Isometric coactions of compact quantum groups on compact quantum metric spaces

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Compact Quantum Groups

Definition (CQG - Woronowicz)

A compact quantum group (CQG) is a pair (A, Δ) where

- A is a unital C*-algebra
- the 'comultiplication' $\Delta : A \to A \otimes A$ is a *-homomorphism

such that

- Δ is 'coassociative': $(\iota \otimes \Delta)\,\Delta = (\Delta \otimes \iota)\,\Delta$
- the sets $\Delta(A)(1 \otimes A)$ and $\Delta(A)(A \otimes 1)$ are dense in $A \otimes A$.

A classical compact group $(G, *)$ is a CQG too: take the unital (commutative) C*-algebra $C(G)$, and define the comultiplication

$$\begin{aligned}\Delta : C(G) &\to C(G) \otimes C(G) \cong C(G \times G)\\ f &\mapsto (G \times G \to \mathbb{C} : (g, h) \mapsto f(g * h))\,.\end{aligned}$$

Quantise metric spaces

Classical definition	C*-algebra's
• Compact space X	• Commutative C*-algebra $C(X)$

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Classical definition

- Compact space X
- Metric $d : X \times X \rightarrow [0, \infty)$

C*-algebra's

- Commutative C*-algebra $C(X)$
- $L_d : C(X) \rightarrow [0, \infty]$

$$L_d(f) = \sup \left\{ \frac{|f(x) - f(y)|}{d(x,y)} \,\middle|\, x, y \in X, x \neq y \right\}$$

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$$L_d(f) = \sup \left\{ \frac{|f(x) - f(y)|}{d(x,y)} \,\middle|\, x, y \in X, x \neq y \right\}$$

- L_d is a seminorm
- $L_d(f) = L_d(\bar{f})$
- $L_d(f) = 0 \Leftrightarrow f$ is a constant function

From L_d to d

Define a metric ρ_{L_d} on the state space $\mathcal{S}(C(X))$:

$$\rho_{L_d}(\mu, \nu) = \sup\{|\mu(f) - \nu(f)| \,|L_d(f) \leq 1\}.$$

On the extreme states $\mu_x : C(G) \to \mathbb{C} : f \mapsto f(x)$ for $x \in X$, this metric coincides with the metric d:

$$\rho_{L_d}(\mu_x, \mu_y) = \sup\{|f(x) - f(y)| \,|L_d(f) \leq 1\} = d(x, y)$$

Compact Quantum Metric Spaces

Definition (CQMS - Rieffel, 1999)

A compact quantum metric space (CQMS) is a pair (B, L) where

- B is a unital C*-algebra
- L is a Lipnorm on B.

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A Lipnorm L on a unital C*-algebra B is a seminorm $L : B \to [0, \infty]$ such that

- $L(b) = L(b^*)$ for every $b \in B$
- $\forall b \in B : L(b) = 0 \Leftrightarrow b \in \mathbb{C}1$

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- $L(b) = L(b^*)$ for every $b \in B$
- $\forall b \in B : L(b) = 0 \Leftrightarrow b \in \mathbb{C}1$
- the ρ_L topology coincides with the weak *-topology on $\mathcal{S}(B)$, where

$$\rho_L(\mu, \nu) = \sup\{|\mu(b) - \nu(b)| \,|b \in B, L(b) \leq 1\}$$

  for every $\mu, \nu \in \mathcal{S}(B)$.

Compact Quantum Metric Spaces

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A compact quantum metric space (CQMS) is a pair (B, L) where

- B is a unital C*-algebra
- L is a Lipnorm on B.

A spectral triple $(A, \mathcal{H}, D)$ is a *-algebra A in $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbertspace, and a Diracoperator D on $\mathcal{H}$.
The closure of A can become a CQMS with $L(a) = |||[D, a]|||$.

Coactions

A classical action of a group $(G, *)$ on a metric space (X, d) is a mapping

$$\cdot : X \times G \to X : (x, g) \mapsto x \cdot g$$

In the C*-algebraic translation, we have

$$\begin{aligned} \alpha : C(X) &\to C(X \times G) \cong C(X) \otimes C(G) \\ f &\mapsto (X \times G \to \mathbb{C} : (x, g) \mapsto f(x \cdot g)) \end{aligned}$$

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- the set $\alpha(B)\,(1 \otimes A)$ is norm dense in $B \otimes A$.

If B has the structure of a CQMS,
when will we call such a coaction 'isometric'?

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Classical case

An action $\cdot$ of a group $(G, *)$ on a metric space (X, d) is called isometric if and only if

$$d(g \cdot x, g \cdot y) = d(x, y) \qquad \forall x, y \in X, \ \ \forall g \in G$$

CQG on finite classical space - T. Banica, 2004

Let $\alpha : C(X) \to C(X) \otimes A$ be a coaction of the CQG (A, Δ) on the finite metric space (X, d) with n points.
For $i, j = 1, \cdots, n$, there are elements a_{ij} in A such that

$$\alpha(\delta_i) = \sum_{j=1}^{n} \delta_j \otimes a_{ji}$$

Notation:

- a is the $(n \times n)$-matrix $(a_{ij})_{i,j}$
- d is the $(n \times n)$-distancematrix $(d(i,j))_{i,j}$

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$$a \cdot d = d \cdot a.$$

CQG on spectral triples - D. Goswami and J. Bhowmick, 2008

A CQG (A, Δ) acts by orientation-preserving isometries on the spectral triple $(B, \mathcal{H}, D)$ if there is a unitary representation U of A on $\mathcal{H}$ such that

- $(\iota \otimes \phi)\alpha_U(b) \in (B)''$ for every state ϕ on A, where $\alpha_U(x) = \tilde{U}(a \otimes 1)\tilde{U}^*$ for $x \in \mathcal{B}(\mathcal{H})$.
- $\tilde{U}$ commutes with $D \otimes 1$.

Here $\tilde{U}$ is the extension to $\mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes A$ from V given by $V(\xi \otimes a) = U(\xi)(1 \otimes a)$ for $\xi \in \mathcal{H}, a \in A$.

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We want to investigate this in the context of Rieffels definition of a CQMS.

Second half-classical case

Let $\alpha : B \to B \otimes C(G)$ be a coaction of the group $(G, \cdot)$ on the compact quantum metric space (B, L).
Identify $B \otimes C(G)$ with $C(G, B)$, by

$$b \otimes f : g \mapsto f(g)b$$

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Definition

A coaction α of a group $(G, \cdot)$ on a CQMS (B, L) is called isometric if and only if

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Different notation: $L((\iota \otimes \omega_g)\alpha(b)) = L(b)$ where ω_g is the state on $C(G)$ evaluating in g:

$$\omega_g : C(G) \to \mathbb{C} : f \mapsto f(g).$$

CQG on CQMS

One can write every linear mapping from $C(G)$ to $\mathbb{C}$ as a linear combination of the mappings ω_g for $g \in G$: $\omega = \sum_{g \in G} \omega(g)\omega_g$.
This means every state is a convex combination of the states ω_g.

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If the equality $L((\iota \otimes \omega_g)\alpha(b)) = L(b)$ holds, we have, for any $\omega \in \mathcal{S}(C(G))$, that

$$L((\iota \otimes \omega)\alpha(b)) \leq \sum_{g \in G} \omega(g) L((\iota \otimes \omega_g)\alpha(b)) = L(b)$$

voor alle $b \in B$.

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voor alle $b \in B$.

Definition (Isometric coaction)

A coaction α of a CQG (A, Δ) on a CQMS (B, L) is called isometric if and only if

$$L((\iota \otimes \omega)\alpha(b)) \leq L(b) \qquad \forall b \in B, \ \forall \omega \in \mathcal{S}(A).$$

Comparison with existing definitions

Theorem

Let (X, d) be a finite metric space with n points and (A, Δ) a CQG working on X by the coaction $\alpha : C(X) \to C(X) \otimes A$.
Take elements a_{ij} in A such that $\alpha(\delta_i) = \sum_{j=1}^{n} \delta_j \otimes a_{ji}$ where δ_j is 1 on the j-th point of X and 0 elsewhere. Write

- $L_d(f) = \sup \left\{ \frac{|f(x)-f(y)|}{d(x,y)} \middle| x, y \in X, x \neq y \right\}$ *for $f \in C(X)$*
- *a for the $(n \times n)$-matrix $(a_{ij})_{i,j=1\cdots n}$*
- *d for the $(n \times n)$-matrix $(d(i,j))_{i,j=1\cdots n}$.*

Then

$$a \cdot d = d \cdot a$$

if and only if

$$L_d((\iota \otimes \omega)\alpha(f)) \leq L_d(f), \text{ for every } f \in C(X) \text{ and every } \omega \in \mathcal{S}(A).$$

$L_d((\iota \otimes \omega)\alpha(f)) \leq L_d(f) \Rightarrow a \cdot d = d \cdot a$

Sufficient to prove: a commutes with the 'colorcomponents' d_γ.

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Sufficient to prove: a commutes with the 'colorcomponents' d_γ.

The colorcomponent d_γ of a distance γ is the matrix which has a 1 on position (i,j) if $d(i,j) = \alpha$ and 0 elsewhere.
Then $d = \sum \alpha d_\gamma$.

$L_d((\iota \otimes \omega)\alpha(f)) \leq L_d(f) \Rightarrow a \cdot d = d \cdot a$

Sufficient to prove: a commutes with the 'colorcomponents' d_γ.

Sufficient to prove: $a_{ij}a_{kl} = 0$ if $d(i,k) \neq d(j,l)$

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Sufficient to prove: $a_{ij} a_{kl} = 0$ if $d(i,k) \neq d(j,l)$

$$(a \cdot d_\gamma)_{ij} = \sum_{\substack{k \\ d(k,j)=\alpha}} a_{ik} = \sum_{\substack{k \\ d(k,j)=\alpha}} a_{ik} \sum_{l} a_{lj} = \sum_{\substack{k \\ d(k,j)=\alpha}} \sum_{\substack{l \\ d(l,i)=\alpha}} a_{ik} a_{lj}$$

$$= \sum_{\substack{l \\ d(l,i)=\alpha}} \sum_{k} a_{ik} a_{lj} = \sum_{\substack{l \\ d(l,i)=\alpha}} a_{lj} = (d_\gamma \cdot a)_{ij}$$

$L_d((\iota \otimes \omega)\alpha(f)) \leq L_d(f) \Rightarrow a \cdot d = d \cdot a$

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Sufficient to prove: $a_{ij}a_{kl} = 0$ if $d(i,k) \neq d(j,l)$

Suppose $d(i,k) \neq d(j,l)$ and $a_{ij}a_{kl} \neq 0$.

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Sufficient to prove: $a_{ij}a_{kl} = 0$ if $d(i,k) \neq d(j,l)$
Suppose $d(i,k) \neq d(j,l)$ and $a_{ij}a_{kl} \neq 0$.

We can take a state ω on A such that $\omega(a_{ij}a_{kl}a_{ij}) \neq 0$.
Then also $\omega(a_{ij}) \neq 0$.
Define a new state on A:

$$\omega_{ij} : A \to \mathbb{C} : x \mapsto \frac{\omega(a_{ij}xa_{ij})}{\omega(a_{ij})}$$

Then we have a state ω_{ij} such that $\omega_{ij}(a_{ij}) = 1$ and $\omega_{ij}(a_{kl}) \neq 0$

$L_d((\iota \otimes \omega)\alpha(f)) \leq L_d(f) \Rightarrow a \cdot d = d \cdot a$

Sufficient to prove: a commutes with the 'colorcomponents' d_γ.
Sufficient to prove: $a_{ij}a_{kl} = 0$ if $d(i,k) \neq d(j,l)$
Suppose $d(i,k) \neq d(j,l)$ and $a_{ij}a_{kl} \neq 0$.
Then we have a state ω_{ij} such that $\omega_{ij}(a_{ij}) = 1$ and $\omega_{ij}(a_{kl}) \neq 0$

To get a contradiction, it is sufficient to prove the following lemma:

Lemma

For every distance α, for every $i,j \in \{1 \cdots n\}$ and for every state ω on A, which is 1 in a_{ij}, we have:

- *If $d(i,k) = \alpha$ and $d(j,l) \neq \alpha$, then $\omega(a_{kl}) = 0$.*
- *If $d(i,k) \neq \alpha$ and $d(j,l) = \alpha$, then $\omega(a_{kl}) = 0$.*

$L_d((\iota \otimes \omega)\alpha(f)) \leq L_d(f) \Rightarrow a \cdot d = d \cdot a$

Sufficient to prove: a commutes with the 'colorcomponents' d_γ.
Sufficient to prove: $a_{ij}a_{kl} = 0$ if $d(i,k) \neq d(j,l)$
Suppose $d(i,k) \neq d(j,l)$ and $a_{ij}a_{kl} \neq 0$.
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For every distance α, for every $i, j \in \{1 \cdots n\}$ and for every state ω on A, which is 1 in a_{ij}, we have:

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- *If $d(i,k) \neq \alpha$ and $d(j,l) = \alpha$, then $\omega(a_{kl}) = 0$.*

We can prove this lemma by induction on the distances γ, starting with the smallest distance $\gamma = 0$ and using the given inequality from the Lipschitznorm.

$a \cdot d = d \cdot a \Rightarrow L_d((\iota \otimes \omega)\alpha(b)) \leq L_d(b)$

Fix a state ω on A and points $x, y \in X$.
Sufficient to prove: there are positive numbers λ_{ij} such that

$$\begin{cases} \sum_j \lambda_{ij} = \omega(a_{xi}) \\ \sum_i \lambda_{ij} = \omega(a_{yj}) \\ \lambda_{ij} = 0 \text{ if } d(i,j) \neq d(x,y). \end{cases}$$

$a \cdot d = d \cdot a \Rightarrow L_d((\iota \otimes \omega)\alpha(b)) \leq L_d(b)$

Fix a state ω on A and points $x, y \in X$.
Sufficient to prove: there are positive numbers λ_{ij} such that $\sum_j \lambda_{ij} = \omega(a_{xi})$, $\sum_i \lambda_{ij} = \omega(a_{yj})$, $\lambda_{ij} = 0$ if $d(i,j) \neq d(x,y)$.

$$\frac{|(\iota \otimes \omega)\alpha(f)(x) - (\iota \otimes \omega)\alpha(f)(y)|}{d(x,y)} = \frac{|\sum_i f(i)\omega(a_{xi} - a_{yi})|}{d(x,y)}$$

$$= \frac{\left|\sum_i f(i)\left(\sum_j \lambda_{ij} - \sum_i \lambda_{ij}\right)\right|}{d(x,y)} = \frac{\left|\sum_{ij} \lambda_{ij}(f(i) - f(j))\right|}{d(x,y)}$$

$$= \frac{\left|\sum_{ij} \lambda_{ij}(f(i) - f(j))\right|}{d(i,j)} \leq L_d(f)$$

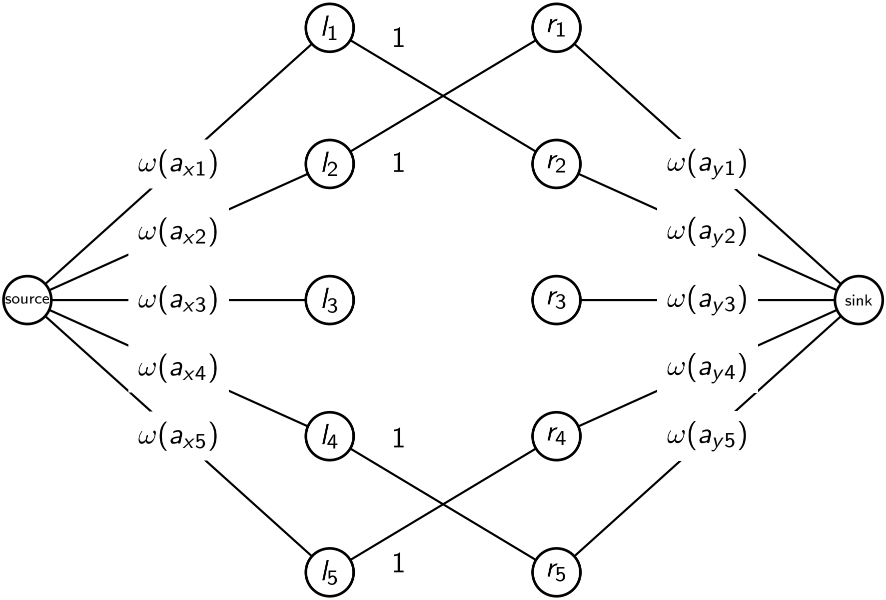
$a \cdot d = d \cdot a \Rightarrow L_d((\iota \otimes \omega)\alpha(b)) \leq L_d(b)$

Fix a state ω on A and points $x, y \in X$.
Sufficient to prove: there are positive numbers λ_{ij} such that $\sum_j \lambda_{ij} = \omega(a_{xi})$, $\sum_i \lambda_{ij} = \omega(a_{yj})$, $\lambda_{ij} = 0$ if $d(i,j) \neq d(x,y)$.

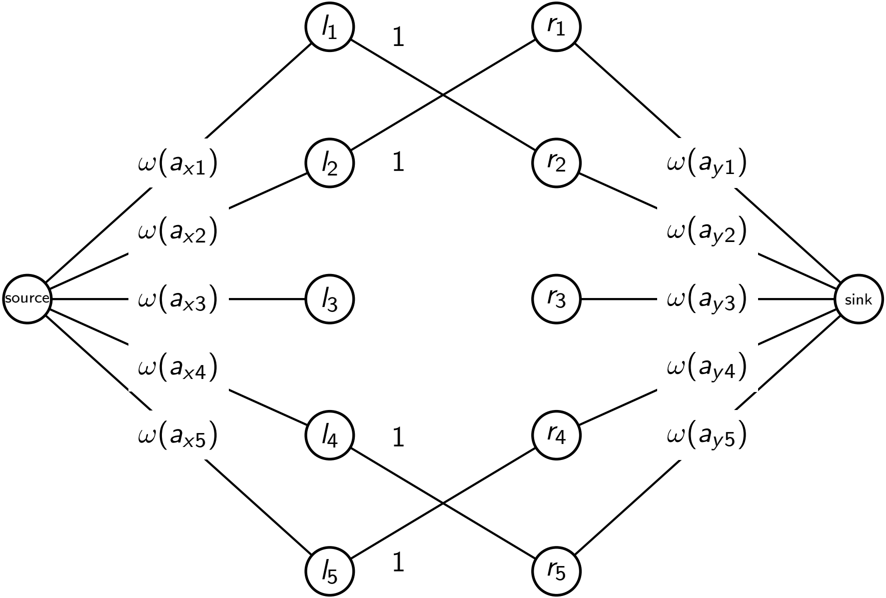
For every subset $Z \subseteq X$, and every $x, y \in X$, we have

$$\sum_{i \in Z} a_{xi} \cdot \sum_{\substack{j \\ \exists i \in Z : d(i,j) = d(x,y)}} a_{yj} = \sum_{i \in Z} a_{xi},$$

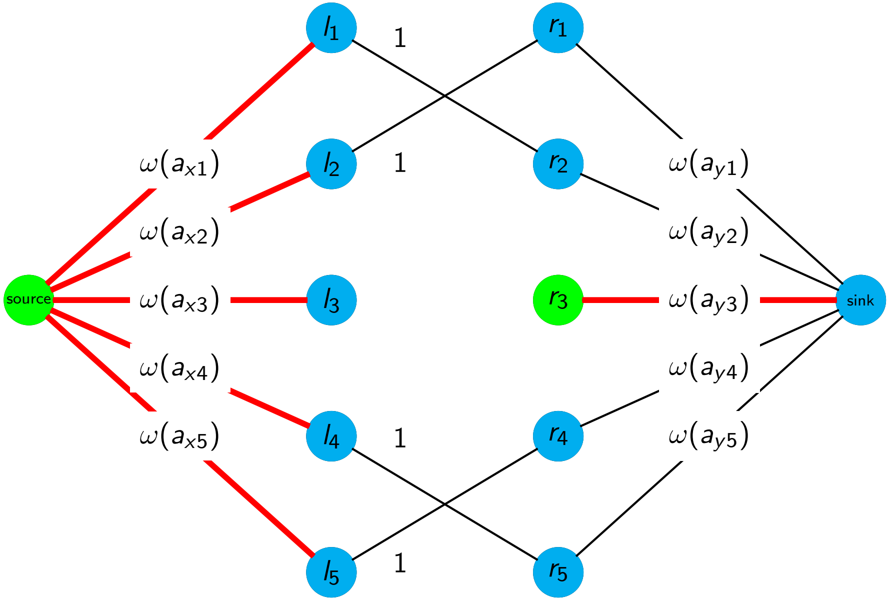
$$\text{so } \sum_{i \in Z} a_{xi} \leq \sum_{\substack{j \\ \exists i \in Z : d(i,j) = d(x,y)}} a_{yj}$$



Edge form l_i to r_j if $d(i,j) = d(x,y)$



The minimal capacity of a cut is equal to the maximal flow



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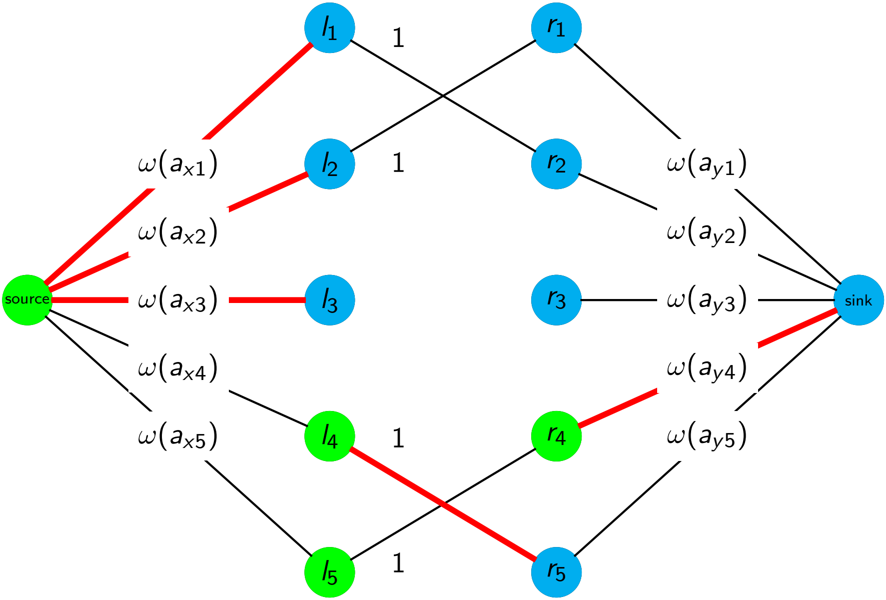

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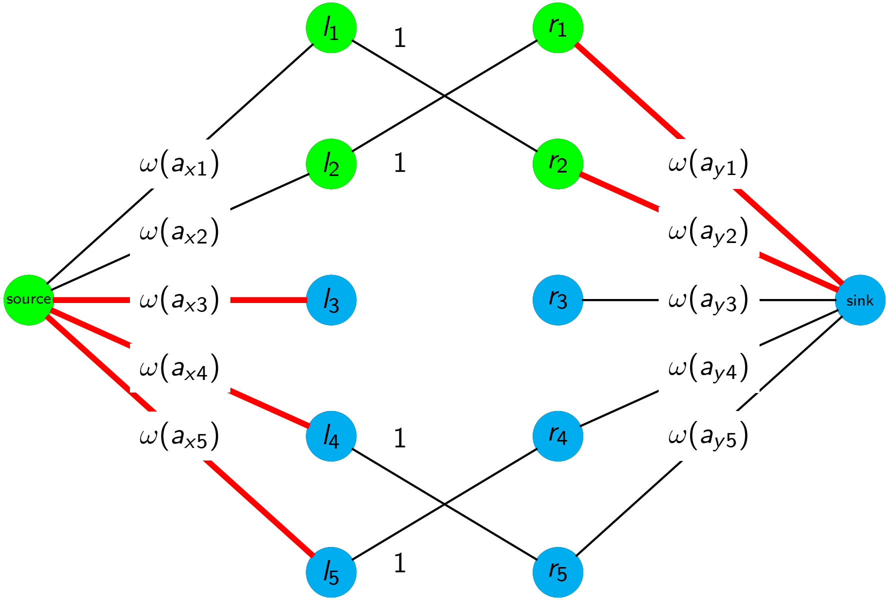

Edge form l_i to r_j if $d(i,j) = d(x,y)$

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Theorem

Let (A, Δ) be a CQG, working on a CQMS (B, L) by a coaction $\alpha : B \to B \otimes A$. Then there exists a coideal $\mathcal{I}$ in A such that $\bar{\alpha} : B \to B \otimes A/\mathcal{I}$ is isometric.

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We call a *-representation $\pi : A \to \mathcal{B}(\mathcal{H})$ on a Hilbertspace $\mathcal{H}$ admissible if

$$L((\iota \otimes \omega\pi)\alpha(b)) \leq L(b) \qquad \forall b \in B, \forall \omega \in \mathcal{S}(\mathcal{B}(\mathcal{H}))$$

$$\Pi = \{\pi : A \to \mathcal{B}(\mathcal{H}) | \mathcal{H} \text{ Hilbertspace}, \pi \text{ admissible *-representation on } \mathcal{H}\}$$

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Let (A, Δ) *be a CQG, working on a CQMS* (B, L) *by a coaction* $\alpha : B \to B \otimes A$. *Then there exists a coideal* $\mathcal{I}$ *in* A *such that* $\bar{\alpha} : B \to B \otimes A/\mathcal{I}$ *is isometric.*

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$$\Pi = \{\pi : A \to \mathcal{B}(\mathcal{H}) | \mathcal{H} \text{ Hilbertspace}, \pi \text{ admissible *-representation on } \mathcal{H}\}$$

The coideal $\mathcal{I}$ of the theorem is given by

$$\mathcal{I} = \{a \in A \mid \pi(a) = 0, \; \forall \pi \in \Pi\}$$