Net cohomology and local charges

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Outline

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- Net cohomology was introduced by Roberts '76 as a cohomological (non-Abelian) approach to the theory of superslection sectors. Many deep applications (developped by Roberts):
  - Equivalence between net cohomology and DHR-analysis.
  - The α-induction.
  - Completness theorem (equivalently non-existence) of DHR-sectors.
  - Attempt of a cohomological description of electromagnetic charges. Byproduct: n-categorical fomulation of non-Abelian cohomology.
- In this talk I describe a recent application of net cohomology: the discovered of charged (superselection) sectors in a curved spacetimes which are affected by the topology of the spacetime [Brunetti & R. '09].

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The simplicial set

Singular n–simplices are order preserving maps

$$x : \widetilde{\Delta}_n \rightarrow K$$

$\widetilde{\Delta}_n$ is the standard n–simplex considered as a poset with respect of inclusion of its subsimplices.
Σ_n denotes the set of *n–simplices* and

$$\partial_i : \Sigma_n \rightarrow \Sigma_{n-1} \quad \textit{face}, \qquad \sigma_i : \Sigma_n \rightarrow \Sigma_{n+1} \quad \textit{degeneracy}.$$

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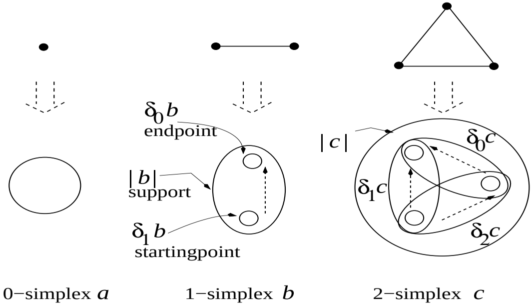
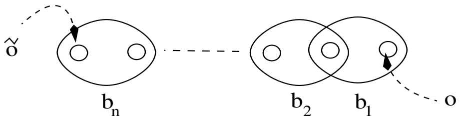


The simplicial set

Composing 1–simplices we get paths. A path $p : o \to \tilde{o}$ is a finite ordered set $b_n * \cdots * b_1$ s.t.

$$\partial_0 b_n = \tilde{o}, \quad \partial_1 b_{i+1} = \partial_0 b_i, \quad \partial_1 b_1 = o,$$



K is pathwise connected: for any pair $o, \tilde{o}$ there is a path $p : o \to \tilde{o}$.

Homotopy equivalence relation $\sim$ on paths with the same endpoints.

This leads to first homotopy group $\pi_1(K, o)$ of K, with base o, and to fundamental group $\pi_1(K)$ since K is pathwise connected. K is simply connected if $\pi_1(K)$ is trivial.

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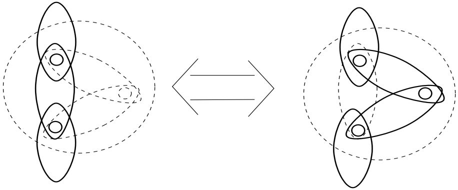

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The simplicial set

K is simply connected if

- either K is upward directed: for any pair o', o'' there is o such that $o', o'' \leq o$,
- or K is dawnward directed, since

$$\pi_1(K) \cong \pi_1(K^\circ)$$

K° is the dual poset of K.

Let K be a base of neighbourhoods of arcwise and simply connected open subsets of a connected topological space M ordered under inclusion.

The fundamental groups of $\pi_1(K)$ and $\pi_1(M)$ are isomorphic.

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1-Cohomology

A **1–cocycle** z with values in $\mathfrak{B}(\mathcal{H})$ is a field

$$\Sigma_1 \ni b \longrightarrow z(b) \in \mathfrak{B}(\mathcal{H})$$

of unitaries operators satisfying the 1-cocycle equation:

$$z(\partial_0 c)\, z(\partial_2 c) = z(\partial_1 c) \ , \qquad c \in \Sigma_2$$

$\mathrm{Z}^1(K, \mathfrak{B}(\mathcal{H}))$ set of 1-cocycles of K taking values in $\mathfrak{B}(\mathcal{H})$.

Any 1–cocycle z defines a unitary representation R_z of $\pi_1(K)$:

$$R_z([p]) := z(p) \ , \qquad [p] \in \pi_1(K, o)$$

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M 4-d globally hyperbolic spacetime. K is the set of diamonds of M ordered under inclusion. In particular, it is base of neighbourhoods of arcwise and simply connected open subsets of M.

The observable net in a reference representation is the correspondence

$$\mathscr{A} : K \ni o \to \mathcal{A}(o) \subseteq \mathfrak{B}(\mathcal{H})$$

$\mathcal{A}(o)$ is the vN-algebra generated by *all* the observables measurable within o

- *isotony* : $o_1 \subseteq o_2 \Rightarrow \mathcal{A}(o_1) \subseteq \mathcal{A}(o_2)$
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Examples: free scalar fields in Hadamard representation: [Verch 97, R. 05]

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The observable net

Difference with respect to Minkowski spacetime

In general $\mathscr{A}$ is not a net: K is not upward directed when either M is multiply connected or M has compact Cauchy surfaces.

Examples of multiply connected spacetimes arises in cosmology: Friedmann-Lamâitre models.

A 1–cocycle z of $Z^1(K, \mathfrak{B}(\mathcal{H}))$ takes values in the observable net $\mathscr{A}$ if

$$z(b) \in \mathcal{A}(|b|)\ , \ \ b \in \Sigma_1\ .$$

1-cocycles taking values in $\mathscr{A}$ define a C^*-category $\mathrm{Z}^1(K, \mathscr{A})$.

$\mathrm{Z}^1_{DHR}(K, \mathscr{A})$ is the full subcategory of $\mathrm{Z}^1(K, \mathscr{A})$ whose objects define trivial representations of the fundamental group of M.

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The program

- The existence of a charge structure in $\mathrm{Z}^1(K, \mathscr{A})$.

  [Guido, Longo, Roberts & Verch '01]
  [Roberts '03]
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- $\mathrm{Z}^1(K, \mathscr{A})$ defines new superselection sectors of $\mathscr{A}$.
- Show the topological content of $\mathrm{Z}^1(K, \mathscr{A})$.
- Physical interpretation.

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Theorem

- $\mathrm{Z}^1(K, \mathscr{A})$ *is a symmetric tensor* C^**-category with conjugation.*
- $\mathrm{Z}^1_{DHR}(K, \mathscr{A})$ *is closed under all operations.*
- *The inclusion* $\mathrm{Z}^1_{DHR}(K, \mathscr{A}) \to \mathrm{Z}^1(K, \mathscr{A})$ *is a full faithful symmetric tensor functor.*

Charge structure

$\otimes$ tensor product	charge composition
ε permutation symmetry	statistics
$z \leftrightarrow \bar{z}$ conjugation	charge-anticharge symmetry

Two invariants (charge quantum numbers) classify statistics

- statistical phase $\kappa \in \{1, -1\}$ (Bose/Fermi)
- statistical dimension $d \in \mathbb{N}$ (order of statistics)

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Idea of the proof. Main problem: K is not upward directed.

- Proceed like in differential geometry.
  - The set of causal punctures $\cup_{x \in M} K_x$ is a covering of K. The problem has a local solution: the category $\mathrm{Z}^1(K_x, \mathscr{A})$ has a charge structure.
  - Restrict $\mathrm{Z}^1(K, \mathscr{A}) \ni z \to z \upharpoonright K_x \in \mathrm{Z}^1(K_x, \mathscr{A})$ for any $x \in M$.
  - Local constructions (tensor product, symmetry, conjugation) made in different points x glue together.
- Charge structure on K_x exists because DHR-like endomorphisms for the net $\mathscr{A} \upharpoonright K_x$ corresponds to 1-cocycles of $\mathrm{Z}^1(K_x, \mathscr{A})$: endomorphisms of the algebra $\mathcal{A}(K_x) := \mathrm{C}^*\{\mathcal{A}(o) \,|\, o \in K_x\}$ localized and transportable in K_x.

  In general K_x is not directed, so how is it possible ?
  - The point x is for K_x "spacelike infinite": let $\mathcal{O}(x) := \{o \in K \,|\, x \in o\}$, for any $o \in K_x$ there is $a \in \mathcal{O}(x)$ s.t. $a \perp o$.
  - Consider the presheaf $\mathcal{O}(x) \ni o \to \mathcal{A}(o)'$. $\mathcal{O}(x)$is downward directed. Then $\mathcal{A}(K_x) = \mathrm{C}^* - \lim_{o \in \mathcal{O}(x)} \mathcal{A}(o)'$.
  - Using this, the spacelike infinite x and punctured Haag duality DHR-endomrphisms are defined using the presheaf.

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The charge structure

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A new quantum number

For any irreducible 1-cocycle z the group von Neumann algebra

$$R_z(\pi_1(M), o) := \{z(p) \,|\, p : o \to o\}''$$

is a factor of type I_n with $n \leq d(z)$.

Definition

The topological dimension $\tau(z)$ of an irr. 1–cocycle z is the dimension of $R_z(\pi_1(M), o)$.

- $\tau(z)$ is an *invariant* of the equivalence class $[z]$
- $\tau(z)$ is *smaller than* $d(z)$
- *stability under conjugation*: $\tau(z) = \tau(\bar{z})$.

The representation R_z of $\pi_1(M)$ satisfies

$$R_z \cong 1 \otimes \sigma$$

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Splitting charge and topological content

Fix $o \in \Sigma_0$, the pole, and choose a path frame $P_o := \{p_{(a,o)} : o \to a \,|\, a \in \Sigma_0\}$. Associate two paths to any 1-simplex b:

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Theorem

Any 1-cocycle z is a composition $z = \chi_z \bowtie \langle z \rangle$, join, of its topological component and its charge component.

In particular we note

- If z is irreducible and has *topological dimension greater than* 1, then $\langle z \rangle$ is reducible: $\langle z \rangle = \oplus_{i=1}^{n} z_i$, where z_i is an irreducible object of $Z^1_{DHR}(K, \mathscr{A})$ with $\kappa(z_i) = \kappa(z)$. So, any such a charge is a finite collection of DHR-charges glued together by a glue of topological nature.
- If z is irreducible and has *topological dimension equals the statistical dimension* $\tau(z) = n = d(z)$, then $\langle z \rangle \cong \bigoplus_{i=1}^{n} u$ where $d(u) = 1$ and $\kappa(u) = \kappa(z)$. In this case the charge is formed by n DHR-charges of the same type.

Theorem (Existence)

For any irreducible finite dimensional and irreducible representation σ of $\pi_1(M)$, there is a 1-cocycle of $Z^1(K, \mathscr{A})$ which defines a repr. of $\pi_1(M)$ equivalent, up to infinite multiplicity, to σ.

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For any irreducible finite dimensional and irreducible representation σ of $\pi_1(M)$, there is a 1-cocycle of $\mathrm{Z}^1(K, \mathscr{A})$ which defines a repr. of $\pi_1(M)$ equivalent, up to infinite multiplicity, to σ.

Splitting charge and topological content

Theorem

Any 1-cocycle z is a composition $z = \chi_z \bowtie \langle z \rangle$, join, of its topological component and its charge component.

In particular we note

- If z is irreducible and has *topological dimension greater than* 1, then $\langle z \rangle$ is reducible: $\langle z \rangle = \oplus_{i=1}^{n} z_i$, where z_i is an irreducible object of $\mathrm{Z}^1_{DHR}(K, \mathscr{A})$ with $\kappa(z_i) = \kappa(z)$. So, any such a charge is a finite collection of DHR-charges glued together by a glue of topological nature.
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Physical interpretation

- Sharp localization of charge
  - $z \in \mathrm{Z}^1(K, \mathscr{A})$, for any $o \in K$ there is a "generalized endomorphism" $\rho^z(o)$ such that

$$\rho^z(o) \upharpoonright \mathcal{A}(\hat{o}) = \mathrm{id}, \qquad \hat{o} \perp o$$

    Cocycles plays the rôle of charge transporters:

$$z(p)\,\rho^z(o) = \rho^z(\tilde{o})\,z(p)\;, \qquad p : o \to \tilde{o}$$

- Analogy with the Aharonov-Bohm effect
  - If $q : o \to \tilde{o}$ *is not homotopic* to p, then

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    i.e. the final state depends on the homotopy class of the path.
    Any 1–cocycle z is a flat connection of a principal bundle over K (Roberts & R. 07):

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Outline

- Example: Massive scalar field in 2-dimension, preprint '08, by Brunetti, Franceschini & Moretti.
- There should be an underlying gauge theory giving rise to the charges $Z^1(K, \mathscr{A})$
  - purely topological
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