} \quad 1 \leq j \leq d, \\ \ \ 0 \ \ = A_1z_d. \end{cases} \tag{1.9}$$

Equation (1.9) can be rewritten as

$$\begin{pmatrix} -A_0 & & & & \\ A_1 & -A_0 & & & \\ & A_1 & -A_0 & & \\ & & \ddots & \ddots & \\ & & & A_1 & -A_0 \\ & & & & A_1 \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ \vdots \\ z_{d-1} \\ z_d \end{pmatrix} = 0. \tag{1.10}$$

Consider the sequence of matrices

$$M_0 = \begin{pmatrix} -A_0 \\ A_1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} -A_0 & \\ A_1 & -A_0 \\ & A_1 \end{pmatrix}, \dots, \quad M_d = \begin{pmatrix} -A_0 & & & & \\ A_1 & -A_0 & & & \\ & A_1 & -A_0 & & \\ & & \ddots & \ddots & \\ & & & A_1 & -A_0 \\ & & & & A_1 \end{pmatrix}$$

where M_d is a matrix with $m(d+2)$ rows and $n(d+1)$ columns. Then a necessary and sufficient condition to find a non zero solution $z(\lambda)$ of (1.8) of degree d is that the rank of M_d be less than $n(d+1)$.

Definition 1.12. We denote by $\mathcal{Z}(A_0, A_1) \subset \mathbb{C}^n[\lambda]$ the set of polynomial solutions of (1.8).

Lemma 1.13. *The set $\mathcal{Z}(A_0, A_1)$ is a $\mathbb{C}[\lambda]$-module. This means that*

$$\begin{aligned} &p_1(\lambda)z^1(\lambda) + p_2(\lambda)z^2(\lambda) \in \mathcal{Z}(A_0, A_1), \\ &\quad \forall z^1(\lambda), z^2(\lambda) \in \mathcal{Z}(A_0, A_1) \quad \text{and} \quad p_1(\lambda), p_2(\lambda) \in \mathbb{C}[\lambda]. \end{aligned}$$

Theorem 1.14. *Let $\langle A_0, A_1 \rangle$ be a nondegenerate singular pencil in $\mathbb{M}_{m\times n}(\mathbb{C})$ and let d be the smallest degree of a non zero polynomial in $\mathcal{Z}(A_0, A_1)$. We have:*

(1) *If $z(\lambda) = z_0 + \cdots + \lambda^d z_d \in \mathcal{Z}(A_0, A_1)$ has minimal degree, then*

$$z_0, \dots, z_d \text{ are linearly independent,}$$
$$A_0 z_1, \dots, A_0 z_d \text{ are linearly independent.}$$

(2) *We can find $\alpha \in \mathbf{GL}(m, \mathbb{C})$, $\beta \in \mathbf{GL}(n, \mathbb{C})$ such that*

$$\alpha(A_0 + \lambda A_1)\beta = \begin{pmatrix} N_d(\lambda) & \\ & A_0' + \lambda {A_1}' \end{pmatrix} \tag{1.11}$$

where $A_0', A_1' \in \mathbb{M}_{(m-d)\times(n-d-1)}(\mathbb{C})$ and

$$N_d(\lambda) = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \end{pmatrix} \in \mathbb{M}_{d\times(d+1)}(\mathbb{C}). \tag{1.12}$$

Proof. Let $z(\lambda) = z_0 + z_1\lambda + \cdots + z_d\lambda^d \in \mathcal{Z}(A_0, A_1)$ be a non zero solution of (1.8), with minimal degree. Note that $d \geq 1$ because $\langle A_0, A_1\rangle$ is nondegenerate, and that $z_0 \neq 0$, because otherwise $\zeta(\lambda) = z_1 + z_2\lambda + \cdots + z_d\lambda^{d-1}$ would be an element of $\mathcal{Z}(A_0, A_1)$, with degree $d-1 < d$.

We divide the proof in several steps.

STEP 1. $A_0z_1, \ldots, A_0z_d$ *are linearly independent in* $\mathbb{C}^n$.

We first show that $A_0z_1 \neq 0$. Otherwise,

$$0 = A_0z_1 = A_1z_0 \Longrightarrow 0 \neq z_0 \in \ker A_0 \cap \ker A_1$$

contradicts the assumption that $\langle A_0, A_1\rangle$ is nondegenerate.

Assume, by contradiction, that $A_0z_1, \ldots, A_0z_d$ are linearly dependent. Then there is a smallest $h \geq 2$ for which A_0z_h is linearly dependent from $\{A_0z_i \mid 1 \leq i < h\}$. Let

$$A_0z_h = a_1A_0z_1 + \cdots + a_{h-1}A_0z_{h-1}.$$

This equation yields

$$A_1z_{h-1} = a_1A_1z_0 + \cdots + a_{h-1}A_1z_{h-2}.$$

Hence, setting

$$z^*_{h-1} = z_{h-1} - a_1z_0 - \cdots - a_{h-1}z_{h-2}$$

we obtain

$$A_1z^*_{h-1} = 0, \quad A_0z^*_{h-1} = A_1(z_{h-2} - a_2z_0 - \cdots - a_{h-1}z_{h-3}) = A_1z^*_{h-2},$$

with

$$z^*_{h-2} = z_{h-2} - a_2z_0 - \cdots - a_{h-1}z_{h-3}.$$

Let us define the chain

$$\begin{cases} z^*_0 = z_0, \\ z^*_{h-j} = z_{h-j} - a_jz_0 - a_{j+1}z_1 - \ldots - a_{h-1}z_{h-j-1} \quad \text{for } 1 \leq j < h. \end{cases}$$

Then

$$\begin{cases} A_0z^*_0 = 0, \\ A_0z^*_j = A_1z^*_{j-1}, \quad \text{for} \quad 1 \leq j \leq h-1, \\ \;\; 0 \;\; = A_1z^*_{h-1}. \end{cases}$$

Indeed we already know that the first and the last equalities hold. For $2 \leq j < h$ we obtain

$$\begin{aligned} A_0z^*_{h-j} &= A_0(z_{h-j} - a_jz_0 - a_{j+1}z_1 - \ldots - a_{h-1}z_{h-j-1}) \\ &= A_1z_{h-j-1} - a_{j+1}A_1z_0 - \ldots - a_{h-1}A_1z_{h-j-2} \\ &= A_1z^*_{h-j-2}. \end{aligned}$$

Hence $z^*(\lambda) = z^*_0 + z^*_1\lambda + \cdots + z^*_{h-1}\lambda^{h-1} \in \mathcal{Z}(A_0, A_1)$, yielding a contradiction, because $h-1 < d$.

STEP 2. $z_0, z_1, \ldots, z_d$ *are linearly independent.*

Indeed, if $a_0z_0 + a_1z_1 + \cdots + a_dz_d = 0$, we obtain $a_1A_0z_1 + \cdots a_dA_0z_d = 0$. Hence $a_1 = 0, \ldots, a_d = 0$ by STEP 1. Then $a_0z_0 = 0$ implies that also $a_0 = 0$, because $z_0 \neq 0$.

STEP 3. If

$$V = \langle z_0, \dots, z_d \rangle \subset \mathbb{C}^n,$$
$$W = \langle A_0 z_1, \dots, A_0 z_d \rangle = \langle A_1 z_0, \dots, A_1 z_{d-1} \rangle \subset \mathbb{C}^m,$$

we have $A(\lambda)(V) = W$ for all $\lambda \in \mathbb{C}$.

Choosing a new basis of $\mathbb{C}^n$ in which the first $d+1$ vectors are $z_0, z_1, \dots, z_d$, and a basis of $\mathbb{C}^m$ in which the first vectors are $A_0 z_1, \dots, A_0 z_d$, we can assume that the matrices A_0 and A_1 have the form

$$A_0 = \begin{pmatrix} J_d & C_0 \\ 0 & B_0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} K_d & C_1 \\ 0 & B_1 \end{pmatrix}$$

where $J_d = (\delta_{i,j-1})_{1 \le i \le d, 1 \le j \le d+1}, K_d = (\delta_{i,j})_{1 \le i \le d, 1 \le j \le d+1} \in \mathbb{M}_{d \times (d+1)}(\mathbb{C})$ are the matrices

$$J_d = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \qquad K_d = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

and $C_0, C_1 \in \mathbb{M}_{d \times (n-d-1)}(\mathbb{C})$, $B_0, B_1 \in \mathbb{M}_{(m-d) \times (n-d-1)}(\mathbb{C})$. Set

$$L(\lambda) = J_d + \lambda K_d, \; C(\lambda) = C_0 + \lambda C_1, \; B(\lambda) = B_0 + \lambda B_1.$$

STEP 4. We conclude now the proof of the theorem.

To this aim we show that *we can find matrices* $X \in \mathbb{M}_{(d+1) \times (m-d-1)}(\mathbb{C})$ *and* $Y \in \mathbb{M}_{d \times (n-d)}(\mathbb{C})$ *such that*

$$\begin{pmatrix} I & Y \\ 0 & I \end{pmatrix} \begin{pmatrix} L(\lambda) & C(\lambda) \\ 0 & B(\lambda) \end{pmatrix} \begin{pmatrix} I & -X \\ 0 & I \end{pmatrix} = \begin{pmatrix} L(\lambda) & 0 \\ 0 & B(\lambda) \end{pmatrix}.$$

This equation is equivalent to

$$\begin{cases} L(\lambda) X = C(\lambda) + Y B(\lambda), \\ X \in \mathbb{M}_{(d+1) \times (m-d-1)}(\mathbb{C}), \; Y \in \mathbb{M}_{d \times (n-d)}(\mathbb{C}). \end{cases} \tag{1.13}$$

Write

$$X = \begin{pmatrix} X_1 \\ \vdots \\ X_{d+1} \end{pmatrix}, \quad \text{with} \quad X_j \in \mathbb{M}_{1 \times (m-d-1)}.$$

Then

$$L(\lambda) X = \begin{pmatrix} X_2 \\ \vdots \\ X_{d+1} \end{pmatrix} + \lambda \begin{pmatrix} X_1 \\ \vdots \\ X_{d-1} \end{pmatrix}.$$

Thus

$$U + \lambda V = \begin{pmatrix} U_1 \\ \vdots \\ U_{d-1} \end{pmatrix} + \lambda \begin{pmatrix} V_1 \\ \vdots \\ V_{d-1} \end{pmatrix} \in L(\lambda) \mathbb{M}_{(d+1) \times (n-d-1)}$$

if and only if

$$\begin{pmatrix} U_1 \\ \vdots \\ U_{d-2} \end{pmatrix} = \begin{pmatrix} V_2 \\ \vdots \\ V_{d-1} \end{pmatrix}.$$

Since

$$\begin{pmatrix} U_1 \\ \vdots \\ U_{d-2} \end{pmatrix} = K_{d-1} \begin{pmatrix} U_1 \\ \vdots \\ U_{d-1} \end{pmatrix} = K_{d-1}U, \qquad \begin{pmatrix} V_2 \\ \vdots \\ V_{d-1} \end{pmatrix} = J_{d-1} \begin{pmatrix} V_1 \\ \vdots \\ V_{d-1} \end{pmatrix} = J_{d-1}V,$$

this equation can be rewritten by

$$(K_{d-2}, -J_{d-1}) \begin{pmatrix} U \\ V \end{pmatrix} = 0.$$

Hence (1.13) admits a solution (X, Y) if and only if the equation[2]

$$J_{d-1}C_1 - K_{d-1}C_0 = K_{d-1}YB_0 - J_{d-1}YB_1, \quad Y \in \mathbb{M}_{d\times n-d} \tag{1.14}$$

is solvable. Let us write

$$Y = \begin{pmatrix} y^1 \\ \vdots \\ y^d \end{pmatrix}$$

with y^j being an $(n-d)$ row matrix. Then

$$K_{d-1}Y = \begin{pmatrix} y^1 \\ \vdots \\ y^{d-1} \end{pmatrix}, \quad J_{d-1}Y = \begin{pmatrix} y^2 \\ \vdots \\ y^d \end{pmatrix},$$

and the right hand side of (1.14) becomes

$$\begin{pmatrix} y^1B_0 - y^2B_1 \\ y^2B_0 - y^3B_1 \\ \vdots \\ y^{d-1}B_0 - y^dB_1 \end{pmatrix}.$$

This can also be written as

$$(y^1, y^2, y^3, \dots, y^{d-1}, y^d) \begin{pmatrix} B_0 \\ -B_1 & B_0 \\ & -B_1 & B_0 \\ & & \ddots & \ddots \\ & & & -B_1 & B_0 \\ & & & & -B_1 \end{pmatrix}. \tag{1.15}$$

By the assumption that d is the minimal degree of a polynomial in $\mathcal{Z}(A_0, A_1)$, the matrix $M_{d-1}(B_0, -B_1)$ in (1.15) has rank $\geq d(n-d-1)$. Thus (1.14), and hence equation (1.13) is solvable for every $C_0, C_1 \in \mathbb{M}_{d\times(n-d-1)}(\mathbb{C})$. This completes the proof. □

[2] We have $K_{d-1}J_d = J_{d-1}K_d = (0_{(d-1)\times 1}, I_{d-1}, 0_{(d-1)\times 1}) \in \mathbb{M}_{(d-1)\times d+1}(\mathbb{C})$.

Example 1.15. Consider the singular pencil of 1.7. We obtain

$$(A_0 - \lambda A_1)(\lambda)(z_0 + \lambda z_1) = 0 \quad \text{with} \quad z_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad z_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Taking in the domain the basis

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

and in the codomain the basis

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

we obtain, in the new basis, the matrix

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & 0 & \lambda \\ 0 & 0 & 1 \end{pmatrix}.$$

Using Theorem 1.14, we obtain

Theorem 1.16. *The* $\mathbb{C}[\lambda]$ *module* $\mathcal{N}(A_0, A_1)$ *is free.*

Proof. The statement means that there exists a set of generators

$$z^1(\lambda), \dots, z^p(\lambda) \in \mathcal{N}(A_0, A_1)$$

such that every element of $\mathcal{N}(A_0, A_1)$ can be expressed in a unique way as a linear combination, with coefficients in $\mathbb{C}[\lambda]$, of $z^1(\lambda), \dots, z^p(\lambda)$. It can be obtained after reducing, by applying several times Theorem 1.14, to the case where

$$A(\lambda) = \begin{pmatrix} N_{d_1}(\lambda) & & & & \\ & N_{d_2}(\lambda) & & & \\ & & \ddots & & \\ & & & N_{d_p}(\lambda) & \\ & & & & B(\lambda) \end{pmatrix}$$

with $B(\lambda)$ nonsingular for columns. □

Definition 1.17 (minimal indices for columns)**.** If $\langle A_0, A_1 \rangle$ is a nondegenerate singular pencil, the degrees

$$d_1 = \deg z^1(\lambda) \le d_2 = \deg z^2(\lambda) \le \cdots d_p = \deg z^p(\lambda).$$

of the elements $z^1(\lambda), \dots, z^p(\lambda)$ of a basis of the free module $\mathcal{N}(A_0, A_1)$, with $\sum_{i=1}^{p} \deg z^i(\lambda)$ minimal are called the *minimal indices for columns* of $\langle A_0, A_1 \rangle$.

Definition 1.18 (minimal indices for rows)**.** If $\langle A_0^t, A_1^t\rangle$ is a nondegenerate singular pencil, the degrees

$$d'_1 = \deg u^1(\lambda) \leq d'_2 = \deg u^2(\lambda) \leq \cdots d'_q = \deg u^q(\lambda).$$

of the elements $u^1(\lambda), \dots, u^q(\lambda)$ of a basis of the free module $\mathcal{N}(A_0^t, A_1^t)$, with $\sum_{i=1}^q \deg u^i(\lambda)$ minimal are called the *minimal indices for rows* of $\langle A_0, A_1\rangle$.

1.3. **Canonical form of a pencil of linear maps.** Let $A_0, A_1 \in \mathbb{M}_{m\times n}(\mathbb{C})$ be linearly independent matrices. By Proposition 1.4, the orbit of $A(\lambda)$ by $\mathbf{G}_{m,n}(\mathbb{C})$ contains a matrix of the form

$$\begin{pmatrix} 0_{\ell\times k} & 0_{\ell\times(n-k)} \\ 0_{(m-\ell)\times\ell} & A'(\lambda) \end{pmatrix} \tag{1.16}$$

with $A'(\lambda) = A'_0 + \lambda A'_1$, and $A'_0, A'_1 \in \mathbb{M}_{(m-\ell)\times(n-k)}$ and $\langle A'_0, A''_1\rangle$ nondegenerate.

Definition 1.19. The minimal indices for columns (respectively. for rows) of $\langle A'_0, A'_1\rangle$ are called the *minimal indices for columns (respectively. for rows)* of the pencil $\langle A_0, A_1\rangle$.

Using Theorem 1.16 and Proposition 1.8 to decompose $A'(\lambda)$, we conclude that the orbit of $A(\lambda)$ by $\mathbf{G}_{m,n}(\mathbb{C})$ contains a matrix of the form

$$\begin{pmatrix} 0 & & & \\ & B(\lambda) & & \\ & & C(\lambda) & \\ & & & D(\lambda) \end{pmatrix}, \tag{1.17}$$

where $0 \in \mathbb{H}_{\ell\times k}(\mathbb{C})$, with $\ell + \dim(\operatorname{Im} A_0 + \operatorname{Im} A_1)$, $k = \dim(\ker A_0 \cap \ker A_1)$, and $B(\lambda)$, $C(\lambda)$, $D(\lambda)$ nondegenerate and, respectively, of the form

$$B(\lambda) = \begin{pmatrix} N_{d_1}(\lambda) & & \\ & \ddots & \\ & & N_{d_p}(\lambda) \end{pmatrix},$$

$$C(\lambda) = \begin{pmatrix} N^t_{d'_1}(\lambda) & & \\ & \ddots & \\ & & N^t_{d'_q}(\lambda) \end{pmatrix},$$

$$D(\lambda) = \begin{pmatrix} J_1(\lambda) & & \\ & \ddots & \\ & & J_r(\lambda) \end{pmatrix},$$

where $(d_1, \dots, d_p)$ are the minimal indices for columns, $(d'_1, \dots, d'_q)$ the minimal indices for rows, and the $J_h(\lambda)$ are described by (1.5) and (1.6).

Definition 1.20. The polynomials $(\lambda - \lambda_h)^{r_h}$, for $J_h(\lambda)$ of the form (1.5), and the rational functions λ^{-r_h} for $J_h(\lambda)$ of the form (1.6), are called the *invariant factors* of $A(\lambda)$.

Theorem 1.21 (Kronecker [3])**.** *The integers* $k = \dim(\ker A_0 \cap \ker A_1)$*,* $\ell = \dim(\mathrm{Im}\, A_0 + \mathrm{Im}\, A_1)$*, the minimal indices for columns and for rows of* $\langle A_0, A_1\rangle$ *and the invariant factors of the pair* (A_0, A_1) *are uniquely determined and characterize the orbit of* $A(\lambda)$ *by* $\mathbf{GL}_{m,n}(\mathbb{C})$*.*

For the complete proof we refer to [2].

2. Pencils of Hermitian matrices

Let $\mathbf{H}_n$ denote the space of $n\times n$ Hermitian symmetric matrices:

$$\mathbf{H}_n = \{H \in \mathbb{M}_{n\times n}(\mathbb{C}) \mid H^* = H\}, \tag{2.1}$$

where, for $H = (h_{i,j})_{1\le i,j\le n}$ the matrix $H^* = (h^*_{i,j})_{1\le i,j\le n}$ has entries $h^*_{i,j} = \bar{h}_{j,i}$. The set $\mathbf{H}_n$ is an n^2-dimensional *real* linear space of complex matrices.

The group $\mathbf{GL}_n(\mathbb{C})$ acts on $\mathbf{H}_n$ by

$$\mathbf{H}_n \times \mathbf{GL}_n(\mathbb{C}) \ni (H, \alpha) \longrightarrow \alpha H \alpha^* \in \mathbf{H}_n.$$

Denote by $\mathfrak{Gr}_k(\mathbf{H}_n)$ the Grassmannian of k-dimensional subspaces of $\mathbf{H}_n$ and by $\mathfrak{O}_k(\mathbf{H}_n)$ the space of orbits of $\mathfrak{Gr}_k(\mathbf{H}_n)$ for the action of $\mathbf{GL}_n(\mathbb{C})$.

Given a k-tuple $(H_1, \dots, H_k)$ of linearly independent Hermitian symmetric forms, we denote by $F =< H_1, \dots, H_k >$ the subspace of $\mathbf{H}_n$ that they generate, and by $[F] \in \mathfrak{O}_k(\mathbf{H}_n)$ the orbit of F.

More generally, we shall use the notation $\mathbf{H}(V)$ for the space of Hermitian symmetric forms on a finite dimensional complex linear space V and $\mathbf{G}_{\mathbb{C}}(V)$ for the group of linear automorphisms of V.

Definition 2.1. A linear subspace $F \subset \mathbf{H}_n$ is called *nondegenerate* if

$$\forall v \in \mathbb{C}^n \setminus \{0\} \;\; \exists H \in F \;\; \text{such that} \;\; Hv \neq 0. \tag{2.2}$$

The subspace F is called *regular* if F contains a nondegenerate form H, and *singular* if all forms $H \in F$ are degenerate.

Remark 2.2. Regular F's are nondegenerate, but a nondegenerate F can be singular if $\dim_{\mathbb{R}} F \ge 2$.

Remark 2.3. For every $H \in \mathbf{H}_n$ we can find $\alpha \in \mathbf{GL}_n(\mathbb{C})$ such that

$$\alpha H \alpha^* = \begin{pmatrix} I_p & & \\ & -I_q & \\ & & 0_{(n-p-q)\times(n-p-q)} \end{pmatrix}, \tag{2.3}$$

where $p+q$ is the rank of H and (p,q) its signature.

Other canonical forms for a Hermitian matrix use the matrices

$$J_d = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix} \in \mathbb{M}_{d\times d}(\mathbb{C}).$$

A matrix H of signature (p,q), with $\dim\ker H = k$ and $\nu = \min(p,q)$, $h = p+q-2\nu$ is equivalent to the matrix

$$\pm\begin{pmatrix} & & J_\nu & \\ & I_h & & \\ I_\nu & & & \\ & & & 0\end{pmatrix} \text{ if } h>1, \quad \pm\begin{pmatrix} J_{p+q} & \\ & 0\end{pmatrix} \text{ if } 0\leq h\leq 1.$$

Note that, for $h=0$, the two matrices with the plus and minus sign are equivalent.

Remark 2.4. If $H_1, H_2 \in \mathbf{H}_n$ and H_1 is positive definite, i.e. if $v^*H_1v > 0$ for all $v \in \mathbb{C}^n \setminus \{0\}$, then there is $\alpha \in \mathbf{GL}_n(\mathbb{C})$ such that

$$\alpha H_2\alpha^* = \begin{pmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n\end{pmatrix}, \text{ with } \lambda_i \in \mathbb{R}, \quad \alpha H_1\alpha^* = I_n.$$

The real numbers $\lambda_1, \dots, \lambda_n$ are the *eigenvalues of H_2 with respect to H_1*, and in fact are the eigenvalues of the endomorphism $H_1^{-1}H_2$. Indeed, from $(\alpha^*)^{-1}H_1^{-1}\alpha^{-1} = I_n$, we obtain that

$$\begin{pmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n\end{pmatrix} = \alpha H_2\alpha^* = (\alpha^*)^{-1}H_1^{-1}\alpha^{-1}\alpha H_2\alpha^* = (\alpha^*)^{-1}(H_1^{-1}H_2)\alpha^*.$$

The general case of a pair (H_1, H_2) of Hermitian symmetric matrices is actually more complicated when no matrix in $F = \langle H_1, H_2\rangle_{\mathbb{R}}$ is positive definite.

The discussion relates to that for a pencil of linear maps by the following

Theorem 2.5. *Let (H_1, H_2) and (K_1, K_2) be two pairs of linearly independent Hermitian symmetric matrices in $\mathbf{H}_n$, and set $H(\lambda) = H_2 + \lambda H_1$, $K(\lambda) = K_2 - \lambda K_1$. Then the following are equivalent:*

(1) $\exists \alpha, \beta \in \mathbf{GL}_n(\mathbb{C})$ *such that* $\alpha H(\lambda)\beta = K(\lambda)$*;*
(2) $\exists \alpha \in \mathbf{GL}_n(\mathbb{C})$ *such that* $\alpha^* H(\lambda)\alpha = K(\lambda)$*.*

Proof. Clearly (2) $\Rightarrow$ (1). Assume vice versa that there exist $\alpha, \beta \in \mathbf{GL}_n(\mathbb{C})$ such that

$$\alpha H(\lambda)\beta = K(\lambda).$$

Since K_1, K_2 are Hermitian symmetric, we also have

$$\beta^* H(\lambda)\alpha^* = K(\lambda).$$

With $\gamma = \alpha^{-1}\beta^* \in \mathbf{GL}_n(\mathbb{C})$, we obtain that

$$\gamma H(\lambda) = H(\lambda)\gamma^*.$$

From this equality, it follows that

$$f(\gamma)H(\lambda) = H(\lambda)f(\gamma^*), \quad \forall f \in \mathbb{C}[\lambda].$$

If we take f with real coefficients, we have $f(\gamma^*) = [f(\gamma)]^*$. Take $f \in \mathbb{R}[\lambda]$ with $f(\gamma) \in \mathbf{GL}_n(\mathbb{C})$. Then we have

$$f(\gamma)H(\lambda)[f(\gamma^*)]^{-1} = H(\lambda)$$

and therefore

$$K(\lambda) = \alpha f(\gamma)H(\lambda)[f(\gamma^*)]^{-1}\beta.$$

We want to prove that f can be chosen in such a way that

$$\alpha f(\gamma) = ([f(\gamma^*)]^{-1}\beta)^* = \beta^*[f(\gamma)]^{-1},$$

i.e.

$$[f(\gamma)]^2 = \alpha^{-1}\beta^* = \gamma.$$

This is always possible, because $\gamma \in \mathbf{GL}_n(\mathbb{C})$. □

This Theorem reduces the classification of the pencils of Hermitian symmetric forms to the results for pencils of linear maps, after recognizing the pairs of (H_1, H_2) corresponding to canonical forms of $H(\lambda) = H_2 + \lambda H_1$. Indeed we have

Corollary 2.6. *Two pairs of Hermitian forms (H_1, H_2) and (K_1, K_2) are equivalent if and only if they have the same invariant factors and minimal indices.*

Remark 2.7. Note that for a pair of Hermitian matrices (H_1, H_2) the rank for rows and the rank for columns coincide.

We have the following (see [1], [5]):

Theorem 2.8. *Let $H_1, H_2 \in \mathbf{H}_n$ and assume that $F = \langle H_1, H_2\rangle_{\mathbb{R}}$ is a two-dimensional nondegenerate linear subspace of $\mathbf{H}_n$.*

Then there is a direct sum decomposition:

$$\mathbb{C}^n = V_1 \oplus V_2 \oplus \cdots \oplus V_m \tag{2.4}$$

having the following properties:

- (i) *V_i is orthogonal to V_j for every $1 \le i \ne j \le m$ and every $H \in F$* (BIORTHOGONALITY)*;*
- (ii) *for each $i = 1, \dots, m$ the subspace V_i is not a direct sum of two nontrivial subspaces that are orthogonal with respect to all the forms $H \in F$* (INDECOMPOSABILITY).

For each $i = 1, \dots, m$ we can find a basis of V_i such that, the restrictions of H_1 and H_2 to V_i has one of the following forms:

$$\text{(I)} \qquad [H_1|_{V_i}] = \epsilon \begin{pmatrix} & & & 1 \\ & & \cdot & \\ & \cdot & & \\ 1 & & & \end{pmatrix}, \qquad [H_2|_{V_i}] = \epsilon \begin{pmatrix} & & & \gamma \\ & & \cdot & 1 \\ & \cdot & \cdot & \\ \gamma & 1 & & \end{pmatrix},$$

which are e-square matrices for some positive integer e, where γ is a real number and $\epsilon = \pm 1$;

$$\text{(II)}\qquad [H_1|_{V_i}] = \left(\begin{array}{c|c} 0_E & \begin{smallmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{smallmatrix} \\ \hline \begin{smallmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{smallmatrix} & 0_E \end{array}\right), \quad [H_2|_{V_i}] = \left(\begin{array}{c|c} 0_E & \begin{smallmatrix} & & \Gamma \\ & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ \Gamma\ 1 & & \end{smallmatrix} \\ \hline \begin{smallmatrix} & & \bar\Gamma \\ & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ \bar\Gamma\ 1 & & \end{smallmatrix} & 0_E \end{array}\right),$$

which are $2E$*-square matrices for a positive integer* E*, and where* Γ *is a complex number with* $\operatorname{Im}\Gamma > 0$*;*

$$\text{(III)}\qquad [H_1|_{V_i}] = \epsilon\begin{pmatrix} & & & 0 \\ & & \cdot & 1 \\ & \cdot & \cdot & \\ \cdot & \cdot & & \\ 0 & 1 & & \end{pmatrix}, \qquad [H_2|_{V_i}] = \epsilon\begin{pmatrix} & & & 1 \\ & & \cdot & \\ & \cdot & & \\ \cdot & & & \\ 1 & & & \end{pmatrix},$$

which are e-square matrices for a positive integer e;

$$\text{(IV)}\qquad [H_1|_{V_i}] = \left(\begin{array}{c|c} 0_{d+1} & \begin{smallmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ 0 & \cdots & 0 \end{smallmatrix} \\ \hline \begin{smallmatrix} 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{smallmatrix} & 0_d \end{array}\right), \quad [H_2|_{V_i}] = \left(\begin{array}{c|c} 0_{d+1} & \begin{smallmatrix} 0 & \cdots & 0 \\ 1 & & \\ & \ddots & \\ & & 1 \end{smallmatrix} \\ \hline \begin{smallmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{smallmatrix} & 0_d \end{array}\right),$$

which are $(2d+1)$*-square matrices of odd size* $2d+1$ *for a positive integer d.*

The decomposition (2.4) is essentially unique: by this we mean that the number and the dimensions of the subspaces involved and also the kind of the blocks and the parameters involved are uniquely determined.

Definition 2.9. Let the basis (H_1, H_2) of F be fixed.

A basis of $\mathbb{C}^n$, subordinated to a biorthogonal decomposition (2.4) and for which the matrices corresponding to H_1 and H_2 are block diagonal with blocks of the types (I), (II), (III) and (IV) described in the theorem, will be called *canonical* for the pair (H_1, H_2).

The parameters $\epsilon = \pm 1$ appearing in the blocks of type (I) and (III) are called *sign* or *inertial signature*.

The parameters $\gamma \in \mathbb{R}$ appearing in the blocks of type (I), Γ and $\bar\Gamma$ appearing in the blocks of type (II), and ∞ if there are blocks of type (III), are called *roots* or *eigenvalues* of the pair (H_1, H_2). The set $\Sigma(H_1, H_2) \subset \mathbb{CP}^1$ of all roots (including ∞ if there are blocks of type (III)), is called the *spectrum* of (H_1, H_2).

The integers d characterizing the blocks of type (IV) are called the *minimal indices* of F. Note that they are independent of the choice of the basis (H_1, H_2).

A block of type (III) for the basis (H_1, H_2) becomes a block of type (I) for the basis (H_2, H_1). Accordingly, we shall distinguish only three types of subspaces in the decomposition (2.4): a subspace V_i will be called of the *real type* if it corresponds to blocks of the types (I) or (III); of the *complex type* if it corresponds to a block of type (II); of the *singular type* if it corresponds to a block of type (IV).

Proof. We note that the minimal indices for rows and columns of (H_1, H_2) coincide, and that also roots and multiplicities of (H_1, H_2) are invariant by conjugation. Then we can construct a Hermitian pair with the correct data using blocks of the types I, II, III, IV,. □

3. The string of data

Having fixed a basis (H_1, H_2) of F, by the preceding theorem we can have, in a canonical basis, a block decomposition of the matrices representing the two forms into blocks of the types (I), (II), (III), (IV). We shall associate to (H_1, H_2) a string of data to describe the different blocks; namely:

(1) we shall list the minimal indices in increasing order:

$$d_1 < d_2 < \cdots < d_s \tag{3.1}$$

and denote by m_i, for $1 \leq i \leq s$, the number of singular subspaces of dimension $2d_i + 1$ of the decomposition (2.4);

(2) we shall list all eigenvalues $\Gamma_1, \ldots, \Gamma_L$ in $\Sigma(H_1, H_2)$ with $\mathrm{Im}\Gamma_I > 0$ for $1 \leq I \leq L$, together with their complex conjugated, and we shall order the sizes $2E_{I,J}$ of the blocks in which appear the parameters $\Gamma_I, \bar{\Gamma}_I$ in decreasing order:

$$E_{I,1} > E_{I,2} > \cdots > E_{I,S_I} \tag{3.2}$$

and denote by $M_{I,J}$ the number of blocks of size $2E_{I,J}$ (for $1 \leq I \leq L$, $1 \leq J \leq S_I$);

(3) finally we list all distinct eigenvalues $\gamma_1, \ldots, \gamma_\ell \in \mathbb{R} \cup \{\infty\}$ in $\Sigma(H_1, H_2)$; for each γ_i, $1 \leq i \leq \ell$, we list in decreasing order the sizes

$$e_{i,1} > e_{i,2} > \cdots > e_{i,s_i} \tag{3.3}$$

of the corresponding blocks of type (I) or (III) with parameter γ_i and denote by $m^+_{i,j}$, $m^-_{i,j}$ the number of those with $\epsilon = 1$, $\epsilon = -1$ respectively; we set $m_{i,j}$ for the sum $m^+_{i,j} + m^-_{i,j}$.

We organize these informations, that describe all the invariants of the pair of Hermitian forms (H_1, H_2), into its *string of data*:

$$\Delta(H_1, H_2) = \begin{pmatrix} d_1, m_1, \ldots, d_s, m_s \\ \Gamma_1, \bar{\Gamma}_1, M_{1,1}, \ldots, M_{1,S_1} \\ \cdots \\ \Gamma_L, \bar{\Gamma}_L, M_{L,1}, \ldots, M_{L,S_L} \\ \gamma_1, e_{1,1}, m^+_{1,1}, m^-_{1,1}, \ldots, e_{1,s_1}, m^+_{1,s_1}, m^-_{1,s_1} \\ \cdots \\ \gamma_\ell, e_{\ell,1}, m^+_{\ell,1}, m^-_{\ell,1}, \ldots, e_{\ell,s_\ell}, m^+_{\ell,s_\ell}, m^-_{\ell,s_\ell} \end{pmatrix} \tag{3.4}$$

3.1. **The effect of a change of basis in F.** We consider the way in which the string of data changes if we change the basis of F. If (H_1', H_2') is another basis, there is a matrix $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathbf{GL}_2(\mathbb{R})$ such that

$$\begin{cases} H_1' = aH_1 + bH_2 \,, \\ H_2' = cH_1 + dH_2 \,. \end{cases} \tag{3.5}$$

The canonical form relative to (H_1', H_2') has blocks of the same type (only those of type (I) and (III) can be changed one into the other), corresponding to a same biorthogonal decomposition (2.4). The string of data $\Delta(H_1', H_2')$ is obtained from $\Delta(H_1, H_2)$ by keeping all the integral parameters fixed, but changing the spectrum: $\Sigma(H_1', H_2')$ is obtained from $\Sigma(H_1, H_2)$ by letting

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} {}^tB^{-1} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix} \tag{3.6}$$

act as a real linear fractional transformation. This indeed trivially follows from

$$H_2' - \lambda' H_1' = (d - \lambda' b)H_2 - (\lambda' a - c)H_1 \,, \tag{3.7}$$

showing that the eigenvalues λ of the pair (H_1, H_2) and λ' of (H_1', H_2') are related by:

$$\lambda = \frac{\lambda' a - c}{d - \lambda' b}, \quad \lambda' = \frac{d\lambda + c}{b\lambda + a} \,. \tag{3.8}$$

The inertial signatures remain the same or change to their opposite.

4. A direct proof of Theorem 2.8

It is convenient to introduce the notation:

$$\begin{aligned} V^0 &= \{w \in \mathbb{C}^n \mid w^* v = 0\}, \\ V^\perp &= [H_1(V)]^0 = \{w \in \mathbb{C}^n \mid v^* H_1 w = 0, \ \forall v \in V\}, \\ V^\dagger &= [H_2(V)]^0 = \{w \in \mathbb{C}^n \mid v^* H_2 w = 0, \ \forall v \in V\}, \\ V^{\perp\!\!\perp} &= V^\perp \cap V^\dagger = \{w \in \mathbb{C}^n \mid v^* H_1 w = 0, \ v^* H_2 w = 0, \ \forall v \in V\}. \end{aligned}$$

4.1. **The subspace N.** Subspaces of the singular type appear in the decomposition (2.4) if and only if F is singular. We have indeed:

Proposition 4.1. *Equation*

$$H(\lambda) z(\lambda) = (H_2 - \lambda H_1) z(\lambda) = 0, \quad z(\lambda) = z_0 + \lambda z_1 + \cdots + z_d \lambda^d \in \mathbb{C}^n[\lambda] \tag{4.1}$$

admits a non trivial solution if and only if $\langle H_1, H_2 \rangle$ is singular.

Proof. The rank of $\langle H_1, H_2\rangle$ equals the rank of $(H_2 - \lambda H_1)$, considered as a matrix with coefficients in the field $\mathbb{C}(\lambda)$ of rational functions. The equation

$$H(\lambda)z(\lambda) = (H_2 - \lambda H_1)z(\lambda), \quad z(\lambda) \in [\mathbb{C}(\lambda)]^n$$

has a non trivial solution if and only if

$$\dim_{\mathbb{C}(\lambda)} \ker H(\lambda) = n - \mathrm{rank}_{\mathbb{C}(\lambda)} H(\lambda) > 0.$$

By chasing denominators, from any non tivial element of $\ker H(\lambda)$ in $[\mathbb{C}(\lambda)]^n$ we obtain non trivial solutions of (4.1). This proves the proposition. □

Remark 4.2. Equation (4.1) is equivalent to

$$\begin{cases} H_2 z_0 = 0, \\ H_2 z_j = H_1 z_{j-1} \quad \text{for} \quad 1 \le j \le m, \\ 0 = H_1 z_d, \end{cases} \tag{4.2}$$

that can also be written as

$$\begin{pmatrix} -H_2 & & & & & \\ H_1 & -H_2 & & & & \\ & H_1 & -H_2 & & & \\ & & \ddots & \ddots & & \\ & & & & H_1 & -H_2 \\ & & & & & H_1 \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ \vdots \\ z_{d-1} \\ z_d \end{pmatrix} = 0 \tag{4.3}$$

where the left hand side is a matrix M_d with $n(d+2)$ rows and and $(d+1)n$ columns. Thus, the necessary and sufficient condition for the existence of a non trivial solution is that $\mathrm{rank}_{\mathbb{C}} M_d < n(d+1)$.

Lemma 4.3. *Let*

$$z(\lambda) = z_0 + \lambda z_1 + \cdots + \lambda^m z_d \in \mathbb{C}^n[\lambda]$$

solve equation (4.1)*. Then the linear subspace* $\langle z_0, z_1, \dots, z_d\rangle$ *generated by the coefficients of* $z(\lambda)$ *is totally isotropic for both* H_1 *and* H_2 *and hence for all* $H \in F = \langle H_1, H_2\rangle$.

Proof. We define z_i for all $i \in \mathbb{Z}$, by setting $z_i = 0$ for $i < 0$ and $i > d$. Then (4.2) can be rewritten as

$$H_1 e_i = H_2 e_{i+1}, \quad \forall i \in \mathbb{Z}.$$

Then we have

$$\begin{aligned} e_i H_1 e_j^* &= e_i^* H_2 e_{j+1} = e_{i-1}^* H_1 e_{j+1} = \cdots = e_{i+j-d}^* H_1 e_d = 0, \\ e_i^* H_2 e_j &= e_{i-1}^* H_1 e_j = e_{i-1}^* H_2 e_{j+1} = \cdots = e_0^* H_2 e_{i+j-d} = 0, \end{aligned}$$

for all $i, j = 0, \dots, d$. □

Definition 4.4. The set $\mathcal{Z}(H_1, H_2) \subset \mathbb{C}^n[\lambda]$ of solutions of (4.1) is a module over the ring $\mathbb{C}[\lambda]$ of complex polynomials in one indeterminate.

Let N be the linear subspace of $\mathbb{C}^n$ generated by the coefficients of the polynomials in $\mathcal{Z}(H_1, H_2)$.

Corollary 4.5. *The subspace N is totally isotropic for all Hermitian matrices $H \in F = \langle H_1, H_2 \rangle$.*

Proof. Let $z'(\lambda), z''(\lambda) \in \mathcal{Z}(H_1, H_2)$. If z' has degree d', we can consider the polynomial $z'''(\lambda) = z'(\lambda) + \lambda^{d'+1} z''(\lambda)$. It is still contained in $\mathcal{Z}(H_1, H_2)$ and therefore the vector space generated by the coefficients of $z'(\lambda)$ and of $z''(\lambda)$, being equal to the vector space generated by the coefficients of $z'''(\lambda)$, is totally isotropic for all $H \in F$ by Lemma 4.3. □

Lemma 4.6. *Let $\langle H_1, H_2 \rangle$ be a nondegenerate singular pencil of Hermitian matrices and let*

$$z(\lambda) = z_0 + \lambda z_1 + \cdots + \lambda^d z_d \in \mathcal{Z}(H_1, H_2)$$

be a non trivial solution of (4.1) *having minimal degree. Then*

$$z_0, z_1, \ldots, z_d, H_2 z_1, \ldots, H_2 z_d$$

are linearly independent.

Proof. Step 1. $H_2 z_1, \ldots, H_2 z_d$ are linearly independent in $\mathbb{C}^n$.

We first show that $H_2 z_1 \neq 0$. Otherwise,

$$0 = H_2 z_1 = H_1 z_0 \Longrightarrow 0 \neq z_0 \in \ker H_2 \cap \ker H_1$$

contradicts the assumption that F is nondegenerate.

Assume, by contradiction, that $H_2 z_1, \ldots, H_2 z_d$ are linearly dependent. Then there is a smallest integer h, with $2 \leq h \leq d$, for which $H_2 z_h$ is linearly dependent from $\{H_2 z_i \mid 1 \leq i < h\}$. Let

$$H_2 z_h = a_1 H_2 z_1 + \cdots + a_{h-1} H_2 z_{h-1}.$$

This equation yields

$$H_1 z_{h-1} = a_1 H_1 z_0 + \cdots + a_{h-1} H_1 z_{h-2}.$$

Hence, setting

$$z'_{h-1} = z_{h-1} - a_1 z_0 - \cdots - a_{h-1} z_{h-2}$$

we obtain

$$H_1 z'_{h-1} = 0, \quad H_2 z'_{h-1} = H_1(z_{h-2} - a_2 z_0 - \cdots - a_{h-1} z_{h-3}) = H_1 z'_{h-2},$$

with

$$z'_{h-2} = z_{h-2} - a_2 z_0 - \cdots - a_{h-1} z_{h-3}.$$

Let us define the chain

$$\begin{cases} z'_0 = z_0, \\ z'_{h-j} = z_{h-j} - a_j z_0 - a_{j+1} z_1 - \ldots - a_{h-1} z_{h-j-1}, \quad \text{for } 1 \leq j < h. \end{cases}$$

Then

$$\begin{cases} H_2 z'_0 = 0, \\ H_2 z'_j = H_1 z'_{j-1}, \quad \text{for} \quad 1 \leq j \leq h-1, \\ \quad 0 \;\; = H_1 z'_{h-1}. \end{cases}$$

Indeed we already know that the first and the last equalities hold. For $2 \leq j < h$ we obtain

$$\begin{aligned} H_2 z'_{h-j} &= H_2(z_{h-j} - a_j z_0 - a_{j+1} z_1 - \ldots - a_{h-1} z_{h-j-1}) \\ &= H_1 z_{h-j-1} - a_{j+1} H_1 z_0 - \ldots - a_{h-1} H_1 z_{h-j-2} \\ &= H_1 z'_{h-j-2}. \end{aligned}$$

Hence $z'(\lambda) = z'_0 + z'_1\lambda + \cdots + z'_{h-1}\lambda^{h-1} \in \mathcal{Z}(H_1, H_2)$, yielding a contradiction, because $h-1 < d$.

STEP 2. $z_0, z_1, \ldots, z_d$ *are linearly independent.*

Indeed, if $a_0 z_0 + a_1 z_1 + \cdots + a_d z_d = 0$, we obtain $a_1 H_2 z_1 + \cdots a_d H_2 z_d = 0$. Hence $a_1 = 0, \ldots, a_d = 0$ by STEP 1. Then $a_0 z_0 = 0$ implies that also $a_0 = 0$, because $z_0 \neq 0$.

STEP 3. $z_0, z_1, \ldots, z_d, H_2 z_1, \ldots, H_2 z_d$ are linearly independent. Assume that

$$a_0 z_0 + a_1 z_1 \cdots + a_d z_d + b_1 H_2 z_1 + \cdots + b_d H_2 z_d = 0.$$

By multiplying to the left by $z_i^* H_2$, for $i = 1, \ldots, d$, we obtain that

$$\sum_{j=1}^{d} b_j z_i^* H_2^2 z_j = 0, \quad \text{for } i = 1, \ldots, d.$$

Since the matrix $(z_i^* H_2^2 z_j)_{i,j=1,\ldots,d}$ is positive definite, this implies that $b_1 = \cdots = b_d = 0$. Then also $a_0 = a_1 = \cdots = a_d = 0$, because $z_0, \ldots, z_d$ are linearly independent. □

4.2. **The smallest minimal index.**

Lemma 4.7. *Let $\langle H_1, H_2 \rangle$ be a nondegenerate singular pencil of Hermitian matrices and let*

$$z(\lambda) = z_0 + \lambda z_1 + \cdots + \lambda^d z_d \in \mathcal{Z}(H_1, H_2)$$

be a non trivial solution of (4.1) *having minimal degree d. Then*

$$z_0, z_1, \ldots, z_d, H_2 z_1, \ldots, H_2 z_d$$

are linearly independent.

Proof. STEP 1. $H_2 z_1, \ldots, H_2 z_d$ *are linearly independent in* $\mathbb{C}^n$.

We first show that $H_2 z_1 \neq 0$. Otherwise,

$$0 = H_2 z_1 = H_1 z_0 \Longrightarrow 0 \neq z_0 \in \ker H_2 \cap \ker H_1$$

contradicts the assumption that F is nondegenerate.

Assume, by contradiction, that $H_2 z_1, \ldots, H_2 z_d$ are linearly dependent. Then there is a smallest $h \geq 2$ for which $H_2 z_h$ is linearly dependent from $\{H_2 z_i \mid 1 \leq i < h\}$. Let

$$H_2 z_h = a_1 H_2 z_1 + \cdots + a_{h-1} H_2 z_{h-1}.$$

This equation yields

$$H_1 z_{h-1} = a_1 H_1 z_0 + \cdots + a_{h-1} H_1 z_{h-2}.$$

Hence, setting

$$z'_{h-1} = z_{h-1} - a_1 z_0 - \cdots - a_{h-1} z_{h-2}$$

we obtain

$$H_1 z'_{h-1} = 0, \quad H_2 z'_{h-1} = H_1(z_{h-2} - a_2 z_0 - \cdots - a_{h-1} z_{h-3}) = H_1 z'_{h-2},$$

with

$$z'_{h-2} = z_{h-2} - a_2 z_0 - \cdots - a_{h-1} z_{h-3}.$$

Let us define the chain

$$\begin{cases} z'_0 = z_0, \\ z'_{h-j} = z_{h-j} - a_j z_0 - a_{j+1} z_1 - \ldots - a_{h-1} z_{h-j-1} \quad \text{for } 1 \leq j < h. \end{cases}$$

Then

$$\begin{cases} H_2 z'_0 = 0, \\ H_2 z'_j = H_1 z'_{j-1}, \quad \text{for} \quad 1 \leq j \leq h-1, \\ \quad 0 \;\; = H_1 z'_{h-1}. \end{cases}$$

Indeed we already know that the first and the last equalities hold. For $2 \leq j < h$ we obtain

$$\begin{aligned} H_2 z'_{h-j} &= H_2(z_{h-j} - a_j z_0 - a_{j+1} z_1 - \ldots - a_{h-1} z_{h-j-1}) \\ &= H_1 z_{h-j-1} - a_{j+1} H_1 z_0 - \ldots - a_{h-1} H_1 z_{h-j-2} \\ &= H_1 z'_{h-j-2}. \end{aligned}$$

Hence $z'(\lambda) = z'_0 + z'_1\lambda + \cdots + z'_{h-1}\lambda^{h-1} \in \mathcal{Z}(H_1, H_2)$, yielding a contradiction, because $h-1 < d$.

STEP 2. $z_0, z_1, \ldots, z_d$ *are linearly independent.*

Indeed, if $a_0 z_0 + a_1 z_1 + \cdots + a_d z_d = 0$, we obtain $a_1 H_2 z_1 + \cdots a_d H_2 z_d = 0$. Hence $a_1 = 0, \ldots, a_d = 0$ by STEP 1. Then $a_0 z_0 = 0$ implies that also $a_0 = 0$, because $z_0 \neq 0$.

STEP 3. Assume that

$$a_0 z_0 + a_1 z_1 \cdots + a_d z_d + b_1 H_2 z_1 + \cdots + b_d H_2 z_d = 0.$$

By multiplying to the left by $z_i^* H_2$, for $i = 1, \ldots, d$, we obtain that

$$\sum\nolimits_{j=1}^{d} b_j z_i^* H_2^2 z_j = 0, \quad \text{for } \; i = 1, \ldots, d.$$

Since the matrix $(z_i^* H_2^2 z_j)_{i,j=1,\ldots,d}$ is positive definite, this implies that $b_1 = \cdots = b_d = 0$. Then also $a_0 = a_1 = \cdots = a_d = 0$ because $z_0, \ldots, z_d$ are linearly independent. □

Proposition 4.8. *Let*

$$z(\lambda) = z_0 + \lambda z_1 + \cdots + \lambda^d z_d \in \mathcal{Z}(H_1, H_2)$$

be a non trivial solution of (4.1) *of minimal degree d. Then we can complete $z_0, \dots, z_d$ to a basis of $\mathbb{C}^n$ in such a way that the matrix representing $H(\lambda)$ in this basis be of the form*

$$\begin{pmatrix} 0 & L^*(\lambda) & 0 \\ L(\lambda) & 0 & 0 \\ 0 & 0 & Q(\lambda) \end{pmatrix}$$

with $Q(\lambda) = Q_2(\lambda) + \lambda Q_1(\lambda)$ for $Q_1, Q_2 \in \mathbf{H}_{n-2d-1}$ and $L_d(\lambda) = J_d - \lambda K_d$, with

$$J_d = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \end{pmatrix}, \ K_d = \begin{pmatrix} 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix} \in \mathbb{M}_{d\times(d+1)}.$$

Proof. Let us set $m = n - 2d - 1$. Let $N_1 = \langle z_0, \dots, z_d \rangle$. We observe that $H_1(N_1) = H_2(N_1)$ and therefore $N_1^{\perp} = N_1^{\dagger} = N_1^{\perp\!\!\perp}$. Choosing a basis $e_1, \dots, e_n$ of $\mathbb{C}^n$ with $e_i = z_{i-1}$ for $1 \le i \le d+1$, $e_1, \dots, e_{n-d}$ a basis of $N_1^{\perp\!\!\perp}$ and $e_i^* H_2 e_{n-d+j} = \delta_{i,j}$ for $1 \le i, j \le d$, we can assume that

$$H(\lambda) = \begin{pmatrix} 0 & 0 & L^*(\lambda) \\ 0 & B(\lambda) & C^*(\lambda) \\ L(\lambda) & C(\lambda) & D(\lambda) \end{pmatrix}$$

with $B(\lambda) = B_2 - \lambda B_1$, $C(\lambda) = C_2 - \lambda C_1$, $D(\lambda) = D_2 - \lambda D_1$ for $B_1, B_2 \in \mathbf{H}_m$, $C_1, C_2 \in \mathbb{M}_{d\times m}(\mathbb{C})$, $D_1, D_2 \in \mathbf{H}_d$.

We divide the proof in several steps.

Step 1. We show first that *the equation*

$$B(\lambda)\theta(\lambda) = 0, \quad \text{with} \quad \theta(\lambda) \in \mathbb{C}^m[\lambda] \tag{4.4}$$

has no non trivial solution of degree less than d.

Set

$$\Phi(\lambda) = (C(\lambda), D(\lambda)), \quad \Psi(\lambda) = \begin{pmatrix} 0 & L_d^*(\lambda) \\ B(\lambda) & D(\lambda) \end{pmatrix}, \quad H(\lambda) = \begin{pmatrix} 0 & \Psi(\lambda) \\ L_d(\lambda) & \Phi(\lambda) \end{pmatrix}.$$

We claim that we can find matrices

$$X \in \mathbb{M}_{(d+1)\times(n-d-1)}(\mathbb{C}) \quad \text{and} \quad Y \in \mathbb{M}_{d\times(n-d)}(\mathbb{C})$$

such that

$$\begin{pmatrix} I_{n-d} & 0 \\ Y & I_d \end{pmatrix} H(\lambda) \begin{pmatrix} I_{d+1} & -X \\ 0 & I_{n-d-1} \end{pmatrix} = \begin{pmatrix} 0 & \Psi(\lambda) \\ L_d(\lambda) & 0 \end{pmatrix}. \tag{4.5}$$

Equation (4.5) is equivalent to

$$L_d(\lambda)X = Y\Psi(\lambda) + \Phi(\lambda). \tag{4.6}$$

Set

$$X = \begin{pmatrix} X_1 \\ \vdots \\ X_{d+1} \end{pmatrix}, \quad \text{with} \quad X_i \in \mathbb{M}_{1\times(n-d-1)}.$$

Then

$$L_d(\lambda)X = \begin{pmatrix} X_2 \\ \vdots \\ X_{d+1} \end{pmatrix} - \lambda \begin{pmatrix} X_1 \\ \vdots \\ X_d \end{pmatrix}.$$

Thus the image of

$T : \mathbb{M}_{(d+1)\times(n-d-1)} \ni X \longrightarrow L_d(\lambda)X \in \mathbb{M}_{d\times(n-d-1)} + \lambda\mathbb{M}_{d\times(n-d-1)} \simeq \mathbb{M}_{2d\times(n-d-1)}$

consists of all

$$U - \lambda V, \quad \text{with} \quad U = \begin{pmatrix} U_1 \\ \vdots \\ U_d \end{pmatrix}, \quad V = \begin{pmatrix} V_1 \\ \vdots \\ V_d \end{pmatrix}, \quad U_i, V_i \in \mathbb{M}_{1\times(n-d-1)},$$

$$\text{with} \quad \begin{pmatrix} U_1 \\ \vdots \\ U_{d-1} \end{pmatrix} = \begin{pmatrix} V_2 \\ \vdots \\ V_d \end{pmatrix}, \quad \text{i.e} \quad K_{d-1}U = J_{d-1}V.$$

Thus (4.6) is solvable if and only if

$$0 = (K_{d-1}, -J_{d-1})Y \begin{pmatrix} \Psi_0 \\ \Psi_1 \end{pmatrix} + (K_{d-1}, -J_{d-1}) \begin{pmatrix} \Phi_0 \\ \Phi_1 \end{pmatrix}, \tag{4.7}$$

where we set $\Psi(\lambda) = \Psi_0 + \lambda\Psi_1$ and $\Phi(\lambda) = \Phi_0 + \lambda\Phi_1$, with $\Psi_0, \Psi_1 \in \mathbb{M}_{(n-d)\times(n-d-1)}(\mathbb{C})$ and $\Phi_0, \Phi_1 \in \mathbb{M}_{d\times(n-d-1)}(\mathbb{C})$.

Set

$$Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_d \end{pmatrix}, \quad \text{with} \quad Y_i \in \mathbb{M}_{1\times(n-d)}(\mathbb{C}).$$

We have

$$(K_{d-1}, -J_{d-1})Y \begin{pmatrix} \Psi_0 \\ \Psi_1 \end{pmatrix} = \begin{pmatrix} Y_1 & -Y_2 \\ Y_2 & -Y_3 \\ \vdots & \vdots \\ Y_{d-1} & -Y_d \end{pmatrix} \begin{pmatrix} \Psi_0 \\ \Psi_1 \end{pmatrix}$$

$$= \begin{pmatrix} Y_1, \dots, Y_d \end{pmatrix} \begin{pmatrix} \Psi_0 & & & \\ -\Psi_1 & \Psi_0 & & \\ & -\Psi_1 & \ddots & \\ & & \ddots & \ddots & \\ & & & \ddots & \Psi_0 \\ & & & & -\Psi_1 \end{pmatrix}$$

Since, by assumption, the equation $\Psi(\lambda)\theta(\lambda)$ has no non trivial solution $\theta(\lambda) \in \mathbb{C}^{n-d-1}[\lambda]$ of degree less than d, the matrix in the right hand side of the last expression has rank larger or equal to $(d-1)(n-d-1)$. Hence the map

$$T' : \mathbb{M}_{(n-d-1)\times(d+1)} \ni Y \longrightarrow (K_{d-1}, -J_{d-1})Y \begin{pmatrix} \Psi_0 \\ \Psi_1 \end{pmatrix} \in \mathbb{M}_{(d-1)\times(n-d-1)}(\mathbb{C})$$

is surjective and (4.7) admits a solution.

Since

$$\begin{pmatrix} 0 & \Psi(\lambda) \\ L_d(\lambda) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & L_d^*(\lambda) \\ 0 & B(\lambda) & C^*(\lambda) \\ L(\lambda) & 0 & 0 \end{pmatrix},$$

if $\theta(\lambda) \in \mathbb{C}^m[\lambda]$ is a solution of (4.4), then

$$\tilde{\theta}(\lambda) = \begin{pmatrix} -X\theta \\ 0 \end{pmatrix} \in \mathbb{C}^n[\lambda]$$

is a solution of (4.1) having the same degree of $\theta(\lambda)$. This yields the statement of step 1.

Step 2. *There is a matrix*

$$A = \begin{pmatrix} I_{d+1} & 0 & 0 \\ \alpha & I_m & 0 \\ \gamma & \beta & I_d \end{pmatrix},$$

with $\alpha \in \mathbb{M}_{m\times(d+1)}, \beta \in \mathbb{M}_{d\times m}, \gamma \in \mathbb{M}_{d\times(d+1)}$, *such that*

$$AH(\lambda)A^* = \begin{pmatrix} 0 & 0 & L^*(\lambda) \\ 0 & B(\lambda) & 0 \\ L(\lambda) & 0 & 0 \end{pmatrix}. \tag{4.8}$$

Equation (4.8) is equivalent to

$$\begin{cases} L_d(\lambda)\alpha^* + \beta B(\lambda) + C(\lambda) = 0, \\ L_d(\lambda)\gamma^* + \gamma L_d^*(\lambda) + \beta B(\lambda)\beta^* + C(\lambda)\beta^* + \beta C^*(\lambda) + D(\lambda) = 0. \end{cases} \tag{4.9}$$

Let us consider the map

$$T : \mathbb{M}_{d\times(d+1)}(\mathbb{C}) \ni \gamma \longrightarrow \gamma L_d^*(\lambda) + L_d(\lambda)\gamma^* \in \mathbf{H}_d + \lambda\mathbf{H}_d \simeq \mathbb{R}^{2d^2}.$$

Its kernel consists of the matrices $\gamma = (\gamma_{i,j})_{\substack{1\le i\le d \\ 1\le j\le d+1}}$ with

$$\begin{cases} \gamma_{i,j} = -\bar{\gamma}_{j,i} & \text{for} \quad 1 \le i, j \le d, \\ \gamma_{i,j+1} = -\bar{\gamma}_{j,i+1} & \text{for} \quad 1 \le i, j \le d. \end{cases}$$

Hence the kernel of T consists of the matrices γ for which

$$\begin{cases} \gamma_{i,j} = \gamma_{r,s} & \text{if} \quad i + j = r + s, \\ \gamma_{i,j} \in i\mathbb{R} & \text{if} \quad 1 \le i \le d,\ 1 \le j \le d + 1. \end{cases}$$

Thus $\dim_{\mathbb{R}} \ker T = 2d$ and therefore

$$\operatorname{rank} T = 2d(d + 1) - 2d = 2d^2$$

proves that T is onto.

Therefore, for any choice of α and β, we can choose γ in such a way that the second equation of (4.9) is satisfied.

Let us show that α, β can be chosen to solve the first equation in (4.9). The argument is very similar to the one used in step 1.

Write

$$\alpha = (\alpha_1, \dots, \alpha_{d+1}), \quad \text{with} \quad \alpha_i \in \mathbb{M}_{m\times 1}.$$

Then

$$L_d(\lambda)\alpha^* = \begin{pmatrix} \alpha_2^* \\ \vdots \\ \alpha_{d+1}^* \end{pmatrix} - \lambda \begin{pmatrix} \alpha_1^* \\ \vdots \\ \alpha_d^* \end{pmatrix}.$$

Thus we can solve the equation

$$U - \lambda V = \begin{pmatrix} U_1 \\ \vdots \\ U_d \end{pmatrix} - \lambda \begin{pmatrix} V_1 \\ \vdots \\ V_d \end{pmatrix} = L_d(\lambda)\alpha^*$$

if and only if

$$K_{d-1}U = \begin{pmatrix} U_1 \\ \vdots \\ U_{d-1} \end{pmatrix} = \begin{pmatrix} V_2 \\ \vdots \\ V_d \end{pmatrix} = J_{d-1}V.$$

Therefore the first equation in (4.9) is solvable if and only if we can find a solution β of

$$(K_{d-1}, -J_{d-1})\beta \begin{pmatrix} B_2 \\ B_1 \end{pmatrix} = (-K_{d-1}, J_{d-1}) \begin{pmatrix} C_2 \\ C_1 \end{pmatrix} \in \mathbb{M}_{(d-1)\times m}(\mathbb{C}).$$

Write

$$\beta = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_d \end{pmatrix}, \quad \text{with} \quad \beta_i \in \mathbb{M}_{1\times m}.$$

We have

$$(K_d, -J_d)\beta \begin{pmatrix} B_2 \\ B_1 \end{pmatrix} = \begin{pmatrix} \beta_1 & -\beta_2 \\ \beta_2 & -\beta_1 \\ \vdots & \vdots \\ \beta_{d-1} & \beta_d \end{pmatrix} \begin{pmatrix} B_2 \\ B_1 \end{pmatrix} = (\beta_1, \dots, \beta_d) \begin{pmatrix} B_2 & & & \\ -B_1 & B_2 & & \\ & \ddots & \ddots & \\ & & -B_1 & B_2 \\ & & & -B_1 \end{pmatrix}.$$

By step 1, we know that the matrix

$$M_{d-1}(B_1, B_2) = \begin{pmatrix} B_2 & & & \\ -B_1 & B_2 & & \\ & \ddots & \ddots & \\ & & -B_1 & B_2 \\ & & & -B_1 \end{pmatrix}$$

has rank greater or equal to $m(d-1)$. Hence the map

$$\mathbb{M}_{1\times dm} \ni (\beta_1, \dots, \beta_d) \to \beta M_{d-1}(B_1, B_2) \in \mathbb{M}_{(d-1)\times m}(\mathbb{C})$$

is surjective. Thus we can solve (4.8). We obtain the thesis by observing that

$$\begin{pmatrix} I_d & 0 & 0 \\ 0 & 0 & I_{d+1} \\ 0 & I_m & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & L_d^*(\lambda) \\ 0 & B(\lambda) & 0 \\ L_d(\lambda) & 0 & 0 \end{pmatrix} \begin{pmatrix} I_d & 0 & 0 \\ 0 & 0 & I_{d+1} \\ 0 & I_m & 0 \end{pmatrix} = \begin{pmatrix} 0 & L_d^*(\lambda) & 0 \\ L_d(\lambda) & 0 & 0 \\ 0 & 0 & B(\lambda) \end{pmatrix}.$$

□

4.3. **The subspace $N^{\perp\!\!\perp}$ and the nonsingular core.**

Proposition 4.9. *Let N be the subspace of Definition 4.4. Then*

$$N \subset N^{\perp\!\!\perp} = N^{\perp} = N^{\dagger}. \tag{4.10}$$

Proof. Since the coefficients of the solutions of (4.1) satisfy (4.2), we obtain $H_1(N) = H_2(N)$. Therefore

$$N^{\perp} = [H_1(N)]^0 = [N_2(N)]^0 = N^{\dagger} \Longrightarrow N^{\perp\!\!\perp} = N^{\perp} \cap N^{\dagger} = N^{\perp} = N^{\dagger}.$$

The inclusion $N \subset N^{\perp\!\!\perp}$ follows, because N is totally isotropic for both H_1 and H_2. □

Since N is totally isotropic for both H_1 and H_2, the Hermitian symmetric forms

$$(v, w) \longrightarrow w^* H_1 v, \quad (v, w) \longrightarrow w^* H_2 v$$

define Hermitian symmetric forms on the quotient $\hat{V} = N^{\perp\!\!\perp}/N$.

Proposition 4.10. *Assume that the pencil $\langle H_1, H_2 \rangle$ is nondegenerate.*

Let V be any linear complement of N in $N^{\perp\!\!\perp}$. Then we can find a complement W of $N^{\perp\!\!\perp}$ in $\mathbb{C}^n$ such that

$$\mathbb{C}^n = N \oplus V \oplus W, \quad N^{\perp\!\!\perp} = N \oplus V, \; V^{\perp\!\!\perp} = N \oplus W, \; W^{\perp\!\!\perp} = V \oplus W. \tag{4.11}$$

Moreover:

(1) *$\mathbb{C}^n = (N \oplus W) \oplus V$ is a biorthogonal decomposition and the restrictions of $\langle H_1, H_2 \rangle$ to the subspaces $N \oplus W$ and V are both nondegenerate.*
(2) *The restriction of $\langle H_1, H_2 \rangle$ to V is regular.*
(3) *No bihortogonal decomposition of $N \oplus W$ for the restriction of $\langle H_1, H_2 \rangle$ contains a subspace on which the restriction of $\langle H_1, H_2 \rangle$ is regular.*

Definition 4.11. The quotient $\hat{V} = N^{\perp\!\!\perp}/N$, or any linear complement V of N in $N^{\perp\!\!\perp}/N$, are called *the nonsingular core* of $\langle H_1, H_2 \rangle$.

To complete the proof of Theorem 2.8 it remains to find the biorthogonal decomposition and the associated canonical forms in the case where $\langle H_1, H_2 \rangle$ is nonsingular.

4.4. **The nonsingular case.** Assume that $F = \langle H_1, H_2 \rangle$ is nonsingular and take a nondegenerate $H = a_1H_1 + a_2H_2 \in F$. Set $L_1 = H^{-1}H_1$, $L_2 = H^{-1}H_2$. Since $a_1L_1 + a_2L_2 = I_n$, the two endomorphisms commute. Moreover, all the elements of $\mathbb{C}[L_1, L_2]$, which are not proportional to the identity, have the same Jordan decomposition

$$\mathbb{C}^n = Q_1 \oplus \cdots Q_s \tag{4.12}$$

into a direct sum of indecomposable subspaces. Indeed:

Proposition 4.12. *Assume that $F = \langle H_1, H_2 \rangle$ is nonsingular. Let $H, K \in F$ be linearly independent, with $H \in F \cap \mathbf{GL}_n(\mathbb{C})$ and set $L = H^{-1}K$. Then the ring $\mathbb{C}[L]$ contains all endomorphisms of the form $A^{-1}B$ with $A, B \in F$ linearly independent, and $A \in F \cap \mathbf{GL}_n(\mathbb{C})$.*

Proof. We can assume for simplicity that $H_1 \in F \cap \mathbf{GL}_n(\mathbb{C})$ and prove the statement for $L = H_1^{-1}H_2$. If $a, b, c, d \in \mathbb{R}$ and $aH_1 + bH_2 \in \mathbf{GL}_n(\mathbb{C})$, we have

$$(aH_1 + bH_2)^{-1}(cH_1 + dH_2) = (aI_n + bL)^{-1}(cI_n + dL)$$

and the thesis follows because[3] $(aI_n + bL)^{-1} \in \mathbb{C}[L]$. □

On each Q_j, either L_1, or L_2 is invertible. Indeed, the restriction of L_i to Q_j is either invertible or nilpotent. If e.g. L_2 is nilpotent on some Q_j, then $a_1 \neq 0$ because H is not proportional to H_2, and $a_1L_1 = I_n - a_2L_2$ shows that the restriction of a_1L_1 to Q_j is the sum of the identity and of a nilpotent element and hence invertible.

Definition 4.13. Take an invertible $H = a_1H_1 + a_2H_2 \in F$ with $a_1 \neq 0$, so that L_2 is not proportional to I_n. If $\lambda'' \in \mathbb{C}$ is an eigenvalue of L_2, then $\lambda' = a_1^{-1}(1 - a_2\lambda'')$ is the corresponding eigenvalue of L_1. The rational value

$$\lambda = \frac{a_1\lambda''}{1 - a_2\lambda''} \in \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$$

is independent of the choice of H and is called a *root*, or *eigenvalue* of the pair (H_1, H_2).

Let $\Sigma \subset \mathbb{CP}^1$ be the set of roots of the pair (H_1, H_2).

Definition 4.14. Let $H = a_1H_1 + a_2H_2 \in F$ be nondegenerate, with $a_1 \neq 0$, and set $L_2 = H^{-1}H_2$. For each $\lambda \in \Sigma \setminus \{\infty\}$ we set

$$E_\lambda = \bigcup\nolimits_{h=1}^{\infty} \ker(\lambda'' I_n - L_2)^h, \quad \text{with} \quad \lambda'' = \frac{\lambda}{a_1 + a_2\lambda}. \tag{4.13}$$

If $\infty \in \Sigma$, we set

$$E_\infty = \bigcup\nolimits_{h=1}^{\infty} \ker(I_n + a_2L_2)^h. \tag{4.14}$$

[3]Indeed, if $A \in \mathbf{GL}_n(\mathbb{C})$ and $f(\lambda) = \det(\lambda I_n - A) = \lambda^n + k_1\lambda^{n-1} + \cdots + k_n$ is its characteristic polynomial, we have $k_n = (-1)^n \det A \neq 0$. Thus

$$I_n = A(-k_n^{-1}[A^{n-1} + k_1A^{n-2} + \cdots + k_{n-1}I_n]) = Ag(A), \text{with} \quad g(\lambda) \in \mathbb{C}[\lambda].$$

Lemma 4.15. *The definition of the subspaces E_λ is independent of the choice of an invertible $H = a_1H_1 + a_2H_2 \in F$ with $a_1 \neq 0$.*

Proof. If $K = b_1H_1 + b_2H_2 \in F$ is nondegenerate with $b_1 \neq 0$ and $T_2 = K^{-1}H$, then we have, for $c_1 = b_1/a_1$, $c_2 = (b_2a_1 - b_1a_2)/a_1$,

$$T_2 = (c_1I_n + c_2L_2) \in \mathbf{GL}_n(\mathbb{C}), \quad T_2 = (c_1I_n + c_2L_2)^{-1}L_2.$$

Then the thesis follows because

$$\begin{aligned}\bigcup\nolimits_{h=1}^{\infty} \ker(\xi I_n - L_2)^h &= \bigcup\nolimits_{h=1}^{\infty} \ker(c_1\xi I_n - c_1L_2)^h \\ &= \bigcup\nolimits_{h=1}^{\infty} \ker(\xi(c_1I_n + c_2L_2) - (c_1 + c_2\xi)L_2)^h \\ &= \bigcup\nolimits_{h=1}^{\infty} \ker\big(\tfrac{\xi}{c_1+c_2\xi}I_n - T_2\big)^h\end{aligned}$$

for all $\xi \in \mathbb{C}$. □

Proposition 4.16. *If F is nonsingular, we have a direct sum decomposition*

$$\mathbb{C}^n = \bigoplus\nolimits_{\lambda\in\Sigma} E_\lambda. \tag{4.15}$$

Lemma 4.17. $E_{\lambda_i} \perp\!\!\!\perp E_{\lambda_j}$ *if* $\lambda_j \neq \bar{\lambda}_i$.

Proof. We observe that a change of the basis of F only changes the labels of the subspaces E_λ by a real fractional transformation of λ. Hence, we can assume in the proof that H_1 is nondegenerate, so that $\infty \notin \Sigma$.

Let $\lambda_i, \lambda_j \in \Sigma$, with $\lambda_j \neq \bar{\lambda}_i$. Let $v_i \in E_{\lambda_i} \setminus \{0\}$, $v_j \in E_{\lambda_j} \setminus \{0\}$, and h_i, h_j be the smallest positive integers such that $(\lambda_iI_n - L)^{h_i}v_i = 0$, $(\lambda_jI_n - L)^{h_j}v_j = 0$. Set $v_{i,a} = (\lambda_iI_n - L)^{h_i-a}v_i$, $v_{j,b}(\lambda_jI_n - L)^{h_j-b}v_j$, for $0 \leq a \leq h_i$, $0 \leq b \leq h_j$. We prove by recurrence that

$$(P_r) \qquad v_{i,a} \perp\!\!\!\perp v_{j,b} \quad \text{if} \quad 1 \leq a \leq h_1, \quad 1 \leq b \leq h_2, \quad \sup\{a + b\} \leq r.$$

We have

$$\begin{cases} v_{i,0} = 0, \\ H_2v_{i,a} = \lambda_iH_1v_{i,a} - H_1v_{i,a-1}, \\ \qquad\qquad \text{if } 1 \leq a \leq h_i, \end{cases} \qquad \begin{cases} v_{j,0} = 0, \\ H_2v_{j,b} = \lambda_jH_1v_{j,b} - H_1v_{j,b-1}, \\ \qquad\qquad \text{if } 1 \leq b \leq h_j. \end{cases}$$

Thus (P_0) and (P_1) are trivially true. Assume that (P_r) holds for some integer r with $1 \leq r < h_1 + h_2$. It suffices to show that $v_{i,a} \perp\!\!\!\perp v_{j,b}$ for every pair of integers (a, b) with $1 \leq a \leq h_1$, $1 \leq b \leq h_2$ and $a + b \leq r + 1$. We have:

$$\begin{aligned} v_{i,a}^*H_2v_{j,b} &= \lambda_jv_{i,a}^*H_1v_{j,b} - v_{i,a}^*H_1v_{j,b-1} = \lambda_jv_{i,a}^*H_1v_{j,b} \\ &= \bar{\lambda}_iv_{i,a}^*H_1v_{j,b} - v_{i,a-1}^*H_1v_{j,b} = \bar{\lambda}_iv_{i,a}^*H_1v_{j,b}. \end{aligned}$$

This implies that $v_{i,a} \perp\!\!\!\perp v_{j,b}$. By recurrence we obtain $(P_{h_1+h_2})$, i.e. $v_i = v_{i,h_1} \perp\!\!\!\perp v_{j,h_2} = v_j$. □

Corollary 4.18. *Assume that F is nonsingular. Then every subspace V_i of a biorthogonal decomposition* (2.4) *is contained either in a generalized eigenspace E_γ, for some $\gamma \in \mathbb{R}$, or in a direct sum $E_\Gamma \oplus E_{\bar{\Gamma}}$, with $\Gamma \in \mathbb{C}$ and* $\operatorname{Im}\Gamma > 0$.

4.4.1. *Real eigenvalues.*

Proposition 4.19. *Let $\gamma \neq \infty$ be a real root of the pair (H_1, H_2).*

Then the restriction of H_1 to E_γ is nondegenerate and we can consider the endomorphism $L \in \mathfrak{gl}_{\mathbb{C}}(E_\gamma)$ defined by

$$w^* H_2 v = w^* H_1 L v, \quad \forall v, w \in E_\gamma. \tag{4.16}$$

Then $(\gamma I_{E_\gamma} - L)$ is nilpotent. Let ν be its nilpotency index. Then there is an L-invariant indecomposable ν-dimensional subspace W of E_γ such that

$$W^{\perp\!\!\perp} = W^{\perp}, \quad \mathbb{C}^n = W^{\perp\!\!\perp} \oplus W \quad \textit{(biorthogonal decomposition)}. \tag{4.17}$$

There exists a Jordan basis $e_1, \ldots, e_\nu$ of W for L, in which the restrictions of H_1 and H_2 to W are represented by matrices of type (I).

Proof. By substituting $H_2 - \gamma H_1$ to H_2, we can assume that $\gamma = 0$.

Let

$$E_0 = W_1 \oplus W_2 \oplus \cdots \oplus W_\ell \tag{4.18}$$

be a Jordan decomposition of E_0, with $\dim W_1 \geq \dim W_2 \geq \cdots \geq \dim W_\ell$. Then $\dim W_1 = \nu$. Let $e_1, \ldots, e_\nu$ be a Jordan basis for W_1. We have:

$$\begin{cases} H_2 e_1 = 0, \\ H_2 e_2 = H_1 e_1, \\ \ \ldots \quad \ldots \\ H_2 e_\nu = H_1 e_{\nu-1}, \\ \quad 0 = H_1 e_\nu. \end{cases} \tag{4.19}$$

Claim: *We can assume that $e_1^* H_1 e_j \neq 0$ for some j with $1 \leq j \leq \nu$.*

Indeed, if $e_1^* H_1 e_j = 0$ for all j with $1 \leq j \leq \nu$, then, by the assumption that H_1 is nondegenerate, there is some element $e_{a,j}$ of a Jordan basis $e_{a,1}, \ldots, e_{a,n_a}$ of some W_a, with $2 \leq a \leq \ell$, such that $e_1^* H_1 e_{a,j} \neq 0$. Set $e_{a,j} = 0$ for $j \leq 0$, and define

$$\begin{cases} e_1' = e_1 + e_{a,n_i-\nu+1}, \\ e_2' = e_2 + e_{a,n_i-\nu+2}, \\ \ldots \quad \ldots \\ e_\nu' = e_\nu + e_{a,n_i}. \end{cases}$$

Then we have

$$\begin{cases} H_2 e_1' = 0, \\ H_2 e_2' = H_1 e_1', \\ \ \ldots \quad \ldots \\ H_2 e_\nu' = H_1 e_{\nu-1}', \\ \quad 0 = H_1 e_\nu', \end{cases}$$

and we can substitute $\langle e_1', \ldots, e_\nu' \rangle$ to W_1 in the Jordan decomposition (4.18). The new W_1 fulfills our claim.

Assume therefore that W_1 has a basis $e_1, \dots, e_\nu$ satisfying (4.19), with $e_1 \notin W_1^\perp$.

If $\nu = 1$, we have $e_1^{\perp\!\!\!\perp} = e_1^\perp$, and then we obtain decomposition (4.20) because $e_1^* H_1 e_1 \neq 0$.

Set $e_i = 0$ for $i \leq 0$ and $i > \nu$. Then we have

$$\begin{aligned} e_i^* H_2 e_j &= \gamma e_i^* H_1 e_j + e_i^* H_1 e_{j-1} \\ &= \gamma e_i^* H_1 e_j + e_{i-1}^* H_1 e_j \\ &\qquad\qquad \forall 1 \leq i, j \leq n. \end{aligned}$$

This implies that $e_i^* H_1 e_j$, for $1 \leq i, j \leq n$, only depends on the sum $(i+j)$. When $i + j \leq n$, we have

$$e_i^* H_1 e_j = e_0^* H_1 e_{i+j} = 0^* H_1 e_{i+j} = 0.$$

In particular, since $e_1^* H_1 e_j = 0$ for $1 \leq j < \nu$, we obtain that $e_1^* H_1 e_\nu \neq 0$ and, in general

$$e_i^* H_1 e_j = e_1^* H_1 e_\nu \neq 0, \quad \forall 1 \leq i, j \leq \nu \quad \text{with} \quad i + j = \nu.$$

A new Jordan basis $v_1, \dots, v_n$ is obtained by choosing complex numbers $k_0, k_1, \dots, k_{n-1}$, with $k_0 \neq 0$, and setting

$$v_j = \sum\nolimits_{\substack{0 \leq a < n, \\ 1 \leq b \leq n, \\ a+b=j}} k_a e_b, \qquad \text{for} \quad j = 1, \dots, n.$$

Having fixed k_0, we can choose $k_1, \dots, k_{n-1}$ in such a way that

$$v_i^* H_1 v_j = 0 \quad \text{if } i + j \neq n + 1. \tag{4.20}$$

We obtain indeed

$$v_i^* H_1 v_j = \sum\nolimits_{\substack{0 \leq a \leq i \\ 0 \leq b \leq j}} \bar{k}_a k_b e_{i-a}^* H_1 e_{j-b}.$$

As, by the first part of the proof, $v_i^* H_1 v_j = 0$ for $i + j \leq n$, it suffices to show that we can choose $k_1, \dots, k_\nu$ in such a way that

$$v_i^* H_1 v_n = \sum\nolimits_{\substack{0 \leq a \leq i \\ 0 \leq b \leq n}} \bar{k}_a k_b e_{i-a}^* H_1 e_{n-b} = 0 \quad \text{for} \quad i = 2, \dots, n. \tag{4.21}$$

Since $e_a^* H_1 e_b = 0$ when $a + b \leq n$, (4.21) can be rewritten by

$$\sum_{\substack{0 \leq a \leq i \\ 0 \leq b \leq n \\ a+b \leq i-1}} \bar{k}_a k_b e_{i-a}^* H_1 e_{n-b} = 0 \quad \text{for} \quad i = 2, \dots, n,$$

yielding

$$\sum\nolimits_{\nu=0}^{i-1} \left(\sum\nolimits_{a+b=\nu} \bar{k}_a k_b\right) e_{i-\nu}^* H_1 e_n = 0. \tag{4.22}$$

Equations (4.22) express each sum $\bar{k}_0 k_{i-1} + \bar{k}_{i-1} k_0$ as a quadratic polynomial in $k_a, \bar{k}_a$ for $1 \leq a < (i-1)$ and therefore can be recursively solved. Finally, we can fix k_0 in such a way that $v_1^* H_1 v_n = \pm 1$. In this way we obtain the canonical expression (I) for H_1, H_2. □

Corollary 4.20. *If $\gamma \in \mathbb{R}$ is a real root of (H_1, H_2), then there is a Jordan decomposition*

$$E_\gamma = W_1 \oplus \cdots \oplus W_\ell$$

of E_γ for $L = H_1^{-1} H_2$, which is also a biorthogonal decomposition of E_γ for F. We can choose a basis in each W_j in which the matrices associated to the restrictions of H_1, H_2 have the form (I).

Proposition 4.21. *If ∞ is a root of (H_1, H_2), then H_2 is nondegenerate on E_∞ and we can define $L \in \mathfrak{gl}_{\mathbb{C}}(E_\infty)$ by setting*

$$w^* H_1 v = w^* H_2 L v, \quad \forall v, w \in E_\infty. \tag{4.23}$$

Then L is nilpotent. Let ν be its nilpotency index. Then there exists an L-invariant indecomposable ν-dimensional subspace of E_∞ such that

$$W^{\perp\!\!\perp} = W^\dagger, \quad \mathbb{C}^n = W^{\perp\!\!\perp} \oplus W \quad \textit{(biorthogonal decomposition)}. \tag{4.24}$$

There exists a Jordan basis $e_1, \ldots, e_\nu$ of W in which the restrictions of H_1, H_2 to W are matrices of type (III).

Proof. After exchanging H_1 and H_2, we repeat the proof of Proposition 4.19. □

Corollary 4.22. *If ∞ is a root of (H_1, H_2), then there is a Jordan decomposition*

$$E_\gamma = W_1 \oplus \cdots \oplus W_\ell$$

of E_∞ for $L = H_2^{-1} H_1$, which is also a biorthogonal decomposition of E_∞ for F. We can choose a basis in each W_j in which the matrices associated to the restrictions of H_1, H_2 have the form (III).

4.4.2. *Complex eigenvalues.*

Proposition 4.23. *Assume that $F = \langle H_1, H_2 \rangle$ is nonsingular. Let $\Gamma \in \mathbb{C}$, with $\operatorname{Im}\Gamma > 0$ be a complex root of (H_1, H_2). Then both H_1 and H_2 are nondegenerate on $E_\Gamma \oplus E_{\bar{\Gamma}}$. Define $L \in \mathfrak{gl}_{\mathbb{C}}(E_\Gamma \oplus E_{\bar{\Gamma}})$ by*

$$w^* H_2 v = w^* H_1 L v, \quad \forall v, w \in E_\Gamma \oplus E_{\bar{\Gamma}}). \tag{4.25}$$

Then E_Γ and $E_{\bar{\Gamma}}$ are L-invariant and $(\Gamma I - L)$ is nilpotent on E_Γ, $(\bar{\Gamma} I - L)$ is nilpotent on $E_{\bar{\Gamma}}$, with the same nilpotency indices.

We have Jordan decompositions

$$E_\Gamma = V_1 \oplus \cdots V_\ell,$$
$$E_{\bar{\Gamma}} = W_1 \oplus \cdots W_\ell$$

of $E_\Gamma, E_{\bar{\Gamma}}$ with respect to L such that

$$E_\Gamma \oplus E_{\bar{\Gamma}} = (V_1 \oplus W_1) \oplus \cdots \oplus (V_\ell \oplus W_\ell) \tag{4.26}$$

is a biorhtogonal decomposition of $E_\Gamma \oplus E_{\bar{\Gamma}}$ for F.

We can find basis of V_j, W_j such that the corresponding matrices of the restrictions of H_1, H_2 to $V_j \oplus W_j$ have the form (II).

Proof. We observe that the Jordan decompositions of E_Γ and $E_{\bar\Gamma}$ with respect to L have the same number of subspaces of the same dimensions, because the two endomorphisms L and $L^* = H_1 L H_1^{-1}$ are conjugate. We also know that E_Γ and $E_{\bar\Gamma}$ are totally isotropic for both H_1 and H_2.

Assume that $\dim V_1 \geq \dim V_2 \geq \cdots \dim V_\ell$. Let $e_1, \ldots, e_\nu$ be a Jordan basis of V_1 for L. We have, with $e_0 = 0$,

$$H_2 e_j = \Gamma H_1 e_j + H_1 e_{j-1}, \quad \text{for} \quad 1 \leq j \leq \nu. \tag{4.27}$$

Since H_1 is, by assumption, nondegenerate, we can assume that $e_1 \notin W_1^\perp$. Let $v_1, \ldots, v_\mu$ be a Jordan basis for W_1. With $v_0 = 0$ we have

$$H_2 v_j = \bar\Gamma H_1 v_j + H_1 v_{j-1}, \quad \text{for} \quad 1 \leq j \leq \mu. \tag{4.28}$$

Thus we obtain, for all $1 \leq i \leq \nu$ and $1 \leq j \leq \mu$,

$$\begin{aligned} v_i^* H_2 e_j &= \Gamma v_i^* H_1 e_j + v_i^* H_1 e_{j-1} \\ &= \Gamma v_i^* H_1 e_j + v_{i-1}^* H_1 e_j. \end{aligned}$$

This implies that

$$v_i^* H_1 e_j = v_h^* H_1 e_k \quad \text{if} \quad 0 \leq i, h \leq \mu,\ 0 \leq j, k \leq \nu.$$

In particular,

$$v_i^* H_1 e_1 = v_0^* H_1 e_{1+i} = 0 \quad \text{if} \quad 1 \leq i \leq \min\{\mu, \nu - 1\}$$

implies that $\mu = \nu$ and that $v_\nu^* H_1 e_1 \neq 0$. By rescaling, we can obtain that

$$v_j^* H_1 e_i = v_\nu^* H_1 e_1 = 1, \quad \text{if} \quad 1 \leq i, j \leq \nu,\ i + j = \nu + 1.$$

We can modify the Jordan basis $e_1, \ldots, e_\nu$ by setting, after fixing complex numbers $k_1, \ldots, k_{\nu-1}$,

$$\begin{cases} u_1 = e_1, \\ u_i = e_i + k_1 e_{i-1} + \cdots + k_i e_1, \quad \text{for } 2 \leq i \leq \nu. \end{cases}$$

Then to require that $v_i^* H_1 u_j = 0$ for $i + j \neq \nu + 1$ is equivalent to

$$v_\nu^* H_1 u_j = v_\nu^* H_1 e_j + k_1 v_\nu^* H_1 e_{j-1} + \cdots + k_{j-2} v_\nu^* H_1 e_2 + k_{j-1} = 0, \quad \text{for } j = 2, \ldots, \nu.$$

This system of linear equations has a unique solution, yielding a basis $u_1, \ldots, u_\nu$ such that the restrictions of H_1, H_2 to $V_1 \oplus W_1$ have in this basis the canonical form (II).

By recurrence we obtain the thesis, by applying the argument of the proof to $(E_\Gamma \oplus E_{\bar\Gamma}) \cap (V_1 \oplus W_1)^{\perp\!\!\!\perp}$. □

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