Quantum groupoids and matched pairs of groupoids

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C^* quantum groupoid $(A, \Gamma, \kappa, \epsilon)$

Definition (Bohm, Nill, Szlachanyi)

- A : finite dim. C^*-algebra $(A = \bigoplus_{i=1,..k} M_{n_i}(\mathbb{C}))$
- $\Gamma : A \rightarrow A \otimes A$: $*$-homom.
  $(\Gamma \otimes i)\Gamma = (i \otimes \Gamma)\Gamma$ ($\Gamma(1) \neq 1 \otimes 1$, in general)
- $\kappa : A \rightarrow A$, linear antimultiplicative :
  $(\kappa \circ *)^2 = i$
  $\varsigma(\kappa \otimes \kappa)\Gamma = \Gamma \circ \kappa$
  $(m(\kappa \otimes i) \otimes i)(\Gamma \otimes i)\Gamma(x) = (1 \otimes x)\Gamma(1)$
  $(\mathrm{where\ m(a \otimes b) = ab})$
- $\epsilon : A \rightarrow \mathbb{C}$: linear s.t. $(\epsilon \otimes i)\Gamma = (i \otimes \epsilon)\Gamma = i$
  $(\epsilon \otimes \epsilon)((x \otimes 1)\Gamma(1)(1 \otimes y)) = \epsilon(xy)$

Bases : $A_t := \{a \in A / \Gamma(a) = (a \otimes 1)\Gamma(1) = \Gamma(1)(a \otimes 1)\}$
$A_s := \{a \in A / \Gamma(a) = (1 \otimes a)\Gamma(1) = \Gamma(1)(1 \otimes a)\}$

Groupoids

- $\forall x \in \mathcal{G}$, there exists x^{-1},
- $\mathcal{G}^0$(Units:) $\subset \mathcal{G}$: $s(x) = x^{-1}x, t(x) = xx^{-1}$
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  $x.y$ composable iff $s(x) = t(y)$
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$$\mathcal{G} = \bigsqcup_{finite} X_\alpha \times X_\alpha \times G_\alpha$$

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Quantum groupoids and subfactors

Theorem

(Nikshych-Vainerman)

$M_0 \subset M_1$: type II_1 subfactors of finite index and depth 2

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- $A = M_0' \cap M_2$, $B = M_1' \cap M_3$: C^*-**weak Hopf algebras** in duality
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If $\mathcal{G}$ group $\quad \gamma_{H,K}(g) \equiv |H \cap K|$, if $H \cap K = \mathcal{G}^0 \quad \gamma_{H,K}(g) \equiv 1$

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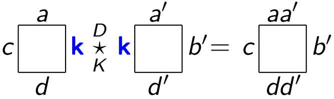

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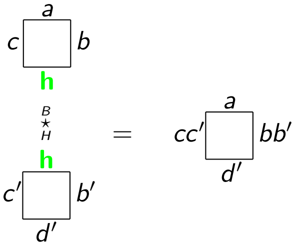

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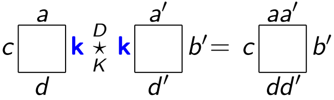

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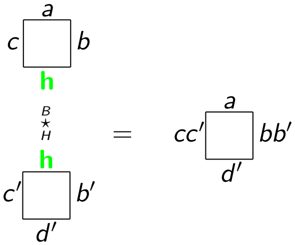

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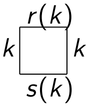

Adapted pairs and C^* quantum groupoids

$\mathcal{T}$ is a groupoid of basis K and a groupoid of basis H

- Horizontal product: $c\,\underset{d}{\overset{a}{\boxed{}}}\,\mathbf{k} \underset{K}{\overset{D}{\star}} \mathbf{k}\,\underset{d'}{\overset{a'}{\boxed{}}}\,b' = c\,\underset{dd'}{\overset{aa'}{\boxed{}}}\,b'$
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Corollary: $\kappa^2(t) = \frac{\ulcorner(t)\lrcorner(t)}{\llcorner(t)\urcorner(t)} t$

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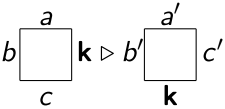

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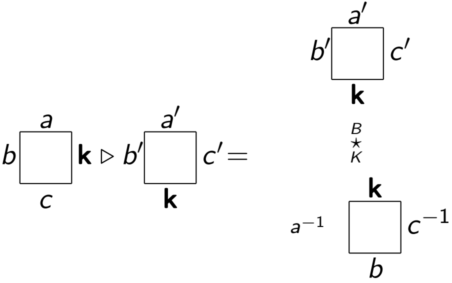

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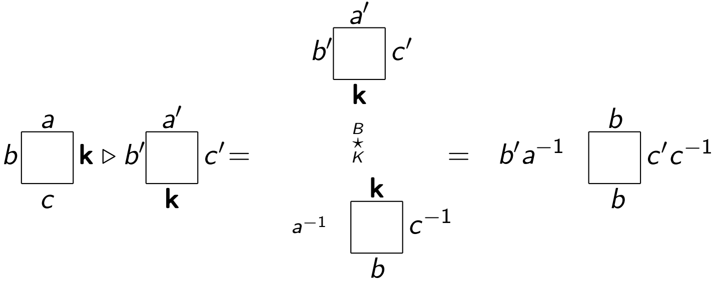

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Measured Quantum Groupoids

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(Nikshych-Vainerman)
$M_0 \subset M_1$: type II_1 subfactors of finite index and depth 2

$M_0 \subset M_1 \overset{e_1}{\subset} M_2 \overset{e_2}{\subset} M_3$....... Jones tower

- $A = M_0' \cap M_2$, $B = M_1' \cap M_3$: C^*-**weak Hopf algebras** in duality
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Answer: A and B are **measured quantum groupoids**

Measured quantum groupoids

Definition (F.Lesieur,M.Enock)

$\mathfrak{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu)$ measured quantum groupoid :

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