Subfactors, quantum groupoids and tensor categories

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September 2, 2009 Vietri sul Mare

A finite group action on a vN algebra M

$$M^G \subset M \subset M \rtimes G \subset (M \rtimes G) \rtimes \hat{G} \subset ....$$

Outer action : $(M^G)' \cap M = \mathbb{C}1 \Longrightarrow$ All vN algebras are factors and $[M : M^G] = [M \rtimes G : M] = ... = |G|$

Galois correspondence: $M^G \subset K \subset M \iff$ subgroups of G.
II_1-factors: (M, τ), where τ faithful normal finite trace $(\tau(ab) = \tau(ba))$.
Example of II_1 factor: $M = \mathcal{L}(H)$, H a discrete ICC group, $\tau(m) =< m\delta_e, \delta_e >$

Fundamental construction

$M_0 \subset (M_1, \tau) \longrightarrow e : L^2(M_1, \tau) \to \overline{M_0}, \quad M_2 = \{M_1, e\}$ - a factor either of type II_1 (finite index case) or II_∞

$$M_0 \subset M_1 \subset M_2 \subset ... \quad \text{Jones tower}$$

$$[M_1, M_0] := \tau(e)^{-1} \in \{4\cos^2\frac{\pi}{n} \ \ |n \geq 3\} \cup [4, \infty]$$

Derived tower:

$$M_0' \cap M_1 \subset M_0' \cap M_2 \subset M_0' \cap M_3 \subset ... \quad \text{Finite dim}$$

Depth k -fundamental construction from step k

Example : $M^G \subset M \subset M \rtimes G \subset (M \rtimes G) \rtimes \hat{G} \subset ....$ - depth 2

Theorem(Longo; Szymanski)
$M_0 \subset M_1$: type II_1 subfactors of finite index λ and depth 2,
$M_0' \cap M_1 = \mathbb{C}1,\ M_0 \subset M_1 \overset{e_1}{\subset} M_2 \overset{e_2}{\subset} M_3$....... Jones tower

- $A = M_0' \cap M_2,\ B = M_1' \cap M_3$: C^*-**Hopf algebras** in duality
- B acts on M_2 and $M_1 = M_2^B, \quad M_3 = \theta(M_2 \rtimes B)$
- $[M_k : M_{k-1}] = \dim A = \dim B \quad (k = 1, 2, ..)$

Duality : $\boxed{< a, b >= \lambda^{-2}\tau(ae_2e_1b).}$

$< \Gamma(a), b \otimes c >=< a, bc >$

$< \kappa(a), b >= \overline{< a^*, b^* >}$

$\epsilon(a) =< a, 1 >$

Motivation: $M_0' \cap M_1 \neq \mathbb{C}1$, non integer index

C^* quantum groupoid $(A, \Gamma, \kappa, \epsilon)$

Definition (Bohm, Nill, Szlachanyi)

- A : finite dim. C^*-algebra ($A = \bigoplus_{i=1,..k} M_{n_i}(\mathbb{C})$)
- $\Gamma : A \to A \otimes A$: $*$-homom.
  $(\Gamma \otimes i)\Gamma = (i \otimes \Gamma)\Gamma$ ($\Gamma(1) \neq 1 \otimes 1$, in general)
- $\kappa : A \to A$, linear antimultiplicative :
  $(\kappa \circ *)^2 = i$
  $\varsigma(\kappa \otimes \kappa)\Gamma = \Gamma \circ \kappa$
  $(m(\kappa \otimes i) \otimes i)(\Gamma \otimes i)\Gamma(x) = (1 \otimes x)\Gamma(1)$
  $(\mathrm{where\ m(a \otimes b) = ab})$
- $\epsilon : A \to \mathbb{C}$: linear
  $(\epsilon \otimes i)\Gamma = (i \otimes \epsilon)\Gamma = i$
  $(\epsilon \otimes \epsilon)((x \otimes 1)\Gamma(1)(1 \otimes y)) = \epsilon(xy)$

$\epsilon^t(b) = (\epsilon \otimes i)(\Gamma(1)(b \otimes 1))$, $\epsilon^s(b) = (i \otimes \epsilon((1 \otimes b)\Gamma(1))$
$A_t = Im\epsilon^t$, $A_s = Im\epsilon^s$
A is a Hopf algebra iff $\Gamma(1) = 1 \otimes 1$. Duals.

Definition

A finite groupoid $\mathcal{G}$ is a finite category for which any morphism is invertible.

- $\forall x \in \mathcal{G}$, there exists x^{-1},
- $\mathcal{G}^0$(Units:) $\subset \mathcal{G}$: $s(x) = x^{-1}x, t(x) = xx^{-1}$
  x= xs(x) =t(x)x
- $\forall x, y \in \mathcal{G}$:
  $x.y$ composable iff $s(x) = t(y)$
  $(xy)z = x(yz)$,

Commutative finite quantum groupoid

$\mathcal{G}$ finite groupoid

- $A = C(\mathcal{G}), \quad A \otimes A = C(\mathcal{G} x \mathcal{G})$
- $\Gamma(f)(x,y) = \begin{cases} f(xy) & \text{x, y composable} \\ 0 & \text{otherwise} \end{cases}$
- $\kappa(f)(x) = f(x^{-1})$
- $\epsilon(f) = \sum_{u \in \mathcal{G}^0} f(u)$
- $\epsilon^t(\delta_g) = \sum_{vv^{-1}=g} \delta_v, \quad \epsilon^s(\delta_g) = \sum_{v^{-1}v=g} \delta_v$

Symmetric quantum groupoids

$\mathcal{G}$ finite groupoid

- $\hat{A} = \mathbb{C}\mathcal{G}$
- $\hat{\Gamma}(\lambda(s)) = \lambda(s) \otimes \lambda(s)$
- $\hat{\kappa}(\lambda(s)) = \lambda(s^{-1})$
- $\hat{\epsilon}(\lambda(s)) = 1$
- $\hat{\epsilon}^t(g) = gg^{-1}, \quad \hat{\epsilon}^s(g) = g^{-1}g$

Quantum groupoid actions

Definitions

Action of $(A, \Gamma, \kappa, \epsilon)$ on a von Neumann algebra M
Linear w-continuous application : $A \otimes M \to M, \quad a \otimes m \mapsto a \triangleright m$

- M a left A-module via $\triangleright$
- $a \triangleright xy = (a_{(1)} \triangleright x)(a_{(2)} \triangleright y) \quad (\Gamma(a) = a_{(1)} \otimes a_{(2)})$
- $(a \triangleright x)^* = \kappa(a)^* \triangleright x^*$
- $a \triangleright 1 = \epsilon^t(a) \triangleright 1$ and $a \triangleright 1 = 0$ iff $\epsilon^t(a) = 0$

Crossed product of A by M: $M \ltimes A = M \underset{A_t}{\otimes} A$:
$\forall z, z \in A_t (:= \{z / \epsilon^t(z) = z\}) \quad m(z \triangleright 1) \otimes a \equiv m \otimes za$
$[m \otimes a][n \otimes b] = [m(a_{(1)} \triangleright n) \otimes a_{(2)} b], \quad [m \otimes a]^* = [a^*_{(1)} \triangleright m^* \otimes a^*_{(2)}]$

Fixed point subalgebra: $M^A = \{m / \forall a \in A, a \triangleright m = \epsilon^t(a) \triangleright m\}$

$$M^A \subset M \subset M \ltimes A$$

Theorem(Nikshych-V)
$M_0 \subset M_1$: type II_1 subfactors of finite index λ and depth 2

$M_0 \subset M_1 \overset{e_1}{\subset} M_2 \overset{e_2}{\subset} M_3$....... Jones tower

- $A = M_0' \cap M_2,\ B = M_1' \cap M_3$: C^*-**weak Hopf algebras** in duality
- B acts on M_2 and $M_1 = M_2^B, \quad M_3 = \theta(M_2 \rtimes B)$
- $[M_k : M_{k-1}] = [B : B_t]$

Duality : $\boxed{< a, b >= d\lambda^{-2}\tau(ae_2e_1Hb).}$
($H \in Z(M_1' \cap M_2)$ s.t. $\tau(Hz) = Tr(z)$)

$< \Gamma(a), b \otimes c >=< a, bc >$

$< \kappa(a), b >= \overline{< a^*, b^* >}$

$\epsilon(a) =< a, 1 >$

Temperley-Lieb algebras

Generators:

$$e_i^2 = e_i = e_i^* : \quad e_i e_{i\pm 1} e_i = \lambda e_i, \quad e_i e_j = e_j e_i$$

if $|i-j| \geq 2$, $(\lambda^{-1} = 4\cos^2 \frac{\pi}{n+3}, \quad n \geq 2; \quad i = 1, 2, \ldots)$

For fixed n, let $A = Alg\{1, e_1, \ldots, e_{2n-1}\}$
$A_t = Alg\{1, e_1, \ldots, e_{n-1}\}, \quad A_s = Alg\{1, e_{n+1}, \ldots, e_{2n-1}\}$

For $n = 2 : A = Alg\{1, e_1, e_2, e_3\} \simeq M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$
$A_t = Alg\{1, e_1\} \simeq \mathbb{C} \oplus \mathbb{C}, \quad A_s = Alg\{1, e_3\} \simeq \mathbb{C} \oplus \mathbb{C},$

$$\lambda^{-1} = 4\cos^2 \frac{\pi}{5}$$

Left coideal $*$-subalgebra: $I \subset B$ s.t $\Gamma(I) \subset B \otimes I$
One can define $M \rtimes I = span\{[m \otimes b] | m \in M, b \in I\} \subset M \rtimes B$

Theorem(Nikshych-V) There is isomorphisms of lattices $l(M_2 \subset M_3)$ of intermediate vN subalgebras $M_2 \subset K \subset M_3$ and $l(B)$ of left coideal $*$-subalgebras of B:

$$K \mapsto \theta^{-1}(M_1' \cap K) \subset B, \quad I \mapsto \theta(M_2 \rtimes I) \subset M_3$$

K is a factor iff I is **connected**: $Z(I) \cap B_s = \mathbb{C}1$.

Observation: If Depth $(M_0 \subset M_1) = n$ and

$$M_0 \subset M_1 \overset{e_1}{\subset} M_2 \overset{e_2}{\subset} M_3 \text{.......is the Jones tower,}$$

then Depth $(M_0 \subset M_k) = 2$ if $k = n - 1$. Then the previous theorem characterizes all finite index and finite depth subfactors.

Relation to tensor categories

Theorem(Nikshych-V) If Depth $(M_0 \subset M_1) = n$, then $Bimod_{M_0 - M_0} \simeq Rep(B^*)$ as tensor categories with $\otimes_{M_0}$ and $V \boxtimes W = \Gamma(1)(V \otimes W)$, resp. $(b \cdot (v \boxtimes w) = b_{(1)} \cdot v \boxtimes b_{(2)} \cdot w)$, where B corresponds to the depth 2 subfactor $M_0 \subset M_{n-1}$.

Fusion category: an abelian semisimple rigid tensor category $(\mathcal{C}, \otimes)$ with finitely many simple objects and finite dim. Hom-spaces

$$V_i \otimes V_j = \bigoplus_k N_{ij}^k V_k \quad \text{(fusion rule)} \quad \text{and} \quad V_i^* = V_{\bar{i}}, \ \ \mathbf{1} = V_{i_0}$$

Examples

- Finite dim. vector spaces (rank 1 cat.)
- $Rep(B)$, where B is a (weak) Hopf algebra
- Yang-Lee fusion category: $Ob(\mathcal{C}) = \{\mathbf{1}, V\}$

$$V \otimes V = \mathbf{1} \oplus V$$

Then $(\dim V)^2 = 1 + \dim V \Longrightarrow \dim V = \frac{1 \pm \sqrt{5}}{2}$
$\Longrightarrow$ Temperley-Lieb quantum groupoid

Reconstruction Theorem (T. Hayashi): Given a fusion category $(\mathcal{C}, \otimes)$, there is a **WHA** B, even with commutative $B_t = \mathbb{C}^n$ where n is the number of simple objects of $\mathcal{C}$, s.t. $\mathcal{C} \simeq Rep(B)$

Tambara-Yamagami fusion category
Simple objects: $X = G \cup \{m\}$, where G is a finite abelian group
Fusion Rules:

$$g \otimes h = gh, \quad g^* = g^{-1}, \quad \mathbf{1} = e \in G,$$

$$g \otimes m = m \otimes g = m = m^*, \quad m \otimes m = \bigoplus_{g \in G} g.$$

$\Longrightarrow$ WHA $\Longrightarrow$ Coideal $*$-subalgebras $\Longrightarrow$ Intermediate subfactors.

$$[B : B_t] = (n + \sqrt{n})^2,$$

where $n = |G|$.

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