# Algebraic conformal QFT

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# 1. Conformal fields in two dimensions

Lect. 1

- Local fields
- Chiral fields
- Local description
- Fourier description
- Algebraic description

# Local fields

- Hilbert space: $\Phi(x)$, $\Phi^*(x)$ operator valued distributions on common invariant dense domain $\mathcal{D} \subset \mathcal{H}$.
- Locality: Commutation at spacelike separation.
- Covariance: Unitary covering representation of the conformal group $x \mapsto g(x)$:

$$U(g)\Phi(f)U(g)^* = \Phi(D(g)f \circ g^{-1})$$

- Vacuum: There is a unique $U$-invariant state $\Omega \in \mathcal{D}$. Ground state for the generator of the time translations, and cyclic in $\mathcal{H}$ for all fields.

From these axioms, the correlation functions

$$(\Omega, \Phi_1(x_1) \cdots \Phi_N(x_n)\Omega)$$

are defined as distributions, and from the knowlegde of all correlation functions, one can recover the fields.

- The main question is: Which realizations of the axioms are there, and how can they be constructed?

These lectures: two spacetime dimensions.

## Local fields (ct'd)

Infinitesimal transformation laws (for scalar fields of scaling dimension $d$)

$$i[P_\mu, \Phi(x)] = \partial_\mu \Phi(x), \quad i[K_\mu, \Phi(x)] = [2x_\mu (x \cdot \partial) - (x^2)\partial_\mu + 2d\, x_\mu]\Phi(x)$$
$$i[D, \Phi(x)] = [(x \cdot \partial) + d]\Phi(x), \quad i[M_{\mu\nu}, \Phi(x)] = [x_\mu \partial_\nu - x_\nu \partial_\mu]\Phi(x).$$

To define finite conformal transformations $g \in SO(2,2)$ in two-dimensional Minkowski spacetime (signature $+-$), one has to extend the latter to the Dirac manifold $=$ projective null cone in four dimensions with signature $++--$

$$\overline{\mathbb{R}^{1,1}} := \{\zeta \in \mathbb{R}^{2,2} : \zeta \cdot \zeta = 0\}/(\zeta \sim \lambda\zeta) \equiv (S^1_t \times S^1_s)/\mathbb{Z}_2.$$

Minkowski spacetime $\mathbb{R}^{1,1}$ is embedded as follows: Choose coordinates $(\sin\tau, \cos\tau)$ for the timelike $S^1_t$ and $(\sin\xi, \cos\xi)$ for the spacelike $S^1_s$. Then

$$(x^0, x^1) = \frac{(\sin\tau, \sin\xi)}{\cos\tau + \cos\xi} \quad \Rightarrow \quad x^0 \pm x^1 = \tan\frac{\tau \pm \xi}{2}.$$

Chiral coordinates $z_\pm = \frac{1+ix_\pm}{1-ix_\pm} = e^{i(\tau\pm\xi)}$:

$$\overline{\mathbb{R}^{1,1}} = S^1_+ \times S^1_-.$$

## Local fields (ct'd)

The conformal group factorizes: $SO(2,2)_0 =$ Möb $\times$ Möb.

The Möbius groups Möb $= SU(1,1)/\mathbb{Z}_2$ act by fractional linear transformations:

$$gz = \frac{\alpha z + \beta}{\bar\beta z + \bar\alpha}.$$

Local fields of scaling dimension $d = h_+ + h_-$, helicity $s = h_+ - h_-$:

$$\hat\Phi(z_+, z_-) = \big(dz_+/dx_+\big)^{-h_+}\big(dz_-/dx_-\big)^{-h_-} \cdot \Phi(x_+, x_-)$$

live on a covering space.

Infinitesimal generators $L_0 = \frac{1}{2}(P+K)$, $L_{\pm1} = \frac{1}{2}(P-K) \pm iD$:

$$[L_m, L_n] = (m-n)L_{m+n}.$$

Positivity of energy $P^0$ is by Lorentz invariance equivalent to positivity of both $P_\pm$, and by conformal invariance to positivity of both $L_{\pm,0}$.

Conformal transformation laws:

$$[L^+_m, \hat\Phi(z_+, z_-)] = z_+^m\big(z_+\partial_{z_+} + (m+1)h_+\big)\hat\Phi(z_+, z_-).$$

# Chiral fields

Symmetric traceless rank $r > 0$ tensor field $A^{\mu\cdots\nu}$ of scaling dimension $d_A = r$:

– are conserved ($\Rightarrow$ generators of local symmetries: gauge, diffeo, ...).

– have only two independent components $A_\pm = A_{0\cdots 00} \pm A_{0\cdots 01}$.

– $A_\pm = A_\pm(x_\pm)$ depend only on one chiral coordinate.

– $A_\pm$ commute with $B_\mp$.

– $A_\pm(x_\pm)$ commute with $B_\pm(y_\pm)$ when $x_\pm \neq y_\pm$.

– vacuum correlation functions factorize: $\langle \prod A_+ \prod A_- \rangle = \langle \prod A_+ \rangle \langle \prod A_- \rangle$.

$\Rightarrow$ pair of decoupled subtheories of local chiral fields, transforming covariantly under one chiral copy of the Möbius group in a positive-energy representation on a cyclic subspace of the Hilbert space.

## Local description

Rank $r = 1$ gives rise to chiral currents of dimension $h = 1$. Most general form of local commutators:

$$[j^a(x), j^b(y)] = i \sum_c f^{ab}_c \, j^c(x) \, \delta(x-y) + i \frac{g^{ab}}{2\pi} \, \delta'(x-y).$$

– Jacobi identity: $f^{ab}_c$ are the structure constants of a Lie algebra $\mathfrak{g}$, and $g^{ab}$, regarded as a metric on $\mathfrak{g}$, is invariant under the adjoint action.

– Hilbert space positivity: $g^{ab}$ positive definite. $\Rightarrow$

– $\mathfrak{g}$ compact, $g^{ab}$ multiple ("level") of Cartan-Killing, or

– $\mathfrak{g}$ abelian, WLOG $g^{ab} = 2\pi\delta^{ab}$.

## Local description (ct'd)

The stress-energy tensor (SET) is an example of rank $r = 2$, with the additional property that its chiral moments are the generators of the conformal group:

$$P = \int T(x)dx, \quad D = \int xT(x)dx, \quad K = \int x^2 T(x)dx.$$

These requirements fix the commutation relations among $T$:

$$[T(x), T(y)] = i(T(x) + T(y))\delta'(x - y) - \frac{ic}{24\pi}\delta'''(x - y),$$

where only the central charge $c > 0$ is a priori undetermined (model specific).

Commutation relations of $T$ with primary fields:

$$[T(x), A(y)] = i\Big[h_A \delta'(x - y) - \delta(x - y)\partial_y\Big] A(y)$$

and similar for 2D fields $\Phi(y_+, y_-)$.

For general covariant fields, there are additional contributions of fields of dimensions $\leq h_A - 2$.

## Summary of the local description:

- Chiral local fields appear in 2D conformal QFT by natural assumptions.
- Their commutator algebra is quite constrained by conformal symmetry.
- Correlation functions can be computed and analysed in terms of partial differential equations.
- Perfect setup to study concrete models.

# Fourier description

Chiral fields as distributions on the circle, via Cayley transformation $z = \frac{1+ix}{1-ix}$:

$$\hat{A}(z) := \left(\frac{dx}{dz}\right)^{h_A} A(x) =: \frac{1}{2\pi i^{h_A}} \sum_n z^{-h_A-n} A_n,$$

($A_n = 2^{1-h_A} \int (1-ix)^{h_A-1-n}(1+ix)^{h_A-1+n} A(x)dx$)

Commutation relations, eg,

$$[j^a_m, j^b_n] = i f^{ab}_c j^c_{m+n} + g^{ab}\, m\delta_{m+n,0}$$

for currents (Kac-Moody algebra),

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0}$$

for the SET (Virasoro algebra).

## Fourier description (ct'd)

For primary fields:

$$[L_m, \hat{A}(z)] = z^m(z\partial_z + (m+1)h_A)\hat{A}(z).$$

This is the variation of $\hat{A}(z) \to (\gamma'(z))^{h_A} \cdot \hat{A}(\gamma(z))$ under an infinitesimal diffeomorphism $\gamma(z) = z + \varepsilon z^{m+1}$, i.e., the Virasoro algebra acts like infinitesimal diffeomorphisms on the primary fields.

Central charge $c > 0$: there can be no diffeomorphism invariant state. The vacuum state is only Möbius invariant.

In terms of Fourier modes:

$$[L_m, A_n] = ((h_A - 1)m - n)A_{m+n}.$$

$\Rightarrow$ $A_n$ are ladder operators for the conformal Hamiltonian $L_0$.

## Summary of the Fourier description:

- Symmetries are emphasized.
- Lie algebra methods are available.
- Spectrum of $L_0$.
- Highest weight representations.
- Classifications, eg,

$$c \geq 1 \qquad \text{or} \qquad c = 1 - \frac{6}{m(m+1)} \quad (m = 3, 4, \cdots),$$

and of primary dimensions if $c < 1$:

$$h = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)} \quad (p = 1, \cdots, m-1; q = 1, \cdots, m).$$

- Drawback: local commutativity is quite obscured in this picture.

# Algebraic description

Example: The algebra of a single $U(1)$ current

$$[j(x), j(y)] = i\,\delta'(x-y)$$

is just a CCR algebra. Written in Weyl form:

$$W(f)W(g) = e^{-i\sigma(f,g)/2} \cdot W(f+g)$$

where $W(f) = e^{ij(f)} \equiv e^{i\int j(x)f(x)}$, and the symplectic form is

$$\sigma(f,g) = \frac{1}{2}\int (f'g - fg')\,dx = \int f'g\,dx.$$

- This form of writing allows to use methods from operator algebras.
- Example: superselection sectors as inequivalent representations: $\longrightarrow\longrightarrow$

## Algebraic description (ct'd)

The conserved charge $Q = \int j(x)\,dx$ commutes with $j(y)$, and hence with the algebra generated by the field. Therefore $\pi(Q) = q \cdot \mathbf{1}$ in every irreducible representation. In the vacuum representation, $q = 0$ because $(\Omega, j(x)\Omega) = 0$.

Consider the map (for a Schwartz function $\lambda$)

$$\rho_\lambda(j(x)) = j(x) + \lambda(x)\mathbf{1}.$$

Automorphism of the CCR algebra, hence $\pi_\lambda = \pi_0 \circ \rho_\lambda$ is a representation.

- Question: is this representation equivalent to the vacuum representation?

Charge $Q$ as an indicator: we have $\pi_\lambda(Q) = q_\lambda \cdot \mathbf{1}$ with $q_\lambda = \int \lambda(x)dx$. If this integral is nonzero, the representation cannot be unitarily equivalent.

To see the equivalence when $q_\lambda = 0$, the Weyl formulation is convenient: Write $\lambda = -h'$ (possible because $q_\lambda = 0$). Then

$$\rho_\lambda(W(f)) = e^{i\int \lambda(x)f(x)dx} \cdot W(f) = e^{-i\sigma(h,f)} \cdot W(f) = W(h)W(f)W(h)^*.$$

## Algebraic description (ct'd)

- It follows that $\pi_\lambda = \mathrm{Ad}_{\pi_0(W(h))}$ is unitarily equivalent to $\pi_0$.
  Similarly, $\pi_{\lambda_1} \sim \pi_{\lambda_2}$ iff $q_{\lambda_1} = q_{\lambda_2}$.
- Question: Are the representations $\pi_\lambda$ diffeomorphism covariant?

Diffeomorphisms act via symplectic maps $(T_\gamma f)(x) = f(\gamma^{-1}(x))$:

$$\alpha_\gamma(W(f)) = W(T_\gamma f).$$

Theory of CCR: A symplectic automorphism is implemented in the GNS representation of a state

$$\omega(W(f)) = e^{-\|f\|^2/2}$$

iff $|T_\gamma| - \mathbf{1}$ is Hilbert-Schmidt in the Hilbert space completion of the symplectic space, defined by the norm $\|f\|^2$.

This condition is sufficiently explicit to be verified for the vacuum state, where $\|f\|^2 = \int_0^\infty k\,dk\,|\hat{f}(k)|^2$.

- $\Rightarrow$ The vacuum representation of the chiral $U(1)$ current is diffeomorphism covariant on the real line (also on the circle). (Möbius covariance was already known because the vacuum state is Möbius invariant.)

## Algebraic description (ct'd)

Next, consider $\pi_\lambda$. A computation gives

$$\pi_\lambda \circ \alpha_\gamma(W(f)) = e^{i\int(\lambda_\gamma(x)-\lambda(x))f(x)dx} \cdot U_0(\gamma)\pi_\lambda(W(f))U_0(\gamma)^*$$

where $\lambda_\gamma(x) = \gamma'(x)\lambda(\gamma(x))$ such that $\int \lambda(x)T_\gamma f(x) = \int \lambda_\gamma(x) f(x)$, and $U_0(\gamma)$ is an implementer of $\gamma$ in the vacuum representation.

Because $q_{\lambda_\gamma} = q_\lambda$, the phase can be produced by the adjoint action of a unitary Weyl operator $W(h_{\lambda,\gamma})$ on $W(f)$, as before. $\Rightarrow$

The diffeomorphisms of $\mathbb{R}$ are implemented by $U_\lambda(\gamma) = U_0(\gamma)\pi_\lambda(W(h_{\lambda,\gamma}))$ in the charged representations $\pi_\lambda$, and so are the diffeomorphisms of $S^1$:

$$\pi_\lambda \circ \alpha_\gamma(\,\cdot\,) = U_\lambda(\gamma)\pi_\lambda(\,\cdot\,)U_\lambda(\gamma)^*.$$

By additivity of spectrum under composition (with the inverse of $\rho_\lambda$): the translations have positive generator for all $q$.

- $\Rightarrow$ A one-parameter family of inequivalent diffeomorphism ($\Rightarrow$ Möbius) covariant positive-energy representations. Described by automorphisms $\rho_\lambda$, and distinguished up to unitary equivalence by a charge $q_\lambda$.

## Summary of the algebraic description:

- Admits powerful methods from operator algebras.
- Superselection sectors = inequivalent representations of the same algebraic structure (generalized charges).
- Locality remains transparent (through support of test functions).

In particular, to accomodate a charge $q$ in the CCR model, one may choose the function $\lambda$ to have compact support in some interval $I$. This implies that $\rho_\lambda$ acts trivially on $W(f)$ is $\mathrm{supp}\, f$ lies outside $I$.

In this sense, $\rho_\lambda$ (or the representation $\pi_\lambda = \pi_0 \circ \rho_\lambda$) is “localized in $I$”.

# 2. Algebraic quantum field theory

- Haag-Kastler axioms
- Chiral AQFT
- Modular Theory
- Diffeomorphisms
- Extension problems
- Back to two dimensions

# Haag-Kastler axioms

A QFT is specified by a net of local algebras. This is an assignment

$$O \mapsto A(O)$$

of a $C^*$ algebra $A(O)$ to every region $O \subset \mathbb{R}^{1,D-1}$, subject to:

- Isotony: $A(O_1) \subset A(O_2)$ if $O_1 \subset O_2$.
- Locality: $A(O_1)$ commutes with $A(O_2)$ if $O_1$ is spacelike from $O_2$. (The statement is meaningful because $A(O_1)$ and $A(O_2)$ are subalgebras of $A(\mathbb{R}^{1,D-1})$.)
- Covariance: There is an action of the Poincaré group by automorphisms on $A(\mathbb{R}^{1,D-1})$ such that
$$\alpha_g(A(O)) = A(gO).$$
- Vacuum: There is a unique Poincaré invariant state on $A(\mathbb{R}^{1,D-1})$, such that the time translations have positive generator in the associated GNS representation.

## Haag-Kastler axioms (ct'd)

Comments:

- Think of $A(O)$ as the algebra generated by unitary operators $e^{i\Phi(f)}$ for hermitean fields and real test functions, with support in $O$. (With caveats!)
- Sufficient to specify $A(O)$ for the open doublecones in $\mathbb{R}^{1,D-1}$, and define $A(\mathbb{R}^{1,D-1})$ is as an inductive limit. Then $A(X)$ for general regions is the smallest subalgebra of $A(\mathbb{R}^{1,D-1})$ that contains all $A(O)$, $O \subset X$.
- Easily adapted to chiral conformal QFT in terms of algebras $A(I)$ for the open intervals $I \subset \mathbb{R}$.
- Complication: to formulate Möbius covariance, one needs the proper intervals on the circle, which do not form a directed set $\Rightarrow$ different definition of $A(S^1)$, the net is rather a "pre-cosheaf".

  Consequence: the vacuum representation is not faithful.
- Example: The Weyl formulation of the chiral free $U(1)$ current. $A(I) =$ CCR subalgebra generated by Weyl operators with test functions $\mathrm{supp}\, f \subset I$.

# Chiral AQFT

General results (Brunetti-Guido-Longo, and many others):

- Reeh-Schlieder property: The vacuum is cyclic and separating for every algebra $A(I)$ ($I$ a proper interval).
- Local normality: Every positive-energy representation is locally normal, ie, the restriction to a local algebra $A(I)$ is unitarily equivalent to the vacuum representation. One may therefore take the vacuum representation as a "reference", and take the weak closure of the local algebras, so as to arrive WLOG at a net of von Neumann algebras on the vacuum Hilbert space.
- Additivity: If $I_1$ and $I_2$ are open intervals such that $I = I_1 \cup I_2$ is another open interval, then $A(I) = A(I_1) \vee A(I_2)$.
- Strong additivity (i.e., the same holds also if $I_1$ and $I_2$ arise by removing an interior point from $I$) is not a consequence of the axioms, but can be established in many models.

## Chiral AQFT (ct'd)

- Bisognano-Wichmann property: the vacuum state restricted to the interval algebra $A(\mathbb{R}_+)$ is a KMS state of inverse temperature $\beta = 2\pi$ w.r.t. the dilation subgroup of Möb. The same is true, by conformal covariance, for every other interval algebra and the corresponding unique conjugate one-parameter subgroup of the Möbius group which preserves $I$.
- Haag duality: $A(I') = A(I)'$. This is stronger than locality ("$\subset$").
- Split property: If $\mathrm{Tr}\, e^{-\beta L_0} < \infty$ for all $\beta > 0$, then for any two intervals $I_1$ and $I_2$ that do not overlap or touch in a point, there is a normal state on $A(I_1) \vee A(I_2)$ such that $\varphi(a_1 a_2) = \varphi(a_1)\varphi(a_2)$. In other words: The algebra $A(I_1) \vee A(I_2)$ is isomorphic to the tensor product $A(I_1) \otimes A(I_2)$. The trace class condition here is not automatic; it can be violated in theories "with too many fields".
- Type: Every interval algebra is a type $III_1$ factor (hyperfinite if split property holds).

These results allow to use powerful methods from Modular Theory: $\longrightarrow\longrightarrow$

# Modular Theory

Given a von Neumann algebra $M$ and a cyclic and separating vector $\Omega$, there is a unique one-parameter group of automorphisms $\sigma_t$ (the modular group) such that the state $\omega = (\Omega, \cdot\Omega)$ on $M$ is a KMS state of temperature $1$ wrt $\sigma_{-t}$. The modular group is unitarily implemented with generator $\log \Delta$, where

$$S = J \cdot \Delta^{\frac{1}{2}}$$

is the polar decomposition of the closable antilinear map

$$S : m\Omega \mapsto m^*\Omega.$$

Moreover, the anti-unitary involution $J$ implements the conjugation $j(M) = M'$.

These data are intrinsic (and functorial) to the pair $(M, \Omega)$. In particular, in ACFT, one can recover the dilation subgroups $U(\Lambda_I(-2\pi t)) = \Delta^{it}_{(A(I),\Omega)}$ and the CPT operators $\Theta_I = J_{(A(I),\Omega)}$ from the local algebras and the vacuum vector.

Further applications of MT later.

## Diffeomorphisms

The stress-energy tensor is the generator of diffeomorphisms. Hence, existence of a SET is formulated as diffeomorphism covariance in ACFT:

There is a projective unitary representation of the orientation preserving diffeomorphism group of the circle in the vacuum Hilbert space such that

$$U(\gamma)A(I)U(\gamma^{-1}) = A(\gamma I),$$

and $U(\gamma)$ acts trivially on $A(I')$ if $\gamma$ is localized in $I$, i.e., $\gamma(x) = x$ for all $x \notin I$.

Haag duality then implies that $U(\gamma) \in A(I)$ if $\gamma$ is localized in $I$.

The implementers of the localized diffeomorphisms therefore generate a covariant net of subalgebras

$$A_0(I) \subset A(I).$$

Think of this as the subtheory generated by the stress-energy tensor. For the $U(1)$ current (CCR), the SET has $c = 1$.

## Extension problems

This scenario gives rise to a classification problem: Fixing $A_0$ (ie, the central charge of the chiral SET), which extensions

$$A_0(I) \subset A(I)$$

exist (conformal field theories which have the same SET)? More generally, one may wish to classify pairs of covariant nets

$$A(I) \subset B(I)$$

with trivial relative commutant ("chiral extension", the same SET), where either $A$ is given (extensions) or $B$ is given (subtheories).

More on this later.

# Back to two dimensions

A pair of chiral conformal QFTs arises in a 2D conformal QFT as subtheories of the generators of local symmetries, eg, diffeomorphisms in the case of the SET, gauge symmetries in the case of currents.

- Question: What is the field content of the 2D QFT beyond the chiral fields, or: which 2D conformal fields do these local symmetries act on?

This is another extension problem, similar to the previous one:

Doublecones in $\overline{\mathbb{R}^{1,1}} \supset \mathbb{R}^{1,1}$ are Cartesian products $O = I \otimes J$ of two chiral intervals. Then

$$A_+(I) \otimes A_-(J) \subset B(O)$$

where the pair of subtheories are embedded as a tensor product into the 2D algebra, because the vacuum correlations of fields from both subtheories factorize due to the cluster property.

# 3. Chiral constructions

Lect. 2

There is a large variety of standard constructions of new models from old ones.

- Nonabelian current algebras.
- Groups and subgroups.
- Tensor products.
- Relative commutants ("cosets").
- Fixpoints ("orbifolds").
- Algebraic constructions.

## Nonabelian current algebras

We wish to construct the local net for currents satisfying

$$[j^a(x), j^b(y)] = i \sum_c f_c^{ab}\, j^c(x)\, \delta(x-y) + i\frac{g^{ab}}{2\pi}\, \delta'(x-y).$$

Start from an auxiliary CAR algebra (free fermions), which possesses a local gauge symmetry, and construct the local observables as unitary operators implementing this symmetry.

Let $\mathcal{H}$ be a complex Hilbert space with an antiunitary involution $\Gamma$. $\mathrm{Cliff}(\mathcal{H}, \Gamma)$ is the unique $C^*$ algebra generated by $f \mapsto B(f)$ linear ($f \in \mathcal{H}$), subject to

$$B(f)^* = B(\Gamma f), \qquad B(f)B(g) + B(g)B(f) = (\Gamma f, g) \cdot \mathbf{1}.$$

Example: $\mathcal{H} = L^2(\mathbb{R}, \mathbb{C}^n \oplus \mathbb{C}^n)$ with $\Gamma(f \oplus g) = \overline{g} \oplus \overline{f}$. The Clifford algebra describes $n$ complex free Fermi fields on the real line, by writing

$$B(f \oplus g) = \int dx \left( \sum_i \psi_i(x) f_i(x) + \psi_i^*(x) g_i(x) \right)$$

(in the sense of operator-valued distrib's), ie, $\{\psi_i(x), \psi_j^*(y)\} = \delta_{ij}\delta(x-y)$ and $\{\psi, \psi\} = 0$.

## Nonabelian current algebras (ct'd)

For a projection $P$ on $\mathcal{H}$ such that

$$\Gamma P \Gamma = \mathbf{1} - P,$$

a quasifree state on the Clifford algebra is given by

$$\omega_P(B(f)B(g)) := (\Gamma f, Pg).$$

In its GNS representation $\pi_P$, one has $\pi_P(B(f)) = 0 = \pi_P(B(\Gamma f))$ if $Pf = 0$. One puts $a(f) := \pi_P(B(\Gamma f))$ and $a(f)^* := \pi_P(B(f))$ for $f \in P\mathcal{H}$, and obtains the CAR

$$\{a(f), a(g)\} = 0, \qquad \{a(f), a^*(g)\} = (f, g) \cdot \mathbf{1} \qquad \text{for} \quad f \in P\mathcal{H}.$$

They define the CAR algebra $\mathrm{CAR}(P\mathcal{H}) := \pi_P(\mathrm{Cliff}(\mathcal{H}, \Gamma))$.

The vacuum state for the Fermi field is obtained by choosing $P =$ projection to positive energy. It has the integral kernel $\Delta(x - y) = \frac{-i}{2\pi(x-y-i\varepsilon)}$.

## Nonabelian current algebras (ct'd)

A unitary operator $u$ on $\mathcal{H}$ that commutes with $\Gamma$ induces an automorphism $\alpha_u$ of the Clifford algebra, and

$$\omega_P \circ \alpha_u = \omega_{u^*Pu}.$$

Representations $\pi_P$ and $\pi_{P'}$ are unitarily equivalent iff the difference $P - P'$ is Hilbert-Schmidt. Hence $\alpha_u$ is implemented in the representation $\pi_P$ if $[u, P]$ is Hilbert-Schmidt.

Gauge transformations are given by the unitary operators $(u(f \oplus g))(x) = U(x)f(x) \oplus U(x)^* g(x)$. The implementability condition is

$$\int dx\, dy\, \mathrm{Tr}\, |U(x) - U(y)|^2 |\Delta(x-y)|^2 < \infty.$$

Obviously, global gauge transformations ($U = const$) are implemented. Convergence requires that $U(x) \to U_\pm$ as $x \to \pm\infty$ with $U_+ = U_-$. One may therefore regard implementable gauge transformations $U$ as elements of the loop group $LSU(n)$.

## Nonabelian current algebras (ct'd)

In particular, gauge transformations $U \in L_I SU(n)$ which are trivial outside an interval $I$, are implementable. The implementing operators $W_U$ are only determined up to a phase, which can be chosen such that

$$W_U W_V = e^{i\omega(f,g)/2} \cdot W_{UV}$$

is equivalent (Carey-Ruijsenaars) to the level $k = 1$ Kac-Moody algebra

$$[j(f), j(g)] = ij([f,g]) - \frac{i}{4\pi n} \oint dz\, g_{\mathrm{CK}}(f'(z), g(z)),$$

where $W_U = e^{ij(f)} = e^{i\oint dz \widehat{j^a}(z) f_a(z)}$ for $U(z) = e^{if_a(z)\tau^a}$.

The corresponding local net is given by $A(I) =$ von Neumann algebra generated by $W_U$ for $U \in L_I SU(n)$ (ie $\operatorname{supp} U \subset I$). Its vacuum representation is the cyclic subrepresentation of the Fock vacuum for the auxiliary Fermi fields.

## Nonabelian current algebras (ct'd)

This net possesses automorphisms acting on $W_U$ for $U(z) = e^{if_a(z)\tau^a}$ as

$$\rho_h(W_U) = e^{-i\varphi_{h,U}} \cdot W_{hUh^{-1}}, \quad \varphi_{h,U} = \frac{1}{4\pi n} \oint dz\, g_{\rm CK}(\partial_z h h^{-1}, hfh^{-1})$$

where $h : S^1 \to SU(n)/\mathbb{Z}_n$ ($\mathbb{Z}_n$ = centre of $SU(n)$), hence representations $\pi_h := \pi_0 \circ \rho_h$.

If $h$ lifts to a loop $V : S^1 \to SU(n)$ of compact support, $\rho_h$ is implemented by $W_V$. $\Rightarrow$ $\pi_{h_1} \sim \pi_{h_2}$ whenever $h_1 h_2^{-1}$ lifts to a loop with compact support.

In particular, $\pi_h$ is diffeomorphism covariant if $(h \circ \gamma)h^{-1}$ lifts to a loop with compact support (for "small" diffeo's $\gamma$); this is the case whenever $h$ is constant outside an interval $I$.

$\Rightarrow$ The unitary equivalence classes of diffeomorphism covariant representations are labelled by the elements of the centre $\mathbb{Z}_n$ of $SU(n)$.

These representations have positive energy (same argument as before).

## Groups and subgroups

$$H \subset G \quad \Rightarrow \quad LH \subset LG.$$

Example:

Level $k > 1$ representations ($g^{ab} = (k/2n) \cdot g^{ab}_{\mathrm{CK}}$ for $SU(n)$) are obtained by embedding $G$ into $G \times \cdots \times G$ "diagonally", ie, via $g \mapsto (g, \ldots, g)$, in particular

$$SU(n) \subset SU(n) \times \cdots \times SU(n) \subset SU(kn).$$

The level $k$ net is generated by $W_U$ for diagonally embedded loops $U$. In terms of fields:

$$J^a(x) = ((j^a(x) \otimes \mathbf{1} \otimes \ldots) + \cdots + (\cdots \otimes \mathbf{1} \otimes j^a(x)).$$

# Tensor products

The tensor product of two chiral nets is defined as

$$A(I) = A_1(I) \otimes A_2(I).$$

Clearly, the implementors of diffeomorphisms also arise by tensoring. This means that the stress-energy tensor is additive:

$$T(x) = T_1(x) \otimes \mathbf{1} + \mathbf{1} \otimes T_2(x),$$

and so is the central charge:

$$c = c_1 + c_2.$$

Representations of the tensor product theory are tensor products of representations of $A_i$.

# Fixpoints ("orbifold construction")

If the net $A$ has some compact symmetry group that preserves all local algebras, one can pass to the fixpoint net

$$A^G(I) = A(I)^G.$$

If the vacuum state is invariant under $G$, then the vacuum representation is reducible for the fixpoint net. More precisely, the irreducible subrepresentations are in 1:1 correspondence with the unitary irreps of $G$, and arise with finite multiplicities equal to the dimension of the irreps.

Examples:

- $U(1)$ current (CCR), even subtheory.
- Kac-Moody models, fixpoints under global $G$.
- Tensor products of identical theories, fixpoints under permutations.

## Relative commutants ("coset construction")

If a net $A$ is contained in a net $B$, one may consider the net of relative commutants:

$$A^c(I) = B(I) \cap A(I)'.$$

- If $A^c(I) = \mathbb{C} \cdot \mathbf{1}$ (then $A$ and $B$ have the same SET), we call $A \subset B$ a "chiral extension" ($\equiv$ "conformal inclusion" if $A$ and $B$ arise from Lie groups)

  Examples: $SU(2)_{10} \subset SO(5)_1, \quad SU(2)_{28} \subset (G_2)_1.$

- Otherwise: $A^c(I)$ defines the coset net. Then $A \otimes A^c \subset B$ is a chiral extension. For the SET this means $T_B = T_A \oplus T_{A^c}$, $c_B = c_A + c_{A^c}$.

  Examples:

  – Level $k$ Kac-Moody $\subset$ $k$-fold tensor product of level $1$ (has a huge coset)

  – $SU(2)_{k+1} \subset SU(2)_k \otimes SU(2)_1 \Rightarrow$ coset $=$ Virasoro with $c = 1 - \frac{6}{(k+2)(k+3)}$.

  – Level-rank duality: $SU(n)_k \otimes SU(k)_n \subset SU(kn)$ (are each other's cosets)

In general, a chiral extension $A \otimes C \subset B$ defines chiral extensions $A \subset C^c$, $C \subset A^c$ such that $C^c \otimes A^c \subset B$ are each other's cosets.

## Algebraic constructions (no direct analogue in terms of fields)

- "Mirror extensions". See below.
- "Half-sided modular factorizations". (Guido-Longo-Wiesbrock): By the Bisognano-Wichmann property, the modular group associated with the pair $(A(I), \Omega)$ coincides with the dilation subgroup $\subset$ Möb preserving $I$. Splitting the circle into three intervals in clockwise order, the inclusions $A(I_i) \subset A(I_{i+1 \mod 3})' = A(I'_{i+1 \mod 3})$ are half-sided modular (hsm), ie, the modular groups $\sigma_t^{(i)}$ of each $A(I'_{i+1 \mod 3})$ are contractions of $A(I_i)$ for positive values of the parameter.

  Conversely, each quadruple of three commuting algebras $M_i$ and a joint cyclic and separating vector, such that $M_i \subset M'_{i+1 \mod 3}$ are hsm, defines a local net as follows:

  The three modular groups satisfy relations (Borchers) which generate the Möbius group. Applying the Möbius group to one of the three algebras, one obtains a Möbius covariant local net $I \mapsto A(I)$ in the vacuum representation, in which $M_i$ are the local algebras for three intervals as before. Positivity of the energy follows from the half-sided modular property.

# 4. Superselection sectors

- Representations and endomorphisms
- Intertwiners
- Conjugates
- Statistics
- Modular tensor category

# Representations and endomorphisms

Superselection (DHR) sectors are inequivalent positive-energy representations of the algebra of observables, ie of the net $I \longrightarrow A(I)$.

They may arise by the branching of the vacuum representation of a larger net ("field algebra") into inequivalent rep'ns of the smaller net ("observables"): The "charged" fields create charged states, while the "neutral" observables cannot change the charge. This mechanism is the generic origin of superselections sectors in four spacetime dimensions, with charge = representation of a compact gauge group (Doplicher-Roberts).

- In low dimensional QFT, not all sectors are generated in this way.
- A general mechanism for the origin of sectors is not known.
- All DHR sectors of a CFT on the circle can be "detected" by inspection of the inclusion

$$A(I_1 \cup I_3) \subset A(I_2 \cup I_4)'$$

  for any splitting of the circle into four intervals (see below). Namely, the larger algebra contains neutral elements which carry some charge in $I_1$ and an opposite charge in $I_3$.
- More general theory: Net cohomology (Roberts, Ruzzi).

## Representations and endomorphisms (ct'd)

A covariant representation of the net is a covariant and compatible collections of representations $\pi_I$ of $A(I)$ on $\mathcal{H}_\pi$.

$A(I)$ factor $\Rightarrow$ $\pi_I$ is normal. Type $III$ $\Rightarrow$ locally equivalent to $\pi_0 = \mathrm{id}$: $\pi_I = \mathrm{Ad}_{V_I}$.

Choose $I_0$, and define for all $I$:

$$\rho_I(a) := Ad_{V_{I_0'}} \circ \pi_I(a).$$

- By construction, $\rho_I$ is trivial for $I \subset I_0'$:

$$\rho_I(a) = a \quad \text{for} \quad a \in A(I),$$

- and by Haag duality, $\rho_I$ is an endomorphism for $I \supset I_0$:

$$\rho_I(A(I)) \subset A(I).$$

In particular, $\rho$ is a localized endomorphism of the net on $\mathbb{R} \subset S^1$.

Examples: automorphisms $\rho_\lambda$ of CCR, $\rho_h$ of current algebras.

# Intertwiners

(Doplicher-Haag-Roberts):

Intertwiners: $t : \rho_1 \to \rho_2 \quad \Leftrightarrow \quad t\rho_1(a) = \rho_2(a)t \quad \Leftrightarrow \quad t \in \mathrm{Hom}(\rho_1, \rho_2)$.

By Haag duality, intertwiners are local operators. Unitary intertwiners are "charge transporters", changing the interval of localization.

By locality, one obtains the structure of a $C^*$ tensor category. The objects are representations (endomorphisms), the arrows are the intertwiners.

- Subrepresentations correspond to subobjects: $\sigma \prec \rho$ iff there is $t : \sigma \to \rho$, $t^*t = \mathbf{1}$, hence $tt^*$ a projection in $\mathrm{Hom}(\rho, \rho)$.
- The product of objects is the composition of endomorphisms. This defines a product of representations ("fusion", much simpler than in the field-theoretic setup)!

# Conjugates

If $\rho$ is an automorphism, such as the automorphisms of CCR and current algebras above), then its inverse is also a covariant representation.

Otherwise ($\rho$ irreducible): $\bar{\rho}$ irreducible is conjugate to $\rho$ if $\mathrm{id} \prec \rho\bar{\rho}$. In other words: the vacuum representation is a subrepresentation of the composition of $\pi$ with $\bar{\pi}$.

If it exists, it is unique up to unitary equivalence.

- generalizes the inverse of automorphisms.
- gives rise to "left inverse" maps $\phi_\rho \circ \rho = \mathrm{id}$ of $\rho$, and conditional expectations $\mu_\rho : A(I) \to \rho(A(I))$.

Useful formula (Longo):

$$\bar{\rho} = j \circ \rho \circ j$$

where $\rho$ is localized in $I \subset \mathbb{R}_+$, $j$ is the CPT transformation ($\to$ modular theory) with $j(A(\mathbb{R}_\pm)) = A(\mathbb{R}_\mp)$, so that $\bar{\rho}$ is localized in $-I \subset \mathbb{R}_-$.

# Statistics

(DHR, Fredenhagen-KHR-Schroer): If $\rho_i$ are localized in disjoint intervals $\subset \mathbb{R}$, then $\rho_1\rho_2 = \rho_2\rho_1$. Since every representation is unitarily equivalent to an endomorphism localized in any interval of choice, in general there are distinguished unitary intertwiners ("statistics operators") $\varepsilon_{\rho_1,\rho_2} : \rho_1\rho_2 \to \rho_2\rho_1$ whenever $I_1, I_2 \subset \mathbb{R}$.

Statistics operators satisfy "coherence relations" with intertwiners. They give rise to representations of the braid group $B_\infty$, and thus turn the sector category into a braided category.

A sector invariant:

- $\rho$ irreducible: "statistics parameter" $\lambda_\rho := \mu(\varepsilon_{\rho,\rho}) \in \mathbb{C}$.
- ($\lambda_\rho \neq 0$, $\kappa_\rho := \lambda_\rho/|\lambda_\rho|$) Spin-Statistics Theorem (Guido-Longo): $U_\rho(2\pi) = \kappa_\rho \cdot 1$.
- ($d_\rho := 1/|\lambda_\rho|$) Index Theorem (Longo): $d_\rho^2 = [A(I) : \rho_I(A(I))]$ (finite iff conjugate exists).
- The statistical dimension $d_\rho \in [1, \infty]$ is additive (under direct sums) and multiplicative (under compositions). $d_\rho = 1$ for automorphisms. In general not an integer.

# Modular tensor category

Sectors = unitary equivalence classes of irreducible representations. Choose $\rho_i$ = representatives of sector $[i]$.

Further invariants:

- "Fusion rules" $N_{ij}^k := \dim \mathrm{Hom}(\rho_i\rho_j, \rho_k)$.
- (KHR, FRS) "Statistics characters" $Y_{ij} = Y_{ji} := d_i d_j \cdot \mu_i\mu_j(\varepsilon_{\rho_i,\rho_j}\varepsilon_{\rho_j,\rho_i}) \in \mathbb{C}$, satisfying

$$Y_{i\ell}Y_{j\ell} = \sum_k N_{ij}^k Y_{k\ell}.$$

  (Related to "charge transport once around the circle").
- If $(Y_{ij})$ is a nondegenerate matrix ("non-degenerate braiding"), then one can construct a matrix representation of $SL(2,\mathbb{Z})$ out of these invariants.
- This property turns the sector category into a modular tensor category.

## Modular tensor category (ct'd)

(Kawahigashi-Longo-Müger): A net $A$ is "completely rational" if it is strongly additive and split, and for any splitting of the circle into four intervals

$$\mu_A := [A(I_2 \cup I_4)' : A(I_1 \cup I_3)] < \infty.$$

(Established in many models, but there are also counter examples.) Then

- $A$ has only finitely many sectors, all of them of finite dimension $d_i$, and
$$\mu_A = \sum d_i^2.$$
(See above: detection of all sectors via the four-interval subfactor.)
- The matrix $Y$ is invertible, hence the category of superselection sectors is a modular tensor category.

Deep miracle: $S =$ representative of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL(2,\mathbb{Z})$ relates $\mathrm{Tr}\, U_i(e^{-\beta L_0})$ to $\mathrm{Tr}\, U_j(e^{-(4\pi^2/\beta)L_0})$ (low $\leftrightarrow$ high temperature). The statistical dimension $d_\rho$ controls the behaviour of $\mathrm{Tr}\, U_i(e^{-\beta L_0})$ as $\beta \to 0$: $d_{\rho_i} = \lim \mathrm{Tr}\, U_i(e^{-\beta L_0})/\mathrm{Tr}\, U_0(e^{-\beta L_0})$.

# 5. Chiral extensions

Lect. 3

- Extensions
- Subfactors
- Invariants
- Classifications
- Mirror extensions

# Extensions

For a given local net $I \mapsto A(I)$ on the circle, we wish to characterize nets $I \mapsto B(I)$ which contain $A$ ("adding more fields"):

$$A(I) \subset B(I), \qquad A(I)' \cap B(I) = \mathbb{C} \cdot \mathbf{1},$$

such that $B$ is relatively local, ie, $B(I_1)$ commutes with $A(I_2)$ if $I_1$ and $I_2$ are disjoint.
($B$ possibly nonlocal.)

We assume that $A(I) \subset B(I)$ is irreducible (trivial relative commutant). If $B$ is local, this means that the coset net is trivial (chiral extension), and $A$ and $B$ have the same SET.

We also assume that $A(I) \subset B(I)$ has finite index (see below, "finitely many charged fields"). This is automatic if $A$ is completely rational (Izumi-Longo-Popa).

For the operator algebraic theory of chiral extensions, some preparation is needed: $\longrightarrow\longrightarrow$

## Subfactors

Let $M$ and $N$ type $III$ factors. If $N \subset M$, the index $[M : N]$ is the inverse of the largest constant $\kappa \geq 0$ for which there is a conditional expectation $\mu : M \to N$ bounded from below:

$$\mu(m^*m) \geq \kappa \cdot m^*m.$$

The index is multiplicative in $N \subset M \subset L$.

A homomorphism $\sigma : N \to M$ gives rise to a subfactor $\sigma(N) \subset M$. Conversely, every subfactor is given by a homomorphism, eg, the injection $\iota : N \to M$ ($=$ identical map $n \mapsto n$).

For homomorphisms $\sigma : N \to M$, one defines intertwiners

$$M \ni t : \sigma_1 \to \sigma_2 \quad \Leftrightarrow \quad t \in \mathrm{Hom}(\sigma_1, \sigma_2) \quad \Leftrightarrow \quad t\sigma_1(n) = \sigma_2(n)t \quad \forall\, n \in N.$$

- $\sigma : N \to M$ is irreducible iff $\mathrm{Hom}(\sigma, \sigma) = \mathbb{C} \cdot \mathbf{1}$ iff $\sigma(N)' \cap M = \mathbb{C} \cdot \mathbf{1}$.
- $\sigma_1 \sim \sigma_2$ iff there is a unitary intertwiner $t : \sigma_1 \to \sigma_2$. With this, one can define equivalence classes (sectors) $[\sigma] \in \mathrm{Sect}(N, M)$.
- $\sigma_1 \prec \sigma_2$ iff there is an isometric intertwiner $t : \sigma_1 \to \sigma_2$. With this, one can define the direct sum of sectors, and decomposition of a sector into irreducibles.
  For $\sigma_1$ irreducible, $\dim \mathrm{Hom}(\sigma_1, \sigma_2) =$ multiplicity of $\sigma_1 \prec \sigma_2$.

## Invariants

- The dimension $d_\sigma := [M : \sigma(N)]^{\frac{1}{2}} \in [1, \infty]$ is a sector invariant. The dimension is additive (under direct sums) and multiplicative (under composition). Quantized below 4 (Jones).
- For $d_\sigma < \infty$, $\bar{\sigma} : M \to N$ is conjugate to $\sigma : N \to M$ if there is a pair of isometric intertwiners $N \ni w : \mathrm{id}_N \to \bar{\sigma}\sigma$, $M \ni v : \mathrm{id}_M \to \sigma\bar{\sigma}$, satisfying relations

$$\sigma(w)^* v = d_\sigma^{-1} \cdot \mathbf{1}_M, \qquad \bar{\sigma}(v)^* w = d_\sigma^{-1} \cdot \mathbf{1}_N.$$

- For $N \subset M$ and $\iota : N \to M$ the injection, call
  - $\gamma = \iota \circ \bar{\iota} : M \to M$ with $\mathrm{id}_M \prec \gamma$ the canonical endomorphism
  - $\theta = \bar{\iota} \circ \iota : N \to N$ with $\mathrm{id}_N \prec \theta$ the (dual canonical endo).
- Q-system (Longo): Let $w : \mathrm{id}_N \to \theta$ and $v : \mathrm{id}_M \to \gamma$ a pair of isometries as before. Then $N \ni w$ and $N \ni x = \bar{\iota}(v) : \theta \to \theta^2$ satisfy

$$x^* w = x^* \theta(w) = d_\theta^{-\frac{1}{2}} \cdot 1_N, \quad xx = \theta(x)x, \quad xx^* = \theta(x^*)x.$$

The triple $(\theta, w, x)$ (data only in $N$) is a complete invariant for $N \subset M$.

## Invariants (ct'd)

- Frobenius reciprocity: For finite index homo's $\eta : N \to P$, $\zeta : P \to M$, $\sigma : N \to M$

$$\dim \mathrm{Hom}(\zeta \circ \eta, \sigma) = \dim \mathrm{Hom}(\eta, \bar{\zeta} \circ \sigma) = \dim \mathrm{Hom}(\zeta, \sigma \circ \bar{\eta}).$$

- Principal graph (Bratteli diagram): Consider
  – $\mathcal{X}_{MM} \subset \mathrm{Sect}(M, M)$ = irreducible subsectors $\eta \prec (\sigma\bar{\sigma})^n : M \to M$
  – $\mathcal{X}_{NM} \subset \mathrm{Sect}(N, M)$ = irreducible subsectors $\zeta \prec (\sigma\bar{\sigma})^n \sigma : N \to M$.

  Two-partite graph with the number of edges given by $\dim \mathrm{Hom}(\zeta, \eta\sigma) = \dim \mathrm{Hom}(\eta, \zeta\bar{\sigma})$.
- Dual principal graph: the same with $\sigma \leftrightarrow \bar{\sigma}$.
- Induced graph: For $\iota : N \to M$ and $\sigma : N \to N$, consider
  – $\mathcal{X}^0 \subset \mathrm{Sect}(N, M)$ = irreducible subsectors $\eta \prec \iota(\sigma\bar{\sigma})^n : N \to M$
  – $\mathcal{X}^1 \subset \mathrm{Sect}(N, M)$ = irreducible subsectors $\zeta \prec \iota(\sigma\bar{\sigma})^n \sigma : N \to M$.

  Two-partite graph with the number of edges given by $\dim \mathrm{Hom}(\zeta, \eta\sigma) = \dim \mathrm{Hom}(\eta, \zeta\bar{\sigma})$.
- By Frobenius Theorem: The incidence matrices of the graphs have norm $\|I\| = d_\sigma$ (because $\eta\sigma = \bigoplus I_{\eta,\zeta} \cdot \zeta \Rightarrow I_{\eta,\zeta} d_\zeta = d_{\eta\sigma} = d_\sigma \cdot d_\eta$ and similarly $I^t_{\zeta,\eta} d_\eta = d_{\zeta\bar{\sigma}} = d_\sigma \cdot d_\zeta$).

## Extensions (resumed)

For a fixed $I_0$, let $\iota : A(I_0) \to B(I_0)$ be the inclusion homomorphism, and $\bar\iota : B(I_0) \to A(I_0)$ its conjugate, $\gamma : B(I_0) \to B(I_0)$ and $\theta : A(I_0) \to A(I_0)$ the canonical and dual canonical endo's.

- (Longo-KHR) By relative locality, $\theta$ extends to a localized endomorphism of the net $A$.
- The reducible superselection sector $[\theta]$ of $A$ is a first invariant for the extension $A \subset B$.
- $\pi_0 \circ \theta$ is the vacuum representation of $B$, as a rep'n of the subtheory $A$: thus $\theta$ specifies the Hilbert space of $B$, and the "charges" of the fields in $B$.
- The "DHR triple" $(\theta, w, x)$ (= Q-system, but $\theta$ regarded as localized endomorphism of $A$, and $w$, $x$ as global intertwiners) is a complete invariant for the extended net $A \subset B$.
- The net $B(I)$ can be reconstructed from these data. It contains "charged fields" $\psi_i \in B$ for each subsector $\rho_i$ of $\theta$, such that $\psi_i a = \rho_i(a)\psi_i$. The multiplication and conjugation of charged fields is determined by the data $(\theta, w, x)$.
- $B$ is local iff $\varepsilon_{\theta,\theta}\, x = x$.

# Classifications

Consider the induced graphs $(\iota\sigma\bar{\sigma}\sigma\ldots)$ for $\iota : A \to B$ and $\sigma$ a localized endomorphism of $A$.

Its incidence matrix has norm $d_\sigma$. If $d_\sigma < 2$, this must be an $A$ or $D$ or $E$ graph.

- This is useful for classification of chiral extensions of Virasoro nets with $c = 1 - \frac{6}{(m+1)(m+2)}$, because these have two independent DHR sectors of dimension $2\cos\frac{\pi}{m+1}, 2\cos\frac{\pi}{m+2} < 2 \Rightarrow$ classification in terms of pairs of $A$-$D$-$E$ graphs with Coxeter numbers $m, m+1$.

Recall that $[\theta]$ is also a DHR sector. Thus, if $\theta$ is known (and the fusion rules of $A$), these graphs can be computed via Frobenius reciprocity: $\dim\mathrm{Hom}(\iota\rho_1, \iota\rho_2) = \dim\mathrm{Hom}(\theta\rho_1, \rho_2)$ gives info about the number of subsectors of $\iota\rho$, their multiplicities, and their distinctness.

- Need to know the admissible dual canonical endo's $\theta$!

This requires more detailed analysis, exploiting the locality properties: $\longrightarrow\longrightarrow$

## Classifications (ct'd)

$\mathrm{DHR}(A) \subset \mathrm{End}(A)$ the DHR localized endomorphisms (positive-energy representations) of the net $A$.

- Induction (Roberts, Longo-KHR, Xu): two functorial prescriptions $\alpha^\pm : \mathrm{DHR}(A) \to \mathrm{End}(B)$ such that $\alpha^\pm_\rho \iota = \iota\rho$ (ie, $\alpha_\rho$ equals $\rho$ on elements of $A$):

$$\alpha^\pm_\rho = \bar{\iota}^{-1} \circ \mathrm{Ad}_{\varepsilon^\pm(\rho,\theta)} \circ \rho \circ \bar{\iota}.$$

- Restriction: $\sigma_\beta = \bar{\iota}\beta\iota \in \mathrm{DHR}(A)$ for $\beta \in \mathrm{DHR}(B)$ ($\Rightarrow$ branching of superselection sectors).

- $\alpha$-$\sigma$-reciprocity (Xu, Böckenhauer-Evans):

$$\dim \mathrm{Hom}(\alpha^\pm_\rho \gamma, \beta) = \dim \mathrm{Hom}(\rho, \sigma_\beta) = \dim \mathrm{Hom}(\alpha^\pm_\rho, \beta)$$

  for $\rho \in \mathrm{DHR}(A)$ and $\beta \in \mathrm{DHR}(B)$ (if $B$ is local). The first equality is just Frobenius reciprocity. Thus, $\mathrm{id}_B \prec \gamma$ is the only irreducible sub-endo of $\gamma$, that can be contained in any $\alpha^\pm_\rho \in \mathrm{End}(B)$ or in any $\beta \in \mathrm{DHR}(B)$.

## Classifications (ct'd)

- Common subendo's of $\alpha_\rho^+$ and $\alpha_\sigma^-$ belong to $\mathrm{DHR}(B)$.
- (Xu, Böckenhauer-Evans-Kawahigashi) The matrix $Z_{\rho,\sigma} := \dim \mathrm{Hom}(\alpha_\rho^+, \alpha_\sigma^-)$ is a modular invariant wrt to the $SL(2,\mathbb{Z})$ of the modular tensor category of $A$.
- If $B$ is local, the "zero column" of $Z$ ($\rho_0 = \mathrm{id} =$ vacuum sector) gives the multiplicities of $\sigma \prec \theta$, ie $\theta = \bigoplus Z_{0,\sigma} \cdot \sigma$.

A classification of modular invariant matrices $Z$ with nonnegative integer entries (eg, Cappelli-Itzykson-Zuber) gives a list of possible dual canonical endomorphisms $\theta$.

- $\Rightarrow$ Complete classification of local extensions of Virasoro nets with $c < 1$ (Kawahigashi-Longo):
  $(A_m, A_{m+1})$, $(A_{4n-4}, D_{2n})$, $(D_{2n}, A_{4n-2})$, $(A_{10}, E_6)$, $(E_6, A_{12})$, $(A_{28}, E_8)$, $(E_8, A_{30})$. Only $D_{\mathrm{even}}$ and $E_{\mathrm{even}}$ occur.
- $(A_{28}, E_8)$ at $c = \frac{144}{145}$ cannot be produced by known field-theoretic methods.

# Mirror extensions

Suppose that two nets $A$ and $C$ arise as each other's cosets (relative commutants) within some net $B$:

$$A(I) \otimes C(I) \subset B(I).$$

Then the dual canonical endomorphism associated with $\iota : A \otimes C \to B$ is of the form

$$\theta = \bigoplus_i \rho_{A,i} \otimes \rho_{C,i}$$

with a bijection between the superselection sectors $[\rho_A]$ of $A$ and $[\rho_C]$ of $C$ (KHR). This bijection is actually part of an isomorphism (with opposite braiding) between the superselection categories (Müger?).

This implies that every DHR triple $(\theta_A, w_A, x_A)$ of $A$ is isomorphic to a DHR triple $(\theta_C, w_C, x_C)$ of $C$, and there is a bijection between local extensions of $A$ and local extensions of $C$ (Xu).

- By this, one can construct local extensions of a coset theory $A^c$, in terms of those of $A$.
- Among others, the one "new" example of the KL classification at $c = \frac{144}{145}$ arises in this way.

## Mirror extensions (ct'd)

The exceptional Virasoro extensions arise in pairs:

- $(E_6, A_{12})$ at $c = \frac{25}{26}$, and $(A_{10}, E_6)$ at $c = \frac{21}{22}$;
- $(E_8, A_{30})$ at $c = \frac{154}{155}$, and $(A_{28}, E_8)$ at $c = \frac{144}{145}$.

The first entry in each pair arises via subgroups and cosets, while the second arises as a mirror:

$$\begin{array}{ccc} SU(2)_{11} & \overset{\text{cosets}}{\otimes} & \mathrm{Vir}(\frac{25}{26}) \\ & \cap & \\ & SU(2)_{10} \otimes SU(2)_1 & \\ & \cap & \\ & SO(5)_1 \otimes SU(2)_1 & \\ & \cup & \\ SU(2)_{11} & \overset{\text{cosets}}{\otimes} & (E_6, A_{12}) \end{array} \qquad \text{versus} \qquad \begin{array}{ccc} & SU(2)_9 \otimes SU(2)_1 & \\ & \cup & \\ SU(2)_{10} & \overset{\text{cosets}}{\otimes} & \mathrm{Vir}(\frac{21}{22}) \\ \cap & & \cap \\ SO(5)_1 & \overset{\text{mirror}}{\longleftrightarrow} & (A_{10}, E_6) \end{array}$$

The inclusions for the second pair look the same, with $SU(2)_{10} \subset SO(5)_1$ replaced by $SU(2)_{28} \subset (G_2)_1$.

# 6. From chiral CFT to two dimensions

Remember the previous question:

What is the field content of a 2D QFT beyond the chiral fields? Extension problem:

$$A_+(I) \otimes A_-(J) \subset B_{\mathrm{2D}}(O).$$

- Classification for $c < 1$
- Direct construction
- Boundary CFT
- Removing the boundary

# Classification for $c < 1$

- 2D local extensions of $\mathrm{Vir}(c) \otimes \mathrm{Vir}(c)$ for $c < 1$ can be classified in terms of "$\iota\sigma$-graphs" (Kawahigashi-Longo), as in the chiral case.
- The classification of maximal extensions is again in terms of pairs of $A$-$D$-$E$ graphs (allowing $D_{\mathrm{odd}}$ and $E_{\mathrm{odd}}$).
- For the uniqueness (for a given pair of graphs), a nontrivial cohomological problem must be solved.
- Each maximal extension has a finite number of intermediate nonmaximal extensions.

## Direct construction

A local extension $A \otimes A \subset B_{\mathrm{2D}}$ is characterized by its DHR triple

$$(\Theta, W, X)$$

where $\Theta \in \mathrm{DHR}(A \otimes A) = \mathrm{DHR}(A) \otimes \mathrm{DHR}(A)$, and $W$, $X$ isometric intertwiners subject to relations (as above).

- (Longo-KHR) There is a "universal" solution (hence a local 2D ACFT) with

  $$\Theta = \bigoplus \rho \otimes \bar{\rho}.$$

  In particular, every chiral sector arises from this 2D extension. (Remember the question about the "origin of sectors" in low-dim QFT!)

- (KHR) For every nonlocal chiral extension $A \subset B$ there is a solution (hence a local 2D ACFT) with

  $$\Theta = \bigoplus Z_{\rho,\sigma} \cdot \rho \otimes \bar{\sigma},$$

  where $Z_{\rho,\sigma} := \dim \mathrm{Hom}(\alpha^+_\rho, \alpha^-_\sigma)$ (constructed using $\alpha$-induction).

Alternative construction "via boundary CFT". $\longrightarrow$

# Boundary CFT

Given $A$ a local chiral net (completely rational), and $A \subset B$ a chiral extension (relatively local but possibly nonlocal).

Define

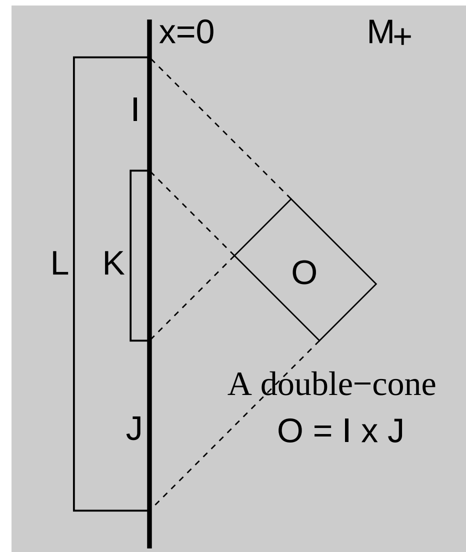
$$B_+(O) := B(K)' \cap B(L).$$

This defines a Haag dual Möbius covariant local net on the halfspace $x > 0$, containing

$$A(I) \vee A(J) \subset B_+(O).$$

Its "restriction" to the boundary (= time axis) gives back the nonlocal chiral net $B$, namely

$$B(L) = B_+(W_L).$$




Explains why local $n$-point correlations behave like nonlocal chiral $2n$-point correlations (Cardy):

$$f(t_1 + x_1, t_1 - x_1, \ldots, t_n + x_m, t_n - x_n).$$

Namely, bulk fields are products of charged chiral fields in $I$ and oppositely charged fields in $J$.

## Removing the boundary

- Pick a doublecone $O_0 = I_0 \times J_0$ in the halfspace.
- Exploit the split isomorphism $A(I_0 \cup J_0) \sim A(I_0) \otimes A(J_0)$ to find a state $\xi$ on $A(I_0) \vee A(J_0)$ such that $\xi(a_I a_J) = \omega(a_I)\omega(a_J)$.
- Extend the state to a state $\hat{\xi}$ on $B_+(O_0)$ by the unique conditional expectation.
- Construct the GNS Hilbert space $\mathcal{H}$ for $\hat{\xi}$.
- Use the modular groups for the subalgebras $A(I_1)$ $(I_1 \subset I_0)$ and $A(J_1)$ $(J_1 \subset J_0)$ to generate a unitary representation of Möb $\times$ Möb on $\mathcal{H}$.
- Define a net $B_{2\mathrm{D}}$ on $\mathcal{H}$ by transporting $B_{2\mathrm{D}}(O_0) := B_+(O_0)$ with Möb $\times$ Möb.
- The 2D vacuum state $\omega_{2\mathrm{D}} = (\Omega_{\hat{\xi}}, \cdot\, \Omega_{\hat{\xi}})$ on $B_{2\mathrm{D}}$ can also be obtained as a limit of the chiral vacuum state $\omega \circ \beta_a$, where $a \in$ Möb $\otimes$ Möb are the spacelike shifts $x \mapsto x + a$, as $a \to \infty$.

This construction gives the same result as the $\alpha$-induction construction, but appears physically more transparent. Interesting question: why does one need the "boundary as a catalyser"?

Generating the boundary CFT with the trivial extension $B = A$, one gets the "universal" 2D theory with $\Theta = \bigoplus \rho \otimes \bar{\rho}$ ("Cardy case").