Net cohomology and local charges

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Outline

1. Introduction

2. Cohomology of a poset
   - The simplicial set
   - 1-Cohomology

3. Net cohomology and superselection sectors
   - The observable net
   - The program
   - The charge structure
   - A new quantum number
   - Splitting charge and topological content
   - Physical interpretation

4. Comments

5. Conclusion
Net cohomology was introduced by Roberts ’76 as a cohomological (non-Abelian) approach to the theory of superselection sectors. Many deep applications (developed by Roberts):

- Equivalence between net cohomology and DHR-analysis.
- The $\alpha$-induction.
- Completeness theorem (equivalently non-existence) of DHR-sectors.

In this talk I describe a recent application of net cohomology: the discovered of charged (superselection) sectors in a curved spacetimes which are affected by the topology of the spacetime [Brunetti & R. ’09].
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Singular $n$–simplices are order preserving maps

$$x : \tilde{\Delta}_n \rightarrow K$$

$\tilde{\Delta}_n$ is the standard $n$–simplex considered as a poset with respect of inclusion of its subsimplices. $\Sigma_n$ denotes the set of $n$–simplices and

$$\partial_i : \Sigma_n \rightarrow \Sigma_{n-1} \quad \text{face}, \quad \sigma_i : \Sigma_n \rightarrow \Sigma_{n+1} \quad \text{degeneracy}.$$
The simplicial set

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The simplicial set

Composing 1–simplices we get paths. A path \( p : o \to \tilde{o} \) is a finite ordered set \( b_n \ast \cdots \ast b_1 \) s.t.

\[
\partial_0 b_n = \tilde{o}, \quad \partial_1 b_{i+1} = \partial_0 b_i, \quad \partial_1 b_1 = o,
\]

\( K \) is **pathwise connected**: for any pair \( o, \tilde{o} \) there is a path \( p : o \to \tilde{o} \).

Homotopy equivalence relation \( \sim \) on paths with the same endpoints.

This leads to first homotopy group \( \pi_1(K, o) \) of \( K \), with base \( o \), and to fundamental group \( \pi_1(K) \) since \( K \) is pathwise connected. \( K \) is simply connected if \( \pi_1(K) \) is trivial.
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The simplicial set

The simplicial set $K$ is simply connected if

- either $K$ is upward directed: for any pair $o', o''$ there is $o$ such that $o', o'' \leq o$,
- or $K$ is dawnward directed, since

$$\pi_1(K) \cong \pi_1(K^\circ)$$

$K^\circ$ is the dual poset of $K$.

Let $K$ be a base of neighbourhoods of arcwise and simply connected open subsets of a connected topological space $M$ ordered under inclusion.

The fundamental groups of $\pi_1(K)$ and $\pi_1(M)$ are isomorphic.
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Let $K$ be a base of neighbourhoods of arcwise and simply connected open subsets of a connected topological space $M$ ordered under inclusion.

The fundamental groups of $\pi_1(K)$ and $\pi_1(M)$ are isomorphic.
A 1–cocycle $z$ with values in $\mathcal{B}(\mathcal{H})$ is a field

$$\Sigma_1 \ni b \mapsto z(b) \in \mathcal{B}(\mathcal{H})$$

of unitaries operators satisfying the 1-cocycle equation:

$$z(\partial_0 c) z(\partial_2 c) = z(\partial_1 c) , \quad c \in \Sigma_2$$

$Z^1(K, \mathcal{B}(\mathcal{H}))$ set of 1-cocycles of $K$ taking values in $\mathcal{B}(\mathcal{H})$.

Any 1–cocycle $z$ defines a unitary representation $R_z$ of $\pi_1(K)$:

$$R_z([p]) := z(p) , \quad [p] \in \pi_1(K, o)$$

for some $p : o \to o$ with $p \in [p]$. 

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**Net cohomology and superselection sectors**

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The observable net

\( M \) 4-d globally hyperbolic spacetime. \( K \) is the set of diamonds of \( M \) ordered under inclusion. In particular, it is base of neighbourhoods of arcwise and simply connected open subsets of \( M \).

The observable net in a reference representation is the correspondence

\[ \mathcal{A} : K \ni o \rightarrow A(o) \subseteq \mathcal{B}(\mathcal{H}) \]

\( A(o) \) is the vN-algebra generated by all the observables measurable within \( o \)

- **isotony**: \( o_1 \subseteq o_2 \Rightarrow A(o_1) \subseteq A(o_2) \)
- **causality**: \( o_1 \perp o_2 \Rightarrow [A(o_1), A(o_2)] = 0 \)
- **Borchers property**: if \( E \) is a projection of \( A(o) \), then \( E \sim 1 \) in \( A(\tilde{o}) \) for any \( \tilde{o} \) with \( o \subset\subset \tilde{o} \).
- **punctured Haag duality**: the restriction of \( \mathcal{A} \) to \( K_x \) satisfies Haag duality for any \( x \in M \), where \( K_x := \{ o \in K \mid cl(o) \in x^\perp \} \) (causal puncture),
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Examples: free scalar fields in Hadamard representation: [Verch 97, R. 05]
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Examples: free scalar fields in Hadamard representation: [Verch 97, R. 05]
The observable net

Difference with respect to Minkowski spacetime

In general \( \mathcal{A} \) is not a net: \( K \) is not upward directed when either \( M \) is multiply connected or \( M \) has compact Cauchy surfaces.

Examples of multiply connected spacetimes arise in cosmology: Friedmann-Lamâitre models.

A 1–cocycle \( z \) of \( Z^1(K, \mathcal{B}(\mathcal{H})) \) takes values in the observable net \( \mathcal{A} \) if

\[
z(b) \in \mathcal{A}(|b|) \ , \ b \in \Sigma_1 .
\]

1-cocycles taking values in \( \mathcal{A} \) define a \( C^\ast \)-category \( Z^1(K, \mathcal{A}) \).

\( Z^1_{DHR}(K, \mathcal{A}) \) is the full subcategory of \( Z^1(K, \mathcal{A}) \) whose objects define trivial representations of the fundamental group of \( M \).
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1-cocycles taking values in $\mathcal{A}$ define a $C^*$-category $Z^1(K, \mathcal{A})$.

$Z^1_{DHR}(K, \mathcal{A})$ is the full subcategory of $Z^1(K, \mathcal{A})$ whose objects define trivial representations of the fundamental group of $M$. 
The existence of a charge structure in $Z^1(K, \mathcal{A})$.

- [Guido, Longo, Roberts & Verch '01]
- [Roberts '03]
- [R. '05].

- $Z^1(K, \mathcal{A})$ defines new superselection sectors of $\mathcal{A}$.
- Show the topological content of $Z^1(K, \mathcal{A})$.
- Physical interpretation.

- [Brunetti & R. 09]
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The charge structure

**Theorem**

- $Z^1(K, \mathcal{A})$ is a symmetric tensor $\mathbb{C}^*$-category with conjugation.
- $Z^1_{DHR}(K, \mathcal{A})$ is closed under all operations.
- The inclusion $Z^1_{DHR}(K, \mathcal{A}) \rightarrow Z^1(K, \mathcal{A})$ is a full faithful symmetric tensor functor.

**Charge structure**

- $\otimes$ tensor product
- $\varepsilon$ permutation symmetry
- $z \leftrightarrow \bar{z}$ conjugation

Two invariants (charge quantum numbers) classify statistics

- Statistical phase $\kappa \in \{1, -1\}$ (Bose/Fermi)
- Statistical dimension $d \in \mathbb{N}$ (order of statistics)
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- $\otimes$ tensor product
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- charge composition
- statistics
- charge-anticharge symmetry

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Charge structure

- Tensor product $\otimes$ charge composition
- Permutation symmetry $\varepsilon$ statistics
- Conjugation $z \leftrightarrow \bar{z}$ charge-anticharge symmetry

Two invariants (charge quantum numbers) classify statistics:
- Statistical phase $\kappa \in \{1, -1\}$ (Bose/Fermi)
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Idea of the proof. Main problem: $K$ is not upward directed.

- Proceed like in differential geometry.
  - The set of causal punctures $\bigcup_{x \in M} K_x$ is a covering of $K$. The problem has a local solution: the category $Z^1(K_x, \mathcal{A})$ has a charge structure.
  - Restrict $Z^1(K, \mathcal{A}) \ni z \mapsto z \upharpoonright K_x \in Z^1(K_x, \mathcal{A})$ for any $x \in M$.
  - Local constructions (tensor product, symmetry, conjugation) made in different points $x$ glue together.

- Charge structure on $K_x$ exists because DHR-like endomorphisms for the net $\mathcal{A} \upharpoonright K_x$ corresponds to 1-cocycles of $Z^1(K_x, \mathcal{A})$: endomorphisms of the algebra $\mathcal{A}(K_x) := C^* \{ \mathcal{A}(o) \mid o \in K_x \}$ localized and transportable in $K_x$.

In general $K_x$ is not directed, so how is it possible?

- The point $x$ is for $K_x$ "spacelike infinite": let $O(x) := \{ o \in K \mid x \in o \}$, for any $o \in K_x$ there is $a \in O(x)$ s.t. $a \perp o$.
- Consider the presheaf $O(x) \ni o \mapsto \mathcal{A}(o)'$. $O(x)$ is downward directed. Then $\mathcal{A}(K_x) = C^* - \lim_{o \in O(x)} \mathcal{A}(o)'$.
- Using this, the spacelike infinite $x$ and punctured Haag duality DHR-endomorphisms are defined using the presheaf.
Idea of the proof. Main problem: $K$ is not upward directed.

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  - Restrict $Z^1(K, \mathcal{A}) \ni z \mapsto z|_{K_x} \in Z^1(K_x, \mathcal{A})$ for any $x \in M$.
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Idea of the proof. Main problem: $K$ is not upward directed.

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A new quantum number

For any irreducible 1-cocycle $z$ the group von Neumann algebra

$$R_z(\pi_1(M), o) := \{z(p) \mid p : o \rightarrow o\}''$$

is a factor of type $I_n$ with $n \leq d(z)$.

Definition

The topological dimension $\tau(z)$ of an irr. 1–cocycle $z$ is the dimension of $R_z(\pi_1(M), o)$.

- $\tau(z)$ is an invariant of the equivalence class $[z]$
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**Theorem**

Any 1-cocycle \( z \) is a composition \( z = \chi_z \bowtie \langle z \rangle \), join, of its topological component and its charge component.

In particular we note

- If \( z \) is irreducible and has topological dimension greater than 1, then \( \langle z \rangle \) is reducible: \( \langle z \rangle = \bigoplus_{i=1}^{n} z_i \), where \( z_i \) is an irreducible object of \( \mathbb{Z}^1_{DHR}(K, \mathcal{A}) \) with \( \kappa(z_i) = \kappa(z) \). So, any such a charge is a finite collection of DHR-charges glued together by a glue of topological nature.

- If \( z \) is irreducible and has topological dimension equals the statistical dimension \( \tau(z) = n = d(z) \), then \( \langle z \rangle \cong \bigoplus_{i=1}^{n} u \) where \( d(u) = 1 \) and \( \kappa(u) = \kappa(z) \). In this case the charge is formed by \( n \) DHR-charges of the same type.

**Theorem (Existence)**

For any irreducible finite dimensional and irreducible representation \( \sigma \) of \( \pi_1(M) \), there is a 1-cocycle of \( \mathbb{Z}^1(K, \mathcal{A}) \) which defines a repr. of \( \pi_1(M) \) equivalent, up to infinite multiplicity, to \( \sigma \).
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- If $z$ is irreducible and has topological dimension equals the statistical dimension $\tau(z) = n = d(z)$, then $\langle z \rangle \cong \bigoplus_{i=1}^{n} u$ where $d(u) = 1$ and $\kappa(u) = \kappa(z)$. In this case the charge is formed by $n$ DHR-charges of the same type.

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For any irreducible finite dimensional and irreducible representation $\sigma$ of $\pi_1(M)$, there is a 1-cocycle of $Z^1(K, \mathcal{A})$ which defines a repr. of $\pi_1(M)$ equivalent, up to infinite multiplicity, to $\sigma$. 
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Any 1-cocycle $z$ is a composition $z = \chi_z \bowtie \langle z \rangle$, join, of its topological component and its charge component.

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Sharp localization of charge

- $z \in Z^1(K, \mathcal{A})$, for any $o \in K$ there is a “generalized endomorphism” $\rho^z(o)$ such that

$$\rho^z(o) \upharpoonright \mathcal{A}(\hat{o}) = \text{id}, \quad \hat{o} \perp o$$

Cocycles plays the rôle of charge transporters:

$$z(p) \rho^z(o) = \rho^z(\tilde{o}) z(p), \quad p : o \to \tilde{o}$$

Analogy with the Aharonov-Bohm effect

- If $q : o \to \tilde{o}$ is not homotopic to $p$, then

$$z(p) \rho^z(o) \neq z(q)\rho^z(o)$$

i.e. the final state depends on the homotopy class of the path.

Any 1–cocycle $z$ is a flat connection of a principal bundle over $K$ (Roberts & R. 07):

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2 Cohomology of a poset
   • The simplicial set
   • 1-Cohomology

3 Net cohomology and superselection sectors
   • The observable net
   • The program
   • The charge structure
   • A new quantum number
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4 Comments

5 Conclusion
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There should be an underlying gauge theory giving rise to the charges $Z^1(K, A)$
- purely topological
- with a “local” action of the gauge group

There arises the question whether more general superselection sectors can be discovered by enhancing the analysis of net cohomology: the physical meaning of 2-cohomology, for instance.
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