A theory of induction and classification of tensor C^*–categories

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Introduction

In the early 70s, Doplicher, Haag and Roberts introduced tensor C^*–categories in QFT.

If we have a group G, intertwiners suffice to reconstruct it we take into account the natural embedding

$$\text{Rep}(G) \to \text{vector spaces}$$

(Tannaka duality)

The categories arising from AQFT are main examples of abstract categories, in the sense that they do not have such an a priori associated embedding.

• Motivated by 4 dim QFT, Doplicher and Roberts studied the structure of abstract tensor C^*–categories with conjugation and permutation symmetry. They showed in 1989 how to construct

$$\mathcal{T} \to \text{Hilbert spaces},$$

making the category $\mathcal{T}$ equivalent to the representation category of a unique compact group.

• An analogous result by Deligne in an algebraic framework is also well known (1990).

A main question is that of understanding the structure of abstract tensor C^*–categories from low dim QFT, where conjugation and unitary braided symmetry are present.

Another problem is that of characterizing the tensor C^*–categories corresponding to the compact quantum groups of Woronowicz (1987). This amounts again to decide whether there is an embedding

$$\mathcal{T} \to \text{Hilbert spaces}$$

by Woronowicz duality

- In 1983 Jones initiated the theory of subfactors.

- Longo showed a connection between Jones theory and conjugation in 1989.

- There are classification results for categories with rigidity and prescribed fusion rules. Any sl_d–category corresponds to a twist of Rep$(U_q(sl_d))$ (Kazhdan–Wenzl, 93).

- Motivated by Jones theory, Longo and Roberts in 1997 studied abstract categories with conjugation (no symmetries)

- There are $G_1 \neq G_2$ finite groups with Rep(G_1) = Rep(G_2) as abstract categories, hence permutation symmetry is essential for uniqueness (Izumi–Kosaki, 02).

Tensor C^*–categories with conjugation

We assume that for any object ρ there is a conjugate $\overline{\rho}$. If ρ is irreducible, $\overline{\rho}$ is characterized uniquely by $\iota < \rho\overline{\rho}$

We always assume

$$(\iota, \iota) = \mathbb{C}$$

Conjugation implies (Longo–Roberts)

- existence of a real–valued dimension function on objects $\rho \to d(\rho)$,

- $\dim(\rho, \sigma) < \infty$

Subfactors

$N \subset M$, II_1 subfactors with $[M : N] < \infty$ (Jones)

Ocneanu's bimodules:

$$X = {}_M M_N, \quad \overline{X} = {}_N M_M,$$

Tensor products

$$X \otimes_N \overline{X} \otimes_M \dots$$

$$\overline{X} \otimes_M X \otimes_N \dots$$

Together with bimodule maps between them, they form the standard invariant, a complete invariant under amenability assumptions (Popa).

For type III subfactors (Longo)

$$d(\rho)^2 = [M : \rho(M)]$$

For II_1 subfactors an analogous relation holds, conjugate equations are solved by module bases.

With a p.s., the embedding is unique. More generally, various abstract duality theorems showing
existence of an embedding are available, modeled around the fundamental property of the regular representation (Baaj–Skandalis multiplicative unitaries) or the related depth 2 property for subfactors (Ocneanu, Cuntz, Longo, Doplicher–P–Roberts).

In general, there are plenty of examples of non-embedable categories:

• A unitary braided symmetry

$$\mathbb{B}_r \to (\rho^r, \rho^r) \Rightarrow \text{amenability}$$

An amenable object ρ generates a non-embedable category if $d(\rho) \notin \mathbb{N}$ (Longo–Roberts).

• A similar result holds for the category arising from an amenable inclusion of II_1 factors $N \subset M$ in the sense of Popa.

(After I gave this talk, Leonid Vainerman pointed out his joint work with Nikshych, generalizing Ocneanu–Longo duality to depth 2 reducible II_1 inclusions with non-integral indices. In their paper, a finite quantum groupoid in the sense of finite weak Hopf algebras of Bohm and Szlachanyi is constructed, cf. Vainerman slides)

• In case of amenability, if an embedding exists (hence $[M : N] \in \mathbb{N}$) the dimension of the Hilbert space is uniquely determined and only compact quantum group of Kac-type appear. General results about existence for categories with infinitely many irreducibles?

If an embedding into the Hilbert spaces exists, it is not unique in general, and one has the problem of classifying them.

• Prime examples: Temperley–Lieb categories: the universal tensor C^*–categories generated by a single selfconjugate object ρ and a single arrow $R \in (\iota, \rho^2)$. These are classified by $\pm d$, with $d = 2\cos\pi/m$ or $d \geq 2$. They are embedable iff $d \geq 2$, embeddings may be classified and correspond to the quantum groups $A_o(F)$.

• In the non-selfconjugate case,

$$R \in (\iota, \overline{\rho}\rho), R \in (\iota, \rho\overline{\rho})$$

embeddings faithful on objects correspond to $A_u(F)$

(Jones, Woronowicz, Wenzl, Popa, Goodman–Wenzl, Wang, Banica, Yamagami, . . . , P–Roberts)

• A characterization of Rep($S_\mu U(d)$) as an abstract category and its simplicity is known (P).

• Classification of tensor C^*–categories with an object ρ and generated by two arrows $S \in (\iota, \rho^d)$ and $E \in (\rho^{d-1}, \rho^{d-1})$ making $\overline{\rho} < \rho^{d-1}$ (or their embeddings into the Hilbert spaces) is not known, I believe, already for $d = 3$. $S_\mu U(3)$ are the prime examples. The category is not expected to be unique.

• Any tensor C^*–category with conjugation is simple. However, this does not imply uniqueness (P–Roberts)

How to describe non-embedable categories with more arrows? We may hope in a complete answer for a category generated by a single object $\rho \simeq \overline{\rho}$ or two objects ρ, $\overline{\rho}$ conjugate of each other.

We wish to develop a general machine.

Our approach is to generalize the methods of DR reconstruction theorem to treat non-permutation symmetric categories with conjugates and infinitely many irreducbles. These methods are quite different from the methods based on the regular representation, and we shortly recall the main ideas.

The DR reconstruction theorem

Several reductions:

–$d(\rho) \in \mathbb{N}$,

–An inductive process and (automatic) amenability allow to reduce to the case where $\mathcal{T}$ is generated by a single endomorphism ρ of a unital C^*–algebra $\mathcal{A}$, with 'det$(\rho) = 1$'.

This implies existence of an embedding Rep$(SU(d)) \subset \mathcal{T}$.

–construction of a cross product C^*–algebra

$$\mathcal{A} \otimes_{\mathsf{Rep}SU(d)} \mathcal{O}_d, \quad d = d(\rho)$$

with an $SU(d)$–action

$$\alpha(g) = \text{trivial}(g) \otimes \text{canonical}(g), \quad g \in SU(d).$$

$$C(\Omega) := Z(\mathcal{A} \otimes_{\mathsf{Rep}SU(d)} \mathcal{O}_d)$$

is $SU(d)$–ergodic, hence

$$\Omega = K \backslash SU(d)$$

with K an isotropy subgroup. With K one constructs

$$\mathcal{T} \to \mathsf{Rep}(K)$$

An alternative viewpoint

Pimsner introduced the universal C^*–algebra $\mathcal{O}_X$ of a Hilbert C^*–bimodule X, generalizing both $\mathcal{O}_d$ and $\mathcal{A} \rtimes_\alpha \mathbb{Z}$

In case of a single object of determinant 1 one can show that

$$\mathcal{A} \otimes_{\mathsf{Rep}SU(d)} \mathcal{O}_d = \mathcal{O}_X,$$

where

$X = \mathbb{C}^d \otimes C(K\backslash SU(d))$ with $SU(d)$–action:

$$\rho(g) := \text{fundamental}(g) \otimes \text{translation}(g)$$

Recall that given $K \subset G$, for $v \in \text{Rep}(K)$,

$$\text{Ind}(v) = \text{right translation}$$

on L^2–completion of the space of continuous functions

$$X_v := \{f : G \to H_v, f(kg) = v(k)f(g)\}$$

Frobenius reciprocity:

$$(u \upharpoonright_K, v) \simeq (u, \text{Ind}(v))$$

We may adopt a geometric viewpoint and regard Ind(v) as a representation on the induced $C(K\backslash G)$–bimodule X_v. Then,

$$\text{Ind}(v) \otimes_{C(K\backslash G)} \text{Ind}(v') = \text{Ind}(v \otimes v')$$

It follows that

$$\text{Ind} : \text{Rep}(K) \to \text{Bimod}(G)$$

is a faithful tensor *–functor with full image.

For the DR bimodule,

$$X \simeq \text{Ind}(\text{fundamental} \restriction_K)$$

In fact, DR embedding is constructed as

$$DR : \mathcal{T} \to \text{Bimod}(SU(d)) \stackrel{\text{Ind}^{-1}}{\to} \text{Rep}(K).$$

$$\rho \to X \to \text{fundamental} \restriction_K$$

We regard the first arrow as abstract induction

We now drop permutation symmetry. The framework becomes noncommutative.

If the category is not embedable, we can not find any quantum subgroup K and hence the concrete induction functor Ind.

This part of DR reconstruction for non-symmetric categories then amounts to construct abstract induction

$$\mathcal{T} \to \text{Bimod}(G)$$

with G a cqg replacing $SU(d)$, that should be produced intrinsically by the category. G will not be unique.

Analogous situation in measurable ergodic theory. Mackey: an ergodic action of a group G on a commutative von Neumann algebra $L^\infty(X, \omega)$ should be regarded as a virtual subgroup, and, as such, we may talk about induction (Ramsay).

Hence, we should be looking for G and also, following Mackey, for an ergodic G–action, now in a topological noncommutative setting.

This viewpoint may be regarded analogous to Connes–Takesaki flow of weights Mod(M) of a type III von Neumann algebra, where an $\mathbb{R}^*_+$–ergodic action on a commutative von Neumann algebra is intrinsically produced by M.

In our setting, both the space and the acting group will have noncommutative structures.

Strategy

• Often, embedable categories appear as subcategories of non-embedable ones $\mathcal{T}$ (e.g. TL category in the category of Ocneanu's bimodules in subfactor theory)

• To facilitate applications, we start with

$$\mathcal{A} \xrightarrow{\mu} \mathcal{T},$$

we regard $\mathcal{A}$ as a universal version of the embedable subcategory and $\mathcal{T}$ as a building block. Fix an embedding of $\mathcal{A}$,

$$\mathcal{A} \xrightarrow{\tau} \text{Hilbert spaces}$$

defining a compact quantum group G by Woronowicz duality.

We thus have a pair of functors

$$\text{Rep}(G) \xleftarrow{\tau} \mathcal{A} \xrightarrow{\mu} \mathcal{T}$$

and look for

• $\mathcal{C}$ (G–ergodic algebra) replacing $C(K\backslash SU(d))$ and depending on μ and τ,

• induced G–bimodules $\text{Ind}(\rho)$ replacing X_{DR}, for objects in the image of μ

Main restriction:

$$d(\rho) \geq 2$$

For $d(\rho) < 2$, one would have to deal with quantum groups at roots of unity, by Jones fundamental result and Wenzl's work.

Ergodic actions

$G = (\mathcal{Q}, \Delta)$ cqg,

$$\alpha : \mathcal{C} \to \mathcal{C} \otimes \mathcal{Q}$$

an ergodic G–action,

$$\mathcal{C}^\alpha := \{c \in \mathcal{C} : \alpha(c) = c \otimes I\} = \mathbb{C}.$$

on a unital C^*–algebra.

Spectral space of $u \in \mathrm{Rep}(G)$:

$$L_u := (H_u \otimes \mathcal{C})^{u \otimes \alpha} =$$

$$\{(c_i) : c_i \in \mathcal{C}, \alpha(c_i) = \sum_k c_k \otimes u^*_{k,i}\}.$$

Their entries (c_i) span a dense *–subalgebra $\mathcal{C}_{\text{spectral}}$ (Podles).

Example K subgroup of G, $\mathcal{C} = C(K \backslash G)$, $\alpha = G$–translation then

$$L_u \simeq \{K - \text{fixed vectors in } H_u\}$$

via

$$k \to (< u(g)\psi_i, k >)$$

Ergodicity implies

- Existence of a unique G–invariant state,

- L_u are Hilbert spaces:

$$< c, d > := \sum c_i^* d_i$$

Spectrum and multiplicities

$$\text{spec}(\alpha) := \{u \in \text{Rep}(G), \text{irreducible} : L_u \neq 0\}$$

$$\text{mult}(u) := \dim(L_u)$$

This is not enough to reconstruct the ergodic C^*–action: an example of Todd (1950) gives two non-conjugate subgroups K_1, K_2 of a finite group G with $K_1\backslash G$ and $K_2\backslash G$ isospectral. (Mackey, 1964)

If G is a compact group:

- the invariant state is a trace and

$$\text{mult}(u) \leq \dim(u) \quad \text{(H–K L S)}$$

• H–K, L, S problem: does any simple compact group act ergodically on $\mathcal{R}$?

• Jones problem: does any classical compact Lie group act ergodically on $\mathcal{R}$?

• $SU(2)$ does not. It acts ergodically only on type I von Neumann algebras (Wassermann). Wasserman's invariant (multiplicity maps) is stronger than the spectrum

• A complete answer for $SU(3)$ is not known

If G is a cqg:

• Haar state is not tracial (Woronowicz), there exist examples of $A_u(n)$ on $\mathcal{R}$ and of $A_u(F)$ type III factors (Wang). Since first examples (Podles quantum spheres), one starts from the spectral spaces

• mult$(u) \leq q-$ dim(u), generalizing inequality involving quantum and integral dimension for a cqg (Boca)

• Tomatsu has classified certain ergodic actions of $S_\mu U(2)$ embedable into the translation action

• Bichon–De Rijdt–Vaes have given examples of $S_\mu U(2)$ actions with

$$\text{mult}(u) > \text{dim}(u)$$

method:
apply Woronowicz' Krein reconstruction to the possible tensor embeddings

$$\text{Rep}(S_\mu U(2)) \to \text{Hilbert spaces}$$

to construct ergodic actions of $S_\mu U(2)$ with

$$L_{u\otimes v} \simeq L_u \otimes L_v.$$

If G is a group,

$$L_{u\otimes v} \simeq L_u \otimes L_v \quad \Leftrightarrow \quad \text{mult}(u) = \text{dim}(u) \forall u$$

For a general ergodic action,

$$(c_i) \in L_u, \quad (d_j) \in L_v \Rightarrow (d_j c_i) \in L_{u\otimes v},$$

We only have isometries

$$S_{u,v} : L_u \otimes L_v \to L_{u\otimes v}.$$

Dual object of an ergodic action

The dual object of an ergodic G–action $(\mathcal{C}, \alpha)$ is the pair (L, S), where L is regarded as a *–functor

$$L : \text{Rep}(G) \to \text{Hilbert spaces}$$

S as a natural transformation

$$S_{u,v} : L_u \otimes L_v \to L_{u\otimes v}$$

Although $u \to L_u$ is not tensor in general, there are coherent rules that govern the behaviour of L under tensor products. Rules later recognized analogous to Popa's commuting squares appearing in Jones index theory:

Proposition Given spectral spaces L_u, L_v, L_w of an ergodic action, the following diagram commutes

$$\begin{array}{ccc} L_u \otimes L_v \otimes L_w & \xrightarrow{1\otimes S} & L_u \otimes L_{v\otimes w} \\ {\scriptstyle S\otimes 1}\downarrow & & \downarrow{\scriptstyle S} \\ L_{u\otimes v} \otimes L_w & \xrightarrow{S} & L_{u\otimes v\otimes w} \end{array}$$

and it is a commuting square in the sense of Popa:

$$E^{L_{u\otimes v\otimes w}}_{L_{u\otimes v}\otimes L_w}(L_u \otimes L_{v\otimes w}) = L_u \otimes L_v \otimes L_w$$

An abstract pair (L, S) with

$$L : \mathcal{S} \to \mathcal{T}, \qquad S_{u,v} \in L_u \otimes L_v \to L_{u \otimes v}$$

satisfying the commuting square condition is a quasitensor functor

More examples

• $\mathcal{T}$ tensor C^*–category;

$$\rho \in \mathcal{T} \to (\iota, \rho) \in \text{Hilbert spaces}$$

is a quasitensor functor (the minimal one)

• In particular for a II_1 inclusion $N \subset M$, we get $M^{\otimes r} \to N' \cap M_{r-1}$.

Quasitensor functors unify Jones and Boca inequalities:

$$\dim N' \cap M_{r-1} \leq [M : N]^r, \quad \text{mult}(u) \leq q\text{–}\dim(u)$$

Duality Theorem

a) The spectral functor (L, S) of an ergodic action allows to reconstruct

($\mathcal{C}_{\text{spectral}}$, action, invariant state, maximal norm)

b) Any quasitensor functor (L, S) between

$$L : \text{Rep}(G) \to \text{Hilbert spaces}$$

is the spectral functor of an ergodic action of G on a unital C^*–algebra.

Remark: part b) generalizes BDV construction.

Corollary

The spectral functor (L, S) is a complete invariant for ergodic C^*–actions of compact quantum groups over amenable algebras:

$$\mathcal{C}_{\text{red}} = \mathcal{C}_{\text{max}}$$

Examples of amenability: classical compact transitive spaces, $S_\mu U(d)$ (Nagy),

of non amenability: $A_o(F)$, $n \geq 3$ (Skandalis) and $A_u(F)$ (Banica)

The ergodic algebra of a pair of functors

Theorem Given a tensor C^*–category $\mathcal{T}$, with

$$\text{Rep}(G) \xleftarrow{\tau} \mathcal{A} \xrightarrow{\mu} \mathcal{T}$$

there is a unital G–ergodic C^*–algebra ${}_\mu\mathcal{C}_\tau$

if τ is injective, we may compose:

$$\text{Rep}(G) \xrightarrow{\tau^{-1}} \mathcal{A} \xrightarrow{\mu} \mathcal{T} \xrightarrow{\text{minimal}} \text{Hilbert spaces}$$

and find a quasitensor functor.

However, injectivity of μ is not necessary. On arrows it is automatic: any tensor C^*–category with conjugation is simple (PR, 08)

G–Bimodules of cqg

The nc analogue of G–equivariant Hermitian bundles over compact spaces.

Let $G = (\mathcal{Q}, \Delta)$ be a compact quantum group acting on a unital C^*–algebra $\mathcal{C}$,

$$\alpha : \mathcal{C} \to \mathcal{C} \otimes \mathcal{Q}$$

and X a Hilbert bimodule over $\mathcal{C}$

A bimodule representation of G on X is a map

$$v : X_v \to X_v \otimes \mathcal{Q},$$

with X_v a Hilbert $\mathcal{C}$–bimodule, such that

$$v(xc) = v(x)\alpha(c), \quad v(cx) = \alpha(c)v(c)$$

$$< v(x), v(x') >= \alpha(< x, x' >)$$

$$v \otimes 1 \circ v = 1 \otimes \Delta \circ v$$

$$v(X_v)1 \otimes \mathcal{Q} \quad \text{dense in} \quad X_v \otimes \mathcal{Q}$$

Example

Given a C^*–action $(\mathcal{C}, \alpha)$ of G and $v \in \mathrm{Rep}(G)$, we may form $X := H_v \otimes \mathcal{C}$ with action

$$v \otimes \alpha \quad \text{replacing} \quad \mathrm{Ind}(v \restriction_K).$$

It is always a representation of G on the free right module $H_v \otimes \mathcal{C}$.

Due to noncommutativity, trivial left $\mathcal{C}$–action

$$c(\psi \otimes c') := \psi \otimes cc'$$

is not a good choice:

• If G is a group, $u \otimes \alpha$ is a bimodule representation. However,

$$(u \otimes \alpha, u' \otimes \alpha) \subset \mathcal{B}(H_u, H_{u'}) \otimes Z(\mathcal{C}).$$

• If G is a quantum group, $u \otimes \alpha$ is not even a bimodule representation.

Example (Quantum quotients) If K is a quantum subgroup of a compact quantum group G then for $v \in \text{Rep}(G)$,

$$\text{Ind}(v \restriction_K) \simeq H_v \otimes C(K \backslash G),$$

There is a good left module structure, non-trivial precisely in the noncommutative cases:

$$< \psi_i \otimes I, c\psi_j \otimes I >= \sum_h v^*_{hi} c v_{hj}.$$

It is not inner for $K \neq$ trivial group.

Example

For some ergodic actions there isn't any: $\mathcal{C} = M_3$, $G = SU(2)$

The induced bimodules

Theorem (P–Roberts, 09)

Given

$$\text{Rep}(G) \xleftarrow{\tau} \mathcal{A} \xrightarrow{\mu} \mathcal{T}$$

with μ surjective on objects, there are

• induced bimodule G–representations $\text{Ind}(\mu_u) \simeq u \otimes \alpha$ over ${}_\mu\mathcal{C}_\tau$ with a 'good' left action,

• a faithful Frobenius tensor *–functor with full image

$$\text{Ind} : \mathcal{T} \to \text{Bimod}(G)$$

The induced bimodule is constructed first algebraically, requiring validity of Frobenius reciprocity for the abstract 'restriction' functor μ:

$$\text{`}(\mu_u, v) \simeq (u, \mathrm{Ind}(v))\text{'}, \quad u \in \widehat{G}$$

This determines the spectral decomposition of $\mathrm{Ind}(v)$. We start with

$$\mathrm{Ind}(v) := \oplus_{u \in \widehat{G}} (\mu_u, v) \otimes \tau_u.$$

‘Good’ left actions

$$c\xi = \xi c, \qquad \xi \in (H_u \otimes \mathcal{C})^{u \otimes \alpha}, \quad c \in \mathcal{C}.$$

We call such bimodule structures full, as they give rise to a full functor:

$$u \otimes \alpha \in \mathsf{Bimod}(G) \to u \otimes \alpha \in \mathsf{Mod}(G)$$

$$(v \otimes \alpha, v' \otimes \alpha)_{\mathsf{Bimod}} = (v \otimes \alpha, v' \otimes \alpha)_{\mathsf{Mod}},$$

a property trivially satisfied in commutative case.

Corollary There is a one-to-one correspondence between isomorphism classes

$$[\text{Rep}(G) \to \mathcal{T}] \to [G, \mathcal{C}]$$

Hence different categories provide non-conjugate ergodic actions.

Which ergodic actions appear?

The ergodic actions from subfactors

If $N \subset M$ is a proper inclusion of II_1 subfactors with finite Jones index,

$$\overline{X} \otimes_M X = {}_N M_N$$

is a real object

$$R = \sum_i u_i \otimes u_i^*,$$

with (u_i) a module basis for M_N.

$$\|R\|^2 = [M : N].$$

Hence there is

$$\text{Rep}(A_o(F)) \xrightarrow{\mu} \text{Ocneanu's bimodules}$$

$$\mu : \sum \psi_i \otimes F\psi_i \to R$$

if

$$\text{Trace}(F^*F) = [M : N], \quad F\overline{F} = I.$$

Hence there is an $A_o(F)$–ergodic C^*–algebra $\mathcal{C}_{N\subset M}$.

The spectral spaces are

$$L_{u^{\otimes r}} = N' \cap M_{r-1}.$$

Generators and relations of $\mathcal{C}_{N\subset M}$:

$$\overline{T}\otimes\xi,\quad T\in N'\cap M_{r-1},\xi\in H^r,$$

$r=0,1,2,\dots,$ and relations,

a) $(\overline{T}\otimes\xi)(\overline{T'}\otimes\xi')=\overline{Tp_{r,s}T'}\otimes\xi\xi'$,

b) $(\overline{T}\otimes\xi_1\dots\xi_r)^*=\overline{T^*}\otimes j\xi_r\dots j\xi_1$,

for $r\geq s$:

c) $\overline{S}\otimes(1_{u^r}\otimes R_u^*\otimes 1_{u^s}\eta)=\lambda\overline{Sp^{(2s)}_{r-s,2}}\otimes\eta$,

c') $\overline{S'}\otimes(1_{u^r}\otimes R_u\otimes 1_{u^s}\eta')=\overline{\lambda E_{r+s}E_{r+s+1}(S'(p^{(2s)}_{r-s,2})^*)}\otimes\eta'$,

Properties

• If $N \subset M$ is amenable in the sense of Popa the $A_o(F)$–ergodic action is not embedable into the translation action for $[M : N] \notin \mathbb{N}$.

• quantum multiplicities are integral

• If $[M : N] = n \in \mathbb{N}$, we may choose $A_o(n)$, of Kac type, and in this case we find an ergodic algebra with a tracial invariant state.

Problems

• Jones: Is the construction of $\mathcal{C}_{N\subset M}$ related to Guionnet–Jones–Shlyakhtenko construction of a subfactor from a planar algebra?

• Determine the type of the associated von Neumann completions in the GNS representation of the invariant state.

• In particular, we get an action of $S_{-\mu}U(2)$ for $\mu+\mu^{-1} = [M : N]$. When is the resulting $\mathcal{C}_{N\subset M}$ amenable? They are perhaps distinguished by the standard invariant of $N \subset M_1$.

Gaps

Not all ergodic actions arise from pairs of tensor functors

$$\text{Rep}(G) \xleftarrow{\tau} \mathcal{A} \xrightarrow{\mu} \mathcal{T}$$

The simplest gaps are the ergodic actions of

• $G = SU(2)$ on M_n, $n \geq 2$,

• finite groups on M_n with full spectrum but low multiplicities

We may include all ergodic actions starting with

$$\text{Rep}(G) \xleftarrow{\tau} \mathcal{A} \xrightarrow{\mu} \mathcal{T}$$

with μ a quasitensor functor.

Rather surprisingly, the C^*–algebra and bimodule construction may be constructed (computations are more complicated) the only difference being that $\mathrm{Ind}(\mu_u)$ is finite projective rather than free.

This generalization shows that there may be different induction theories on the same noncommutative ergodic space, when we vary μ. As, if we have a tensor embedding

$$\mu : \mathrm{Rep}(G) \to \mathcal{T}$$

we also have a quasitensor one

$$\mathrm{Rep}(G) \to \mathcal{T} \stackrel{\text{minimal}}{\to} \text{Hilbert spaces}$$

giving rise to an inner left module structure.

However,
constructing directly examples of quasitensor functors to non-embedable categories

$$\mathrm{Rep}(G) \to \mathcal{T}$$

seems difficult. Perhaps insight gained from their structure might shed light.

Embedding a TL subcategory

$TL_{\pm d}$:=The universal tensor *−category with objects $\mathbb{N}_0$ and arrows generated by a single $R \in (0, 2)$,

$$R^* \otimes 1 \circ 1 \otimes R = \pm 1,$$

$$d = R^* R$$

is the Temperley-Lieb category, the categorical counterpart of the TL algebras

(often defined without reference to the *−operation).

• $TL_{\pm d}$ is simple except for $d = 2\cos\pi/m$, when it has a single non-zero proper tensor ideal $\mathcal{J}$.

• $TL_{\pm d}$ for $d \geq 2$ and $TL_{\pm d}/\mathcal{J}$ for $d = 2\cos\pi/m$, are tensor C^*−categories (Goodman–Wenzl).

Generalization to the non self-conjugate case

Consider the universal tensor *–category $\mathcal{T}_d$ generated by two objects x, $\overline{x}$ and two arrows $R \in (\iota, \overline{x} \otimes x)$, $\overline{R} \in (\iota, x \otimes \overline{x})$ s.t.

$$\overline{R}^* \otimes 1_x \circ 1_x \otimes R = 1_x, \quad R^* \otimes 1_{\overline{x}} \circ 1_{\overline{x}} \otimes \overline{R} = 1_{\overline{x}},$$

$$R^*R = \overline{R}^*\overline{R} = d.$$

Goodman–Wenzl's theorem extends to $\mathcal{T}_d$

- For $d = 2\cos\pi/m$, the quotient categories of $TL_{\pm d}$ and $\mathcal{T}_d$ can not be embedded into the Hilbert spaces, as these values are not taken by Woronowicz quantum dimension

- For $d \geq 2$, $TL_{\pm d}$ and $\mathcal{T}_d$ are embedable.

More precisely: For any $F \in M_n$ satisfying

$$\text{Tr}(FF^*) = \text{Tr}((FF^*)^{-1}) = d,$$

(and $F\overline{F} = \pm I$, resp.) there is an isomorphism

$$\mathcal{T}_d \to \text{Rep}(A_u(F))$$

$$(TL_{\pm d} \to \text{Rep}(A_o(F)), \text{resp.})$$

Summary

Theorem Let $\mathcal{T}$ be a tensor C^*–category with a real or pseudoreal generating object ρ with $d(\rho) \geq 2$. Then for any invertible matrix F s.t.

$$\text{Trace}(FF^*) = d(\rho), \quad F\overline{F} = \pm I,$$

there is a Frobenius tensor *–isomorphism

$$\mathcal{T} \to \text{Bimod}(A_o(F))$$

Similar conclusions if the objects of $\mathcal{T}$ are generated by two conjugate objects ρ, $\overline{\rho}$ with $d(\rho) \geq 2$. The cqg is now $A_u(F)$.

The above result sheds some light on the problem of recognizing which non-permutation symmetric tensor categories are embedable into the Hilbert spaces:

This problem is related to the H–K L S and Jones problems.

After Takesaki and H–K L S:

Theorem If we have Rep$(G) \to \mathcal{T}$ with G a compact Lie group, and if $\mathcal{C}''$ is of type I then there is a tensor embedding

$$\mathcal{T} \to \text{Hilbert spaces}$$

as a full subcategory of Rep(K), with K a closed subgroup of G.

Remark: It is proved by classifying the full bimodule representations of G.

After Wassermann:

Corollary A tensor C^*–category $\mathcal{T}$ with a distinguished pseudoreal generating object ρ with $d(\rho) = 2$ admits

$$\mathcal{T} \to \text{Hilbert spaces}$$

image is now a full subcategory of Rep(K) with $K < SU(2)$.

Concluding remarks

• In a work in progress we are considering a theory of induction for a pair of quasitensor functors

$$\mathcal{S} \xleftarrow{\tau} \mathcal{A} \xrightarrow{\mu} \mathcal{T}.$$

This more general setting is perhaps helpful to relate to JGS construction from a planar algebra.

• I have tried to described a map

$$\{\text{certain tensor } C^*\text{–categories}\} \to$$

$$\{\text{nc } G\text{–spaces and their Hermitian } G\text{–bundles}\}$$

The description of these spaces naturally emphasizes a spectral viewpoint. It would be interesting to try to pursue the geometric aspect.