Quantum Dynamics and Ramanujan Petersson conjectures

M v. N. algebra

- $\Phi_t : M \to M$ so-continuous

semigroup of unital CP maps ([Davies, Lindblad, ...

Def CP map. $\boxed{\left(\Phi(x_i^* x_j)\right)_{ij} \geq 0}$ $\forall x_i \in M$

Generator $\Leftarrow$ $\Delta(x) = \frac{d}{dt} \Phi_t(x)$

(on a dense domain)

$\boxed{\mathcal{E}(x,y) = \tau(L(x,y)) = \text{Dirichlet forme} \quad \tau \text{ trace on } M}$

$L(x,y) = \Delta(xy) - x\Delta(y) - \Delta(x)y$

CARRÉ DE CHAMP

$\left(L(x_i^*, x_j)\right)_{ij} \leq 0 \implies$ square root of laplacian theory Sauvageot, Cipriani, ..

Ex (Model Sauvageot, Arveson, Bhat..)

$N \subseteq M$ $\quad \boxed{N = pMp}$ $\quad E : M \to N$ $\boxed{E(x) = pxp}$

(Θ_t) semigroup of endomorphisms of M, $\Theta_t(p) \geq p$

$\Downarrow$

$\boxed{\Phi_t(x) = E_N(\Theta_t(x)) = p\,\Theta_t(x)\,p}$ $\quad$ [semigroup property]

$\Phi_t(\Phi_s(x)) = p\,\Theta_t(p\,\Theta_s(x)\,p)\,p = p\Theta_t(p)\Theta_{s+t}(x)p\Theta_t(p)$

②. Two variables (Chapman - Kolmogorov relations)

$(\mathcal{A}_s)$ family of v. N.a. , $(\Phi_{st})_{s \geq t}$ c.P maps

with ~~...~~ traces τ

$$\begin{tikzcd} \mathcal{A}_t \arrow[r, "\Phi_{ts}"] \arrow[rr, bend right, "\Phi_{tr}"'] & \mathcal{A}_s \arrow[r, "\Phi_{sr}"] & \mathcal{A}_r \end{tikzcd}$$

Addl ass. : $(\forall)\, s \geq t$ $(\exists)$ $x \underset{ts}{*} y \in \mathcal{A}_t$

$(\forall)\, x, y \in \mathcal{A}_t$

s.t. $\boxed{\Phi_{st}(x)\Phi_{st}(y) = \Phi_{st}(x \underset{ts}{*} y)}$ $\in \mathcal{A}_t$

Analogue of L : $\boxed{c_t(x,y) = \frac{d}{ds}\Big|_{s=t} x \underset{ts}{*} y}$ $\in \mathcal{A}_t$

Still $\boxed{\left(c_t(x_i^*, x_j)\right)_{ij} \geq 0}$ (coboundary condition)

assoc $\Rightarrow$ $\boxed{x c_t(y,z) - c_t(xy,z) + c_t(x,yz) - c_t(x,y)z = 0}$

$\varepsilon_t(x,y) =$

$\tau(c_t(x,y))$ not suff. to characterize $(\Phi_{ts})_{t,s}$

If $\boxed{\varepsilon_t(x,y) = \tau(c_t(x,y)) = \langle x, Ly \rangle}$,

$\boxed{c'_t(x,y) = c_t(x,y) - (L(xy) - xL(y) - L(x)y)}$ $\in \mathcal{A}_t$

$\Rightarrow$ $\Psi(x,y,z) = \tau(c'_t(x,y)z)$ cyclic [illegible] May be non-trivial

(3) Examples (Quantization of $\mathbb{H}$, $\mathbb{H}/\Gamma$ Berezin)

$\mathbb{H} = \{ z \mid \Im z > 0 \}$

$PSL_2(\mathbb{R})$ acts on $\mathbb{H}$ via Möbius transformation

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}}$$

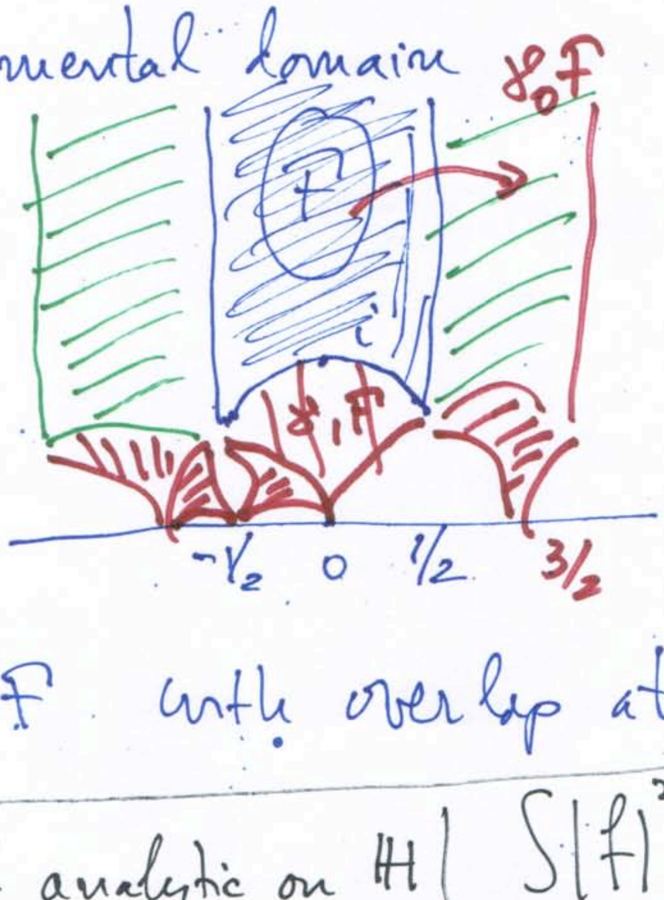
$\Gamma = PSL_2(\mathbb{Z})$

F = fundamental domain

$\gamma_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$V_0 = (\Im z)^{-2} dz\, d\bar{z}$ invariant measure

$$\boxed{\Delta = (\Im z)^{-2} \frac{\partial \partial}{\partial z \partial \bar{z}}}$$ invariant Laplacian



$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\mathbb{H} = \bigcup_{\gamma \in \Gamma} \gamma F$ with overlap at boundaries

$$\boxed{H_n = \{ f \text{ analytic on } \mathbb{H} \mid \int |f|^2 \underbrace{(\Im z)^{n-2} dz\, d\bar{z}}_{d\nu_n} < \infty \}}$$

$\pi_n : PSL_2(\mathbb{R}) \to \mathcal{U}(H_n)$ Discrete series of $PSL_2(\mathbb{R})$

$f \in H_n$; $\boxed{(\pi_n(g) f)(z) = f(g^{-1} z)\, (j(g,z))^{-n}}$

$\boxed{(cz+d)^{-2n}}$

$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

④ Berezin symbol

$A \in B(H_n) \rightsquigarrow \boxed{\tilde{k}(z,\bar{\eta})}$ reproducing kernel

$\Uparrow$ Not square summable

normalized symbol $\to \dfrac{k(z,\bar{\eta})}{(\bar{z}-\eta)^n}$

$$\boxed{(Af)(z) = \int_{\mathbb{H}} \frac{k(z,\bar{\eta})}{(\bar{z}-\eta)^n} f(\eta)\, d\nu_n(\eta)}$$

$\rightsquigarrow k(\bar{z},z)$ function on $\mathbb{H}$.

Ex. $\pi(g)A\pi(g)^{-1}$ has normalized symbol $k(gz,\overline{g\eta})$

$g \in PSL_2(\mathbb{R})$

In particular let $\boxed{\mathcal{A}_n = \{\pi_n(\Gamma)\}' \subseteq B(H_n)}$

Then (Jones) $\mathcal{A}_n$ are II_1 factors

$$\boxed{\dim_{\{\pi_n(\Gamma)\}''} H_n = \frac{n-1}{12}}$$

$A \in \mathcal{A}_n$ its normalized symbol k has property

$\boxed{k(\gamma z, \overline{\gamma\eta}) = k(z,\bar{\eta})}$ $(\forall)\ \gamma \in \Gamma$

$$\boxed{\tau_{\mathcal{A}_n}(k) = \frac{1}{\text{const}} \int_F k(z,\bar{z})\, d\nu_0(z)}$$

(5) $\Gamma = PSL_2(\mathbb{Z})$, $H^2(PSL_2(\mathbb{Z}), \mathbb{R}) = 0$

So one can avoid the logarithmic coboundaries when you do $\log(cz+d)$ and hence the construction can be extended to $n \in (1, \infty)$

$$L^2(\mathcal{A}_t, \tau) \cong \{ f \in L^2(F) \mid \langle f, B_t(\Delta) f \rangle_{L^2(F, \nu_0)} < \infty \}$$

$$\boxed{\|A\|^2_{2,\mathcal{A}_t} = \|B_t(\Delta)^{1/2} f\|^2_{L^2(F,\nu_0)}}$$

$\tau(A^*A)$

where $f(z) = k(z, \bar{z})$, k = symbol of A

$B_t(\Delta)$ some positive fct of invariant Laplacian Δ

~~A, B have csy~~

What are $\mathcal{A}_t \xrightarrow{\Phi_{t,s}} \mathcal{A}_s$

$\{\pi_t(\Gamma)\}'$

$\Phi_{t,s}$ = symbol map

$\boxed{\Phi_{t,s}(A) = \text{operator in } \mathcal{A}_s \text{ that has same symbol as } A}$

$A \to$ symbol k

$B \to$ symbol l

~~$k *_t l$ = symbol of product AB~~ is

$$z, \xi \to \int_{\mathbb{H}} k(z, \bar{\eta}) \, l(\eta, \bar{\xi}) \, d\nu_t(\eta)$$

(6)

symbol of $AB \in \mathcal{A}_t$

four point function(s)

$$\boxed{(k *_t l)(z,\bar{\xi}) = \mathrm{cst}_t \int_{\mathbb{H}} k(z,\bar{\eta})\, l(\bar{\eta},\xi)\, [z,\eta,\bar{\eta},\xi]^t \, d\nu_0(\eta)}$$

$$\boxed{C_t(k,l)(z,\bar{\xi}) = \mathrm{cst} \int_{\mathbb{H}} k(z,\bar{\eta})\, l(\bar{\eta},\xi)\, [z,\eta,\bar{\eta},\xi]^t \ln(\ ) \, d\nu_0(\eta)}$$

$\longrightarrow (6\frac{1}{2})$

~~...~~ If $g_1 \dots g_n \in PSL_2(\mathbb{R})$

~~Fb~~

then the CP map on $B(H_t)$

$$\boxed{\Psi(A) = \sum \pi(g_i)\, A\, \pi(g_i)^*}, \ (\text{if } H \text{ invariates } \mathcal{A}_t)$$

at the lebel of the identification

of $L^2(\mathcal{A}_t)$ with $(L^2(F), B_t(\Delta))$

is $$\boxed{f(z) \to \sum_{i=1}^{n} f(g_i^{-1} z)}$$

$\mathcal{A}_t$

For which $g_1, \dots g_n$ is $\Psi(A) \in \{\pi_t(\Gamma)\}'$

for all $A \in \{\pi_t(\Gamma)\}'$

Hecke operators

$\underline{6\tfrac{1}{2}}$

$$\Psi_t(k,l,m)$$

$$\tau\left(c^0_t(k,l)\,m\right) = \cot_t \tau(klm) + \chi(kl, m) - \chi(k, lm) + \chi(mk, l)$$

for k, l, m in a non unital <u>dense subalgebra</u> of $\mathcal{A}_t$

χ antisymmetric form, corresponding to antisymmetric op (unbounded) with non equal def indices

(7) p a prime $\sigma_p = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \in PGL_2(\mathbb{Q}) \supseteq PSL_2(\mathbb{Z})$ $\Leftarrow$ $\boxed{\text{determinant } p}$

If $\boxed{\Gamma \sigma_p \Gamma = \bigcup_{i=1}^{n} \Gamma \underbrace{\sigma_p s_i}_{g_i}}$ (s_i are reps for $\Gamma / \Gamma \cap \sigma_p^{-1} \Gamma \sigma_p$)
(right cosets)

then the corresponding Ψ is intertwining $\{\pi_t(\Gamma)\}'$

$$\boxed{\Psi_{\sigma_p} : \mathcal{A}_t \to \mathcal{A}_t \text{ is a CP map}}$$

Can be done for all n instead of p

$$\boxed{\Psi_{\sigma_n} \Psi_{\sigma_m} = \sum_{\substack{k = n-m \\ k \equiv n-m \ (\mathrm{mod}\ 2)}}^{n+m} c^{k}_{nm} \Psi_{\sigma_k}}$$

Hecke relations

$$\boxed{\left(\sigma_p s_i \right) = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & d \\ 0 & p \end{pmatrix}} \quad d = 0, \dots, p-1$$

Hecke operator at level of symbols is

$$\boxed{f(z) \xrightarrow{T_p} f(pz) + \sum_{d=0}^{p-1} f\left(\frac{z+d}{p} \right)} \qquad T_p : L^2(F) \to L^2(F)$$

- T_p commutes with Δ

8 Ramanujan Petersson

$$\boxed{\sigma_{disc}\left(T_p\Big|_{L^2(F)\ominus\mathbb{C}1}\right) \subseteq [-2\sqrt{p}, 2\sqrt{p}]}$$

$T_p : L^2(F,\nu_0) \to L^2(F,\nu_0)$

Because of $[T_p, \Delta] = 0$ we can also use the Hilbert space norm $\langle f, f\rangle_t = \langle B_t(\Delta) f, f\rangle$.

so this is equivalent to study

$$\boxed{\sigma_{disc}\left(\Psi_p\Big|_{L^2(\mathcal{A}_t)\ominus\mathbb{C}1}\right) \subseteq [-2\sqrt{p}, 2\sqrt{p}]}$$

From here one $t = 13$, so $\frac{t-1}{12} = 1$ so $\mathcal{A}_t \cong L(\Gamma)$

Observation. Let $\boxed{k = \frac{p-1}{2}}$. Then in F_k free group with k gen.

the T_{p^n} have the same multiplication relation as $\boxed{\chi_n = \sum_{|w|=n} w} \in \mathbb{C}(F_k)$

and Pytlik has shown that $\sigma_{C^*_{red}(F_k)}(\chi_1) \subseteq [-2\sqrt{p}, 2\sqrt{p}]$

so R.P equivalent to

$C^*_{redid} \ni \boxed{\chi_{p^n} \to T_{\sigma_{p^n}}}$ extends to a C^*-algebra isomorphism

$\subseteq$ abelian v. Hecke

(9)

$H_{\Gamma} = \ell^2(\Gamma)$

Automorphic forms of degree n = M

= "fixed points" of $\pi_n(\Gamma)$ acting on H_n (acting H_n)

$f(\gamma^{-1}z) = f(z)(cz+d)^{-2n}$

precisely $\boxed{\{ f \text{ analitic on } H \mid \underbrace{\pi(\gamma) f = f}, \gamma \in \Gamma \quad \int_F |f(z)|^2 d\nu_n < \infty \}}$

Finite dimi space of dimension $\left[\frac{n-1}{12}\right]$ or $\left[\frac{n-1}{12}\right]+1$

and $\boxed{\widetilde{T}_p f = \sum_{i=1}^{n} \pi_n(\sigma_p s_i) f}$

~~R.P.~~

Th (Deligne). R.P. holds for $\widetilde{T}_p$

Ideea now. Find a dilation theory for the semigroup like algebra

$$\boxed{\Psi_n \cdot \Psi_m = \sum_{k=n-m}^{n+m} c_{nm}^{k} \Psi_k}$$

10. Vershik quoting Krein

"When I see a positive definite fct I jump on like a ~~war~~ horse"

Generalized Ramanujan-Petersson and Hecke-algebras.

$\Gamma \subseteq G$, Γ, G discrete i.c.c., Γ almost normal that is $\boxed{\Gamma_\sigma = \Gamma \cap \sigma\Gamma\sigma^{-1} \subseteq \Gamma}$ finite index has finite index for all $\sigma \in G$. (Example $\boxed{\Gamma = PSL_2(\mathbb{Z}) \subseteq PSL_2(\mathbb{Q}) = G}$)

Fix $\boxed{\Gamma\sigma\Gamma = \bigcup_{i=1}^{n} \Gamma\sigma s_i}$, for $\sigma \in G$.

$\sigma\Gamma_{\sigma^{-1}} = \Gamma_\sigma \sigma$

Assume • tinobularity, that is

for all $\boxed{\sigma_1, \sigma_2 \in G \; (\exists)\sigma_3 \text{ s.t. } \Gamma_{\sigma_1} \cap \Gamma_{\sigma_2} \supseteq \Gamma_{\sigma_3}}$

• $\Gamma\sigma\Gamma = \Gamma\sigma^{-1}\Gamma$ ($\Rightarrow$ commutativity of Hecke algebra)

11 Definition ~~Assumption~~ $\mathcal{V} \supseteq H$, V vector space

H - Hilbert space, π a representation of G on V

that acts unitarily on H

let $\boxed{V^{\Gamma_\sigma} = \{v \in V \mid v \text{ is fixed by } \Gamma_\sigma\}}$

and assume there are $\underline{H^{\Gamma_\sigma}} \subseteq V^{\Gamma_\sigma}$ (with a Hilbert structure different from H, but compatible with $\underline{H^{\Gamma_{\sigma_1}} \subseteq H^{\Gamma_{\sigma_2}}}$ if $\Gamma_{\sigma_1} \supseteq \Gamma_{\sigma_2}$)

Generalized Hecke operator

$\Gamma_\sigma = \Gamma \cap \sigma\Gamma\sigma^{-1}$

$\Gamma_{\sigma^{-1}} = \Gamma \cap \sigma^{-1}\Gamma\sigma$

$$\begin{tikzcd}
H^{\Gamma_{\sigma^{-1}}} \arrow[r, "\pi_\sigma"] & H^{\Gamma_\sigma} \arrow[dl, "\mathrm{proj}"] \\
H^{\Gamma} \arrow[u, "\mathrm{ind}"] &
\end{tikzcd}$$

$$\mathrm{proj}\, v = \sum_{i=1}^{n} s_i v$$

reps for Γ_σ in Γ

$$\boxed{T_\sigma(v) = \mathrm{Proj}^{H^{\Gamma_\sigma}}_{H^{\Gamma}}(\sigma v) = \sum_i \sigma s_i v}$$

↑ depends only on $[\Gamma\sigma\Gamma]$

(12) Examples

1/ $\boxed{\mathcal{V} = \text{fcts on } G}$, $\boxed{H = \ell^2(G)}$

$\boxed{\mathcal{V}^\Gamma = \text{fcts on } G/\Gamma}$, $\boxed{H^\Gamma = \ell^2(G/\Gamma)}$

(H^{Γ_σ} is then $\ell^2(G/\Gamma_\sigma)$ normalized so that $\boxed{\ell^2(G/\Gamma) \subseteq \ell^2(G/\Gamma_\sigma) \text{ is isometry}}$:

possible since $[\Gamma:\Gamma_\sigma] = [\Gamma:\Gamma_{\sigma^{-1}}]$))

Then $$\boxed{T_{[\Gamma\sigma_1\Gamma]} T_{[\Gamma\sigma_2\Gamma]} = \sum_{\sigma_3}{}' \, n_{\sigma_1,\sigma_2}^{\sigma_3} T_{[\Gamma\sigma_3\Gamma]}}$$

$$\boxed{[\Gamma\sigma_3\Gamma] \subseteq [\Gamma\sigma_1\Gamma][\Gamma\sigma_2\Gamma]}$$

$$\boxed{T_{\Gamma\sigma\Gamma}[\Gamma\sigma_1] = \sum_{\Gamma\sigma_2 \subseteq \Gamma\sigma\Gamma\sigma_1}{}' [\Gamma\sigma_2]}$$

We get an algebra structure on $\boxed{\mathcal{H}_0 = \mathrm{Sp}\{[\Gamma\sigma\Gamma] \mid \sigma \in G\}}$ together with a representation of $\mathcal{H}_0$ on $\ell^2(G/\Gamma)$

$$\boxed{\mathcal{H} = \overline{\mathcal{H}_0}^{\|\ \|} = \text{reduced } C^*\text{-Hecke algebra}}$$

Vector $[\Gamma]$ gives a trace vector on $\mathcal{H}$

(because of $[\Gamma:\Gamma_\sigma] = [\Gamma:\Gamma_{\sigma^{-1}}]$

Generalized Ramanujan-Petersson.

Assume $\mathcal{H}$ is abelian:

Given $\mathcal{V}, H$, and $\mathcal{V}^\Gamma, H^\Gamma$

~~2023~~ Generalized R.P. If ξ in H^Γ is a joint eigenvector for all $T_{[\Gamma\sigma\Gamma]}$, that is $\boxed{[T_{\Gamma\sigma\Gamma}]\,\xi = \lambda_{[\Gamma\sigma\Gamma]}\,\xi}$, then $\boxed{[\Gamma\sigma\Gamma] \to \lambda_{[\Gamma\sigma\Gamma]}}$ should extend to a character of $\mathcal{H}$

representation of $GL_2(\mathbb{Q})$ on $\bigcup H^{\Gamma_i}$ is ~~[illegible]~~ weakly contained in regular rep

Why is this a generalization of R.P. $\boxed{\Gamma = PSL_2(\mathbb{Z}),\ G = PGL_2(\mathbb{Q})}$

Ex 2. $\boxed{\mathcal{V} = \text{functions on } \mathbb{H} \text{ measurable}}$; $\boxed{H = L^2(\mathbb{H}, \nu_0)}$

$H^\Gamma = L^2(F, \nu_0)$

$\boxed{\mathcal{V}^\Gamma = \Gamma \text{ invariant functions}}$ $\boxed{H^\Gamma = L^2(F, \nu_0)}$

Since reduced C^* Hecke algebra of Hecke operators corresponding to p^n is isomorphic for fixed p to radial algebra $L^1_{rad}(F_{p+1})$, as we proved before an eigenvector for T_{σ_p} is in $[-2\sqrt{p}, 2\sqrt{p}] \Leftrightarrow$ the character given by eigenvector extends to radial

$14^{1/2}$. Main theorem

Calkin algebra

$$\boxed{[\Gamma\sigma\Gamma] \longrightarrow \left[T_{[\Gamma\sigma\Gamma]}\right]_{Q(\ell^2(F,\nu_0))}}$$

$\widehat{B_{13}(\Delta)}^{1/2}$

~~xxxx~~ has a continuous extension to

the reduced C^*- Hecke algebra.

15 Ex3 $\boxed{\mathcal{V} = ~~xxx~~ \text{ analytic fcts on } \mathbb{H}}$

$\boxed{\pi = \pi_n \text{ on } H_n \text{ coming}}$

~~xxx~~ from discrete series rep

Then $\boxed{H_n^\Gamma = \text{ automorphic forms}}$

G.R.P $\Rightarrow$ Deligne theorem.

Ex4 II_1 factors Ramanujan Petersson

Assumption $\Gamma \subseteq G$ as above (with $[\Gamma:\Gamma_\sigma] = [\Gamma:\Gamma_{\sigma^{-1}}]$

and $\mathcal{H}$ abelian. Assume $\boxed{(\exists)\ \pi_0 : G \to \mathcal{U}(\ell^2(\Gamma))}$

a unitary representation extending left regular representation λ_Γ on $\ell^2(\Gamma)$ (e.g. π_{13} for $\Gamma = PSL_2(\mathbb{Z})$ by Jones theorem /

Let $\boxed{\mathcal{V} = B(\ell^2(\Gamma))}$ $\boxed{\pi(g) = Ad(\pi_0(g))}$

$\boxed{\mathcal{V}^\Gamma = (\mathcal{L}(\Gamma))'}$ $\boxed{H = C_2(\ell^2(\Gamma))}$

$\boxed{\mathcal{V}^{\Gamma_\sigma} = (\mathcal{L}(\Gamma_\sigma))' \subseteq B(\ell^2(\Gamma))}$ $\boxed{H^{\Gamma_\sigma} = L^2_{\sigma}((\mathcal{L}(\Gamma_\sigma))')}$

Note that $\boxed{\cup \mathcal{V}^{\Gamma_\sigma}}$ is a II_1 factor (not acting on $\ell^2(\Gamma)$)

$\boxed{\Psi_\sigma(x) = \sum_i \pi(\sigma s_i)\, x\, \pi(\sigma s_i)^*}$, $x \in \mathcal{L}(\Gamma)'$

(16) **Prop.** For $\Gamma = PSL_2(\mathbb{Z}) \subseteq G = GL_2(\mathbb{Q})$

G.R.P ex 4 $\Leftrightarrow$ R.P. in ex 2.

Proof. Via quantization map

Data $\Psi_{[\Gamma\sigma\Gamma]}$ completely positive maps on $\underline{R(\Gamma) \cong \mathcal{L}(\Gamma)}$

s.t. $$\boxed{\Psi_{[\Gamma\sigma_1\Gamma]}\Psi_{[\Gamma\sigma_2\Gamma]} = \sum \Gamma^{\sigma_3}_{\sigma_1\sigma_2}\Psi_{[\Gamma\sigma_3\Gamma]}}$$

structure constants from Hecke algebra

Think "a dilation":

Theorem. Given $\Gamma \subseteq G$ almost normal and such that $(\exists)\ \pi : G \to \mathcal{M}(\ell^2(\Gamma))$ extending left regular representation ~~w~~ there exists $\boxed{\rho : \mathcal{H} \to \mathcal{L}(G)}$ an ~~isomorphism~~ embedding unital and $\boxed{\text{trace preserving}}$

s.t. $\boxed{\rho([\Gamma\sigma\Gamma]) \in \ell^2(\Gamma\sigma\Gamma) \cap C^*_{red}(G)}$

such that

$$\boxed{\Psi_{\Gamma\sigma\Gamma}(x) = E^{\mathcal{L}(G)}_{\mathcal{L}(\Gamma)}\left(\rho(\Gamma\sigma\Gamma)\, x\, \rho(\Gamma\sigma\Gamma)\right)}$$

for all x in $\mathcal{L}(\Gamma)$

(Here we identified $\mathcal{L}(\Gamma)$ with $\mathcal{R}(\Gamma)$)

Corollary 1) If $(\lambda_{\Gamma\sigma\Gamma})$ is an eigenvalue then $\boxed{[\Gamma\sigma\Gamma] \to \lambda_{\Gamma\sigma\Gamma}[\Gamma\sigma\Gamma]}$ is CP on $\mathcal{H}$.

2) $\boxed{[\Gamma\sigma\Gamma] \to \frac{1}{[\Gamma:\Gamma_\sigma]}[\Gamma\sigma\Gamma] \otimes [\Gamma\sigma\Gamma]}$ is CP from $\mathcal{H}$ into $\mathcal{H} \otimes \mathcal{H}$.

Z-Algebra
VERSHIK

17 1/2

Construction of ϱ.

Fix ξ in $\mathcal{H}_{13}$ a cyclic trace vector

then $\varrho(\Gamma\sigma) = \sum_i \langle \pi(\gamma_i^\sigma)\xi, \xi \rangle \gamma_i^\sigma$
(matrix coef of G corresponding to ξ

Essential property: $\varrho(\sigma_1 \Gamma)\,\varrho(\Gamma\sigma_2) = \varrho(\sigma_1 \Gamma \sigma_2)$ $*$

$$\varrho(\Gamma\sigma\Gamma) = \sum_i \varrho(\Gamma\sigma s_i)$$

and $(*) \Rightarrow \varrho(\Gamma\sigma\Gamma)\,\varrho(\Gamma\sigma_1) = \sum_{\Gamma\sigma_2 \subseteq \Gamma\sigma\Gamma\sigma_1} \varrho(\Gamma\sigma_2)$

(18) Case $\Gamma = PSL_2(\mathbb{Z})$ $\quad G = GL_2(\mathbb{Q})$

Recall Akemann Ostrand that

$$(\exists)\ \pi : \boxed{C^*\left(C^*_{\lambda,red}(\Gamma),\ C^*_{\rho,red}(\Gamma)\right)/K(\ell^2(\Gamma))}$$

$$\cong \boxed{C^*_{red}(\Gamma \times \Gamma^{op})}$$

Enlarge this to

$$\boxed{P^{\ell^2(G)}_{\ell^2(\Gamma)}\, C^*\left(C^*_{\lambda,red}(G),\ C^*_{\rho,red}(G)\right)\, P^{\ell^2(G)}_{\ell^2(\Gamma)}}$$

Then relative AO for $\Gamma \subseteq G$ is that there exists an $\underbrace{\text{isomorphism}}_{\text{(modulo the compacts)}}$ from this algebra onto $e_\Gamma\left(C^*_{red}((G \times G) \times X_\Gamma)\right)e_\Gamma$ where $\boxed{X_\Gamma}$ is the space of characteristic of cosets $\boxed{\Gamma_\sigma g}$ with $\tau(\chi_{\Gamma_\sigma g}) = \dfrac{1}{[\Gamma : \Gamma_\sigma]}$

$18\frac{1}{2}$

$\overline{C^*_{red}((G\times G)\rtimes \Gamma)}^{w}$ is the

quantum double algebra from subfactor

theory $\mathcal{L}(G)\underset{e_\Gamma}{\boxtimes}\mathcal{R}(G)$ (Popa, Longo ...)

associated to $\mathcal{L}(\Gamma)\subseteq \mathcal{L}(G)$ of infinite

index

(19) By Ozawa this equivalent to the fact that $G \times G$ acting as a groupoid on $\partial^\beta \Gamma$ (Stone Čech boundary) ($\gamma \in \mathrm{hom}(g_1, g_2)$ if $g_1 \gamma g_2 \in \Gamma$) is ameanable

(AO $\Leftrightarrow$ $\Gamma \times \Gamma$ acts ameanably on $\partial^\beta \Gamma$)

Thus modulo the compacts,

$$\boxed{[\Gamma \sigma \Gamma] \longrightarrow [\Psi_\sigma]_{Q(\ell^2(\Gamma_\sigma))}}$$

is a $\overbrace{\cancel{\text{map for}}}^{\text{trace}}$ preserving map from $\mathcal{H}$ into the II_1 factor

$$\boxed{e_\Gamma\,(C^*_{red}(G \times G, X_\Gamma))\,e_\Gamma}$$

Corollary. $[\Gamma_0\Gamma] \to [\Psi_\sigma]_{Q(\ell^2(\Gamma))}$

extends to a continuous representation of the Hecke algebra.

Hence

Theorem. The R.P conjectures hold true, with the exception of a finite number of eigenvalues.

Question What is the role
$\Phi_{ts}\Phi_{ts}^* = B_{t,s}(\Delta)$. Is there
a similar dilation theory: Can one
estimate discrete spectrum of Δ bellow $\frac{1}{4}$?

Γ_σ