Canonical Weyl Operators on κ-Minkowski Spacetime

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Vietri sul mare, September 4, 2009

On the occasion of the 70^{th} birthday of John Roberts.

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Why coordinate quantisation?

First proposal: Snyder ('49). Motivations: to mimick lattice regularisation in a Lorentz covariant way. Superseded by the success of the Renormalisation Programme.

Known letters between Heisenberg and Pauli, never published (probably because they where not keen to break Lorentz covariance). Motivations: why should spacetime remain classical? (general philosophy).

Doplicher et al ('94). Motivations: to enforce spacetime stability under localisation *alone*! Energy transfer to geometric background due to localisation could produce a closed horyzon preventing localisation itself (paradoxical).

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κ-Minkowski relations

There's another model on market: the κ-minkowski spacetime.
We will discuss quantised coordinates $q^0, \ldots, q^d$ fulfilling

$$[q^0, q^j] = \frac{i}{\kappa} q^j, \quad [q^j, q^k] = 0.$$

This model known as κ-Minkowski Spacetime: first proposed by Lukierski Ruegg ('91), Majid Ruegg ('94).

- Original motivation: quest for Hopf-algebraic deformations of group Lie algebras (quantum groups)
- Renewed interest: as a toy model in the framework of spacetime quantisation (towards Quantum Gravity?).

This model however analysed mainly from the algebraic viewpoint. What about C*-algebras? Representations?
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One step back: Quantum Mechanics and Dear Old Weyl Quantisation

With the canonical commutation relations

$$[P, Q] = -i\hbar I,$$

we need a quantisation prescriptions from functions $f = f(p, q)$ of the canonical coordinates (p, q) of classical phase space.
Weyl solution:

$$f(P, Q) = \int d\alpha \, d\beta \, \check{f}(\alpha, \beta) e^{i(\alpha P + \beta Q)},$$

where

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Merits of Weyl Prescrition

- $\bar{f}(P,Q) = f(P,Q)^*$ and in particular the quantisation of a real function is selfadjoint.
- if f is a function of p alone, Weyl prescription is the same as the replacement $p \to P$ in the sense of functional calculus (hence spectral mapping). Analogously for Q

Note that $e^{i(\alpha P+\beta Q)}$ is precisely the Weyl quantisation of $e^{i(\alpha p+\beta q)}$ (internal consistency).
The product defined implicitly by

$$(f \star g)(P,Q) = f(P,Q)g(P,Q)$$

can be explicitly computed from the Weyl relations:

$$e^{i(\alpha P+\beta Q)}e^{i(\alpha' P+\beta' Q)} = e^{i(\alpha\beta'-\alpha'\beta)/2}e^{i((\alpha+\alpha')P+(\beta+\beta')Q)}$$

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Representations of κ-Minkowski (1+1)

We fix $d = 2$, $\kappa = 1$ in absolute units. (T="time", X="space")
Def. $[T, X] = iX$ in the regular (i.e. Weyl) form if

$$e^{i\alpha T} e^{i\beta X} = e^{i\beta e^{-\alpha} X} e^{i\alpha T}, \quad \alpha, \beta \in \mathbb{R}. \tag{1}$$

Prop. Let P, Q be Schröd. ops on $\mathbb{R}$. The universal representation

$$(T, X) = (T_-, X_-) \oplus (T_0, X_0) \oplus (T_+, X_+),$$

where

$$(T_+, X_+) = (P, e^{-Q}), \quad (T_0, X_0) = (Q, 0), \quad (T_-, X_-) = (P, -e^{-Q})$$

contains (up to multiplicities) any regular representation of (1).
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Weyl Operators?

Problem: Given (T, X) reg. rep., find explicit form of the Weyl Operators

$$W(\alpha, \beta) = e^{i(\alpha T + \beta X)}.$$

Strategy: They provide the unique solution of:

$$W(\alpha, 0) = e^{i\alpha T}, \quad W(0, \beta) = e^{i\beta X}, \tag{2}$$

$$W(\alpha, \beta)^{-1} = W(\alpha, \beta)^{*}, \tag{3}$$

$$W(\lambda\alpha, \lambda\beta)W(\lambda'\alpha, \lambda'\beta) = W((\lambda + \lambda')\alpha, (\lambda + \lambda')\beta). \tag{4}$$

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Explicit Solution !

$$e^{i\alpha T + i\beta X} = e^{i\alpha T} e^{i\beta \frac{e^{\alpha}-1}{\alpha} X}. \tag{5}$$

Check: Compute derivatives (Stone-von Neumann thm).
With $T_{\pm} = P, X_{\pm} = \pm e^{-Q}$,

$$(e^{i\alpha T_{\pm} + \beta X_{\pm}} \xi)(s) = (e^{i(\alpha P \pm \beta e^{-Q})} \xi)(s) = e^{\pm i\beta \frac{1-e^{-\alpha}}{\alpha} e^{-s}} \xi(s+\alpha), \quad \xi \in L^2($$

With $T_0 = Q, X_0 = 0$,

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Rem:P, Q are not quantum mechanical momentum and position, only mathematical analogy:
$Q|s\rangle = s|s\rangle \Leftrightarrow X_{\pm}|s\rangle = \pm e^{-s}|s\rangle$.

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“Keisenberg” Group

The product of two $W(\alpha, \beta)$ is again such (not up to a constant as in the CCR case). They form a subgroup of the unitary group and $\mathbb{R}^2$ inherits a group law:

$$(\alpha_1, \beta_1)(\alpha_2, \beta_2) = (\alpha_1 + \alpha_2, w(\alpha_1 + \alpha_2, \alpha_1)e^{\alpha_2}\beta_1 + w(\alpha_1 + \alpha_2, \alpha_2)\beta_2),$$

where

$$w(\alpha, \alpha') = \frac{\alpha(e^{\alpha'} - 1)}{\alpha'(e^{\alpha} - 1)}. \tag{6}$$

Rem:
$w(\alpha, \alpha') > 0, \quad w(0, 0) = 1, \quad w(\alpha_1, \alpha_2)w(\alpha_2, \alpha_3) = w(\alpha_1, \alpha_3).$
By construction $W(\alpha, \beta)$ provide a strongly continuous unitary representation of the resulting "Keisenberg" group H (faithful if (T, X) is not trivial).

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Unveiling the "Keisenberg" Group

H is isomorphic to the so called "$ax + b$" group, or to

$$\left\{ \begin{pmatrix} e^a & 0 \\ b & 1 \end{pmatrix} : (a, b) \in \mathbb{R}^2 \right\} \subset GL(2).$$

The (real) Lie algebra of H has two generators u, v fulfilling

$$[u, v] = -v.$$

For every unitary representation W of H, (T, X) defined by

$$W(\mathrm{Exp}\{\lambda u\}) = e^{i\lambda T}, \;\; W(\mathrm{Exp}\{\lambda v\}) = e^{i\lambda X}$$

are a regular representation of the κ-Minkowski relations.

This choice of Weyl operators is canonical in the sense that it does not depend on any choice of order of operator products (e.g. "time-first", [Agostini,...]).

The explicit form allows to go beyond formal computations based on BCH or "standalone" theories of star product [Agostini et al, Gracia-Bondia et al, Kosinski et al].

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Quantisation *à la Weyl*

$W(\alpha, \beta)$ are the quantised "plane waves". Following Weyl we define the quantisation

$$f(T, X) = \int d\alpha \; d\beta \; \check{f}(\alpha, \beta) e^{i(\alpha T + \beta X)},$$

where

$$\check{f}(\alpha, \beta) = \frac{1}{(2\pi)^2} \int dt \; dx \; f(t, x) e^{-i(\alpha t + \beta x)},$$

and (T, X) is the universal representation of the relations (1). Notation for the components:

$$f(T, X) = f(T_-, X_-) \oplus f(T_0, X_0) \oplus f(T_+, X_+).$$

This quantisation is "good" (in the previously discussed sense).

Twisted Products

The operator product of quantised symbold induced a twisted product on the symbols themselves:

$$f(T, X)g(T, X) = (f \star g)(T, X)$$

provides $\star$-product of admissible symbols, explicitly given by

$$(f \star g)\check{}(\alpha, \beta) = \int d\alpha' d\beta' \; w(\alpha - \alpha', \alpha)\check{f}(\alpha', \beta')$$
$$\check{g}(\alpha - \alpha', w(\alpha - \alpha', \alpha)\beta - w(\alpha - \alpha', \alpha')e^{\alpha - \alpha'}\beta').$$

It's hard to study $\star$ directly. In the case of the CCR the best way is to realize that $f \mapsto f(P, Q)$ relates to the representation of the group algebra of the Heisenberg group.

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Haar Magic

It is not so in our case, because the “Keisenberg” group H is not unimodular. its left Haar measure and modular function are

$$d\mu(\alpha,\beta) = \frac{e^\alpha - 1}{\alpha} d\alpha\, d\beta, \quad \Delta(\alpha,\beta) = e^\alpha.$$

Something magic happens: with the bijective isometry

$$u : L^1(\mathbb{R}^2) \to L^1(H), \quad (u\varphi)(\alpha,\beta) = \frac{\alpha}{e^\alpha - 1}\varphi(\alpha,\beta),$$

the *-representation

$$\pi(\varphi) := \int d\mu(\alpha,\beta)\varphi(\alpha,\beta)W(\alpha,\beta), \quad \varphi \in L^1(H),$$

of the group algebra $L^1(H)$ fulfils

$$f(T,X) = \pi(u\check{f}) \quad \text{but} \quad \neq \pi(\check{f}).$$

This makes $\star$ O.K.:

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Haar Magic

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$$d\mu(\alpha,\beta) = \frac{e^\alpha - 1}{\alpha} d\alpha\, d\beta, \quad \Delta(\alpha,\beta) = e^\alpha.$$

Something magic happens: with the bijective isometry

$$u : L^1(\mathbb{R}^2) \to L^1(H), \quad (u\varphi)(\alpha,\beta) = \frac{\alpha}{e^\alpha - 1}\varphi(\alpha,\beta),$$

the *-representation

$$\pi(\varphi) := \int d\mu(\alpha,\beta)\varphi(\alpha,\beta)W(\alpha,\beta), \quad \varphi \in L^1(H),$$

of the group algebra $L^1(H)$ fulfils

$$f(T,X) = \pi(u\check{f}) \quad \text{but} \quad \neq \pi(\check{f}).$$

This makes $\star$ O.K.:

$$(f \star g)(T,X) = \pi(u\check{f})\pi(u\check{g}).$$

Schrödinger Op's Forever!

We note that $X_\pm$ looks like the quantisation of the restriction of x to $\pm(0, \infty)$.

- If $f(\cdot, x) = g(\cdot, x), \quad \pm x \in (0, \infty)$, then $f(T_\pm, X_\pm) = g(T_\pm, X_\pm)$.
- If $f(\cdot, 0) = g(\cdot, 0)$, then $f(T_0, 0) = g(T_0, 0)$ in the sense of functional calculus,
- the map

$$(\gamma_\pm f)(t, x) = \int d\alpha \;\, e^{i\alpha t}\, f^{\check{}\otimes \mathrm{id}}\left(\alpha, \pm\frac{e^{\alpha/2} - e^{-\alpha/2}}{\alpha} e^{-x}\right)$$

fulfils

$$(\gamma_\pm f)(P, Q) = f(T_\pm, X_\pm).$$

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Trace and C*-Algebra

Big deal! Just using the properties of the CCR Weyl quantisation, and positivity and cyclicity the operator trace *Tr*: if $\gamma_{\pm} f \in L^1(\mathbb{R}^2)$, then $f(T, X)$ is trace class and

$$Tr\, f(T, X) = \tau(f), \quad \text{where} \quad \tau = \tau_- + \tau_+$$

and

$$Tr\, f(T_\pm, X_\pm) =: \tau_\pm(f) = (\gamma_\pm f)\check{}\,(0,0) = \int dt\, dx\, f(t, \pm e^{-x}).$$

If $\sigma = \tau, \tau_-, \tau_+$,

$$\sigma(\bar{f} \star f) \geqslant 0, \quad \sigma(f \star g) = \sigma(g \star f).$$

(If $f(\cdot, 0) \neq 0$, $f(T, X)$ is not trace class).
We can also determine that the C*-algebra of the relations (1) is

$$\mathcal{A} = \mathcal{K} \oplus \mathcal{C}_\infty(\mathbb{R}) \oplus \mathcal{K},$$

where $\mathcal{K}$ is the algebra of compact operators.

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Large κ Limit

Another consequence:
the classical limit of each $\pm$ component is the same limit as the small $\hbar$ limit of the CCR up to restrictions to open halflines, which then are kept separated from each other and the origin.
Said differently, X has continuous spectrum $\mathbb{R}\backslash\{0\}$ and pure point $\{0\}$ for all κ.
This survives and the large κ (classical) limit of $d = 2$ κ-Mikowski is

$$\mathbb{R} \times \tilde{\mathbb{R}},$$

where

$$\tilde{\mathbb{R}} = (-\infty, 0) \sqcup \{0\} \sqcup (0, \infty).$$

Uncertainty Relations

$$\Delta_\omega(T)\Delta_\omega(X) \geqslant \frac{1}{2}\omega(|[T,X]|) = \frac{1}{2}\omega(|X|)$$

for a state ω with $\omega(X)$ small give no obstructions to have small product of uncertainties.
Indeed we actually found, for every $\varepsilon, \eta > 0$, a non trivial pure vector state Ψ in the domain of T, X, such that

$$\Delta_\Psi(T) < \varepsilon, \quad \Delta_\Psi(X) < \eta.$$

This means that there is no limit on the localisation precision of all the spacetime coordinates, at least in the region close to the space origin (which thus is asymptotically classical at small distances from the origin).
This is in plain contrast with the standard motivations for spacetime quantisation, namely to prevent the formation of closed horizons as an effect of localisation *alone* (see [DFR]).

Conclusions

On the mathematical side, we found an explicit quantisation prescription of κ-Minkowski, which realises precisely the underlying relations, its C*-algebra, and computed the trace. Instead regarding interpretation, we find some problematic features:

- The main motivation for spacetime quantisation, namely to prevent arbitrarily precise localisation (which could lead to horizon formation) is lost for this model.
- On the contrary, the noncommutativity grows ad large distances.
- The macroscopic limit exhibits a pathological (?) topology which should manifest itself as 'impenetrable barriers' breaking the flat Minkowski spacetime in disconnected regions.
- Recall also the well known fact that Lorentz and translation covariance are broken.

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Bibliography

- A. Agostini, *κ-Minkowski representations on Hilbert spaces*, J. Math. Phys. **48**, 052305 (2007).
- A. Agostini, F. Lizzi and A. Zampini, *Generalized Weyl Systems and κ-Minkowski space*, Mod. Phys. Lett. A **17** 2105-2126 (2002).
- S. Doplicher, K. Fredenhagen and J. E. Roberts, *The quantum structure of spacetime at the Planck scale and quantum fields*, Commun. Math. Phys. **172** 187–220 (1995).
- J. M. Gracia Bondía, F. Lizzi, G. Marmo and P. Vitale, *Infinitely many star products to play with*,
- P. Kosinski, J. Lukierski and P. Maslanka, *Local Field Theory on κ-Minkowski Space, Star Products and Noncommutative Translations*, Czech. J. Phys. **50**, 1283 (2000).
- J. Lukierski, A. Nowicki, H. Ruegg and V. N. Tolstoy, *q-Deformation of Poincaré algebra*, Phys. Lett. B **268**, 331–338 (1991).
- J. Lukierski, A. Novicki and H. Ruegg, *New quantum Poincaré algebra and κ-deformed field theory*, Phys. Lett. B **293**, 344–352 (1992).
- S. Majid and H. Ruegg, *Bicrossproduct structure of κ-Poincaré group and noncommutative geometry*, Phys. Lett. B **334**, 348–354 (1994).
- J. von Neumann, *Uber die Eindeutigkeit der Schrödingerschen Operatoren*, Math. Annalen **104**, 570–578 (1931).
- H. Weyl, *Gruppentheorie und Quantenmechanik*, Hirzel, Leipzig, 1928.